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Research Article

The Stability of Multi-Coefficients Pexider Additive Functional Inequalities in Banach Spaces

Yang Liu,[1] **Gang Lyu**,[2] **Yuanfeng Jin**,[1] **and Jiangwei Yang**[2]

[1]*Department of Mathematics, Yanbian University, Yanji 133001, China*
[2]*School of General Education, Guangzhou College of Technology and Business, Guangzhou 510850, China*

Correspondence should be addressed to Gang Lyu; lvgang@gzgs.edu.cn and Yuanfeng Jin; yfkim@ybu.edu.cn

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The Hyers–Ulam stability of multi-coefficients Pexider additive functional inequalities in Banach spaces is investigated. In order to do this, the fixed point method and the direct method are used.

1. Introduction and Preliminaries

For an object possessing some properties only approximately in mathematics and in many other scientific investigations, can one find the special object satisfied them truly? One of the effective methods to solve this problem is to use the concept of generalized Hyers–Ulam stability.

Let us review the definition of Hyers–Ulam stability. In a class of mappings, if each mapping of this class fulfilling the equation approximately is "near" to its real solution or stable approximate solution, then the equation is said to be Hyers–Ulam stability.

The stability problem of functional equations is from a question of Ulam [1] in 1940, that is, the stability of metric group homomorphisms. In 1941, Hyers [2] gave the first affirmative answer to the question of Ulam for Banach spaces about the Cauchy functional equation. Hyers' method of proof is called the "direct method." The functional equation

$$f(x+y) = f(x) + f(y) \tag{1}$$

is called an additive functional equation. More generalizations and applications of the Hyers–Ulam stability to a number of functional equations and mappings can be found in [3–10].

In 2013, Li et al. [11] investigated the generalized Hyers–Ulam stability of the following function inequalities:

$$\begin{aligned}
&\|\mathrm{af}(x) + \mathrm{bf}(y) + \mathrm{cf}(z)\| \leq \left\|\mathrm{Kf}\left(\frac{\mathrm{ax}+\mathrm{by}+\mathrm{cz}}{K}\right)\right\|, \quad (0 < |K| < |a+b+c|),\\
&\|\mathrm{af}(x) + \mathrm{bf}(y) + \mathrm{Kf}(z)\| \leq \left\|\mathrm{Kf}\left(\frac{\mathrm{ax}+\mathrm{by}}{K} + z\right)\right\|, \quad (0 < K \neq 2),
\end{aligned} \tag{2}$$

in quasi-Banach spaces. In the paper, assume that X is a linear space over the field $\mathbb{F}$, and Y is a linear space over the field $\mathbb{K}$. Let $a, b \in \mathbb{F}$ and $A, B \in \mathbb{K}$ be given scalars.

The functional equation

$$f(x+y) = g(x) + h(x) \tag{3}$$

is called a Pexider additive functional equation (for more details, see [12–23]). In the paper, we introduce and investigate the following functional equation:

$$f(\mathrm{ax}+\mathrm{by}) = \mathrm{Ah}(x) + \mathrm{Bg}(y), \quad \forall x, y \in X, \tag{4}$$

where f, g, h: $X \longrightarrow Y$. The stability problems of several functional inequalities have been extensively investigated by a number of authors (see [24–47]).

In order to find the stability of (4), the following fixed point theory would be applied.

Theorem 1 (see [48, 49]). *Let (X, d) be a complete generalized metric space and let J: $X \longrightarrow X$ be a strictly contractive mapping with Lipschitz constant $L < 1$. Then for each given element $x \in X$, either*

$$d\left(J^n x, J^{n+1} x\right) = \infty, \tag{5}$$

for all nonnegative integers n or there exists a positive integer n_0 such that

(1) $d(J^n x, J^{n+1} x) < \infty$, for all $n \geq n_0$

(2) The sequence $\{J^n x\}$ converges to a fixed point y^* of J

(3) y^* is the unique fixed point of J in the set $Y = \{y \in X \mid d(J^{n_0} x, y) < \infty\}$

(4) $d(y, y^*) \leq 1/1 - Ld(y, Jy)$ for all $y \in Y$

2. Hyers–Ulam Stability of Functional Inequality (4): A Fixed Point Method

Theorem 2. *Suppose that Y is a Banach space and φ: $X^2 \longrightarrow [0, \infty)$ is a function such that there exists an $L < 1$ with*

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2} \varphi(x, y), \quad x, y \in X. \tag{6}$$

If f, h, g: $X \longrightarrow Y$ are mappings satisfying $g(0) = h(0) = 0$ and

$$\|f(\mathrm{ax}+\mathrm{by}) - \mathrm{Ah}(x) - \mathrm{Bg}(y)\| \leq \varphi(x, y), \quad x, y \in X, \tag{7}$$

then there exists a unique solution H: $X \longrightarrow Y$ of (4) such that

$$\|f(x) - H(x)\| \leq \frac{L}{2(1-L)} \left\{ \varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) \right\}, \quad x \in X; \tag{8}$$

$$\left\| h(x) - \frac{1}{A} H(\mathrm{ax}) \right\| \leq \frac{1}{|A|} \frac{L}{2(1-L)} \left\{ \varphi\left(x, \left(\frac{a}{b}\right)x\right) + \varphi(x, 0) + \varphi\left(0, \left(\frac{a}{b}\right)x\right) \right\} + \frac{1}{|A|} \varphi(x, 0), \quad x \in X; \tag{9}$$

$$\left\| g(x) - \frac{1}{B} H(\mathrm{bx}) \right\| \leq \frac{1}{|B|} \frac{L}{2(1-L)} \left\{ \varphi\left(\left(\frac{b}{a}\right)x, x\right) + \varphi\left(\left(\frac{b}{a}\right)x, 0\right) + \varphi(0, x) \right\} + \frac{1}{|B|} \varphi(0, x), \quad x \in X. \tag{10}$$

Proof. Letting $x = y = 0$ in (7), we get $f(0) = 0$. Letting $x = 0$ in (7), we obtain

$$\|f(\mathrm{by}) - \mathrm{Bg}(y)\| \leq \varphi(0, y), \tag{11}$$

for all $y \in X$. Thus,

$$\left\| f(y) - \mathrm{Bg}\left(\frac{y}{b}\right) \right\| \leq \varphi\left(0, \frac{y}{b}\right), \quad y \in X. \tag{12}$$

Letting $y = 0$ in (7), we have

$$\|f(\mathrm{ax}) - \mathrm{Ah}(x)\| \leq \varphi(x, 0), \quad x \in X. \tag{13}$$

Thus,

$$\left\| f(x) - \mathrm{Ah}\left(\frac{x}{a}\right) \right\| \leq \varphi\left(\frac{x}{a}, 0\right), \quad x \in X. \tag{14}$$

Next, replacing y by y/b and x by x/a in (7), we get

$$\left\| f(x+y) - \mathrm{Ah}\left(\frac{x}{a}\right) - \mathrm{Bg}\left(\frac{y}{b}\right) \right\| \leq \varphi\left(\frac{x}{a}, \frac{y}{b}\right). \tag{15}$$

Thus,

$$\begin{aligned} \|f(x+y) - f(x) - f(y)\| &\leq \left\| f(x+y) - \mathrm{Ah}\left(\frac{x}{a}\right) - \mathrm{Bg}\left(\frac{y}{b}\right) \right\| \\ &\quad + \left\| f(x) - \mathrm{Ah}\left(\frac{x}{a}\right) \right\| + \left\| f(y) - \mathrm{Bg}\left(\frac{y}{b}\right) \right\| \\ &\leq \varphi\left(\frac{x}{a}, \frac{y}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{y}{b}\right), \quad x, y \in X. \end{aligned} \tag{16}$$

Letting $x = y$ in (16), we get

$$\left\|f(2x) - 2f(x)\right\| \le \varphi\left(\frac{x}{a}, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right), \quad (17)$$

for all $x \in X$.

Consider the set

$$S := \{h: X \longrightarrow Y, h(0) = 0\}, \quad (18)$$

and introduce the generalized metric d on S:

$$d(p,q) = \inf\left\{\mu \in [0,\infty]: \left\|p(x) - q(x)\right\| \le \mu\left(\varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, \frac{x}{b}\right)\right), \quad \forall x \in X\right\}. \quad (19)$$

Then (S,d) will be proved to be complete. Let $d(p,q) = \mu_1$ and $d(p,h)$; by the definition of d and property of infimum, d satisfies the triangle inequality. Suppose that $\{f_n\}$ is d-Cauchy sequence on S. That is, for any $\tau > 0$, $\exists n_0$, $n > m > n_0$, such that $d(f_n, f_m) < \tau$. By the definition of d, it is easy to see that $\{f_n(x)\}$ is a Cauchy sequence in Y. Since Y is complete, there exist $\{f_0(x)\} \subseteq Y$ and $\{f_n(x)\} \longrightarrow \{f_0(x)\}$. Taking the limit as $m \longrightarrow \infty$, we get $d(f_n(x), f_0(x)) < \tau$, for all $n \ge n_0$, such that $\{f_n\}$ is d-convergent, i.e., (S,d) is a complete generalized metric (for more details, we refer to [48]).

Now, we consider the linear mapping $J: S \longrightarrow S$ such that

$$\mathrm{Jp}(x) := 2p\left(\frac{x}{2}\right), \quad (20)$$

for all $x \in X$.

Let $p, q \in S$ be given such that $d(p,q) = \varepsilon$. Then,

$$\left\|p(x) - q(x)\right\| \le \varepsilon\left(\varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, \frac{x}{b}\right)\right), \quad (21)$$

for all $x \in X$. Hence,

$$\begin{aligned} \|\mathrm{Jp}(x) - \mathrm{Jq}(x) &= \left\|2p\left(\frac{x}{2}\right) - 2q\left(\frac{x}{2}\right)\right\| \le 2\varepsilon\left\{\varphi\left(\frac{x}{2a}, 0\right) + \varphi\left(0, \frac{x}{2b}\right) + \varphi\left(\frac{x}{2a}, \frac{x}{2b}\right)\right\} \\ &\le 2\varepsilon\frac{L}{2}\left\{\varphi\left(0, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(\frac{x}{a}, \frac{x}{b}\right)\right\} = L\varepsilon\left\{\varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, \frac{x}{b}\right)\right\}, \end{aligned} \quad (22)$$

for all $x \in X$. So, $d(p,q) = \varepsilon$ implies that $d(J_p, J_q) \le L_\varepsilon$. This means that

$$d(\mathrm{Jp}, \mathrm{Jq}) \le \mathrm{Ld}(p,q), \quad (23)$$

for all $p, q \in S$.

It follows from (17) that

$$\left\|f(x) - 2f\left(\frac{x}{2}\right)\right\| \le \frac{L}{2}\left\{\varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, \frac{x}{b}\right)\right\}, \quad (24)$$

for all $x \in X$. So, $d(f, \mathrm{Jf}) \le L/2$.

By Theorem 1, there exists a mapping $H: X \longrightarrow Y$ satisfying the following:

(1) H is a fixed point of J, i.e.,

$$H(x) = 2H\left(\frac{x}{2}\right), \quad (25)$$

for all $x \in X$. The mapping H is a unique fixed point of J in the set

$$M = \{p \in S: d(f,p) < \infty\}. \quad (26)$$

This implies that H is a unique mapping satisfying (45) such that there exists a $\mu \in (0,\infty)$ satisfying

$$\left\|f(x) - H(x)\right\| \le \mu\left\{\varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, \frac{x}{b}\right)\right\}, \quad (27)$$

for all $x \in X$.

(2) $d(J^n f, H) \longrightarrow 0$ as $n \longrightarrow \infty$. This implies the equality

$$\lim_{n\longrightarrow\infty} 2^n f\left(\frac{x}{2^n}\right) = H(x), \quad (28)$$

for all $x \in X$.

(3) $d(f,H) \le 1/1 - Ld(f, \mathrm{Jf})$, which implies

$$\begin{aligned} &\left\|f(x) - H(x)\right\| \\ &\le \frac{L}{2(1-L)}\left\{\varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{x}{b}\right) + \varphi\left(\frac{x}{a}, \frac{x}{b}\right)\right\}, \end{aligned} \quad (29)$$

for all $x \in X$.

It follows from (16) and (28) that

$$\begin{aligned}&\left\|H(x+y)-H(x)-H(y)\right\|\\&=\lim_{n\longrightarrow\infty}2^{n}\left\|f\left(\frac{x+y}{2^{n}}\right)-f\left(\frac{x}{2^{n}}\right)-f\left(\frac{y}{2^{n}}\right)\right\|\\&\leq\lim_{n\longrightarrow\infty}2^{n}\left\{\varphi\left(\frac{x}{2^{n}a},0\right)+\varphi\left(0,\frac{x}{2^{n}b}\right)+\varphi\left(\frac{x}{2^{n}a},\frac{x}{2^{n}b}\right)\right\}\\&=0,\quad\forall x,y\in X.\end{aligned}\tag{30}$$

So, the mapping $H\colon X\longrightarrow Y$ is additive. Next, by (8), (29) can be proved. Similarly, we can obtain inequalities (9) and (10). □

Corollary 3. *Let $r>1$ and θ be nonnegative real numbers and $f,h,g\colon X\longrightarrow Y$ be mappings satisfying*

$$\left\|f(\mathrm{ax}+\mathrm{by})-\mathrm{Ah}(x)-\mathrm{Bg}(y)\right\|\leq\theta\left(\|x\|^{r}+\|y\|^{r}\right),\tag{31}$$

for all $x,y\in X$ and $h(0)=g(0)=0$. Then there exists a unique additive mapping $H\colon X\longrightarrow Y$ such that

$$\begin{aligned}&\left\|f(x)-H(x)\right\|\leq\frac{\theta L}{1-L}\frac{|a|^{r}+|b|^{r}}{|ab|^{r}}\|x\|^{r};\\&\left\|h(x)-\frac{1}{A}H(\mathrm{ax})\right\|\leq\frac{\theta}{|A|}\frac{L}{1-L}\frac{|a|^{r}+|b|^{r}}{|b|^{r}}\|x\|^{r}+\frac{\theta}{|A|}\|x\|^{r};\\&\left\|g(x)-\frac{1}{B}H(\mathrm{bx})\right\|\leq\frac{\theta}{|B|}\frac{L}{1-L}\frac{|a|^{r}+|b|^{r}}{|a|^{r}}\|x\|^{r}+\frac{\theta}{|B|}\|x\|^{r},\quad\forall x\in X.\end{aligned}\tag{32}$$

Theorem 4. *Let $\varphi\colon X^{2}\longrightarrow[0,\infty)$ be a function such that there exists an $L<1$ with*

$$\varphi(2x,2y)\leq L\varphi(x,y),\tag{33}$$

for all $x,y\in X$. Let $f,h,g\colon X\longrightarrow Y$ be mappings satisfying (7) for all $x,y\in X$ and $h(0)=g(0)=0$. Then there exists a unique additive mapping such that

$$\left\|f(x)-H(x)\right\|\leq\frac{1}{2(1-L)}\left\{\varphi\left(\frac{x}{a},\frac{x}{b}\right)+\varphi\left(\frac{x}{a},0\right)+\varphi\left(0,\frac{x}{b}\right)\right\},\quad x\in X;\tag{34}$$

$$\left\|h(x)-\frac{1}{A}H(\mathrm{ax})\right\|\leq\frac{1}{|A|}\frac{1}{2(1-L)}\left\{\varphi\left(x,\left(\frac{a}{b}\right)x\right)+\varphi(x,0)+\varphi\left(0,\left(\frac{a}{b}\right)x\right)\right\}+\frac{1}{|A|}\varphi(x,0),\quad x\in X;\tag{35}$$

$$\left\|g(x)-\frac{1}{B}H(\mathrm{bx})\right\|\leq\frac{1}{|B|}\frac{1}{2(1-L)}\left\{\varphi\left(\left(\frac{b}{a}\right)x,x\right)+\varphi\left(\left(\frac{b}{a}\right)x,0\right)+\varphi(0,x)\right\}+\frac{1}{|B|}\varphi(0,x),\quad x\in X.\tag{36}$$

Corollary 5. *Let $r>1$ and θ be nonnegative real numbers and $f,h,g\colon X\longrightarrow Y$ be mappings satisfying (31) for all $x,y\in X$ and $h(0)=g(0)=0$. Then there exists a unique additive mapping $H\colon X\longrightarrow Y$ such that*

$$\begin{aligned}&\left\|f(x)-H(x)\right\|\leq\frac{\theta}{1-L}\left(\frac{1}{|a|^{r}}+\frac{1}{|b|^{r}}\right)\|x\|^{r};\\&\left\|h(x)-\frac{1}{A}H(\mathrm{ax})\right\|\leq\frac{\theta}{|A|(1-L)}\frac{|b|^{r}+|a|^{r}}{|b|^{r}}\|x\|^{r}+\frac{\theta}{|A|}\|x\|^{r};\\&\left\|g(x)-\frac{1}{B}H(\mathrm{bx})\right\|\leq\frac{\theta}{|B|(1-L)}\frac{|a|^{r}+|b|^{r}}{|a|^{r}}\|x\|^{r}+\frac{\theta}{|B|}\|x\|^{r},\quad\forall x\in X.\end{aligned}\tag{37}$$

3. Hyers–Ulam Stability of Functional Inequality (4): A Direct Method

Using the direct method, we prove the Hyers–Ulam stability of functional inequality (4).

Theorem 6. *Assume that Y is a Banach space and $f, g, h: X \longrightarrow Y$ with $g(0) = h(0) = 0$ satisfy the inequality*

$$\|f(\mathrm{ax} + \mathrm{by}) - \mathrm{Ah}(x) - \mathrm{Bg}(y)\| \leq \varphi(x, y), \tag{38}$$

where $\phi: X^2 \longrightarrow [0, \infty)$ satisfies

$$\widetilde{\phi}(x, y) := \sum_{j=0}^{\infty} \left(\frac{1}{2}\right)^j \phi\left(2^j x, 2^j y\right) < \infty, \tag{39}$$

for all $x, y \in X$. Then there exists a unique additive mapping $F: X \longrightarrow Y$ such that

$$\begin{aligned} &\|f(x) - F(x)\| \leq \widetilde{\varphi}\left(\frac{x}{a}, \frac{x}{B}\right) + \widetilde{\varphi}\left(0, \frac{x}{B}\right) + \widetilde{\varphi}\left(\frac{x}{a}, 0\right), \quad \forall x \in X, \\ &\left\|h(x) - \frac{1}{A} F(\mathrm{ax})\right\| < \frac{1}{|A|}\left\{\widetilde{\phi}\left(x, \frac{a}{\mathrm{bx}}\right) + \widetilde{\phi}\left(0, \frac{a}{\mathrm{bx}}\right) + 3\widetilde{\phi}(x, 0) + \widetilde{\phi}(2x, 0)\right\}, \quad \forall x \in X, \\ &\left\|g(x) - \frac{1}{B} F(\mathrm{bx})\right\| < \frac{1}{|B|}\left\{\widetilde{\phi}\left(x, \frac{b}{\mathrm{ax}}\right) + \widetilde{\phi}\left(0, \frac{b}{\mathrm{ax}}\right) + 3\widetilde{\phi}(x, 0) + \widetilde{\phi}(2x, 0)\right\}, \quad \forall x \in X. \end{aligned} \tag{40}$$

Proof. Letting $x = y = 0$ in (38), we get $\|f(0)\| \leq \varphi(0, 0)$. So, $f(0) = 0$.

Letting $y = 0$ in (38), we get

$$\|f(\mathrm{ax}) - \mathrm{Ah}(x)\| \leq \varphi(x, 0), \tag{41}$$

for all $x \in X$. Thus,

$$\left\|f(x) - \mathrm{Ah}\left(\frac{x}{a}\right)\right\| \leq \varphi\left(\frac{x}{a}, 0\right), \tag{42}$$

for all $x \in X$.

Letting $x = 0$ in (38), we get

$$\|f(\mathrm{by}) - \mathrm{Bg}(y)\| \leq \varphi(0, y), \tag{43}$$

for all $y \in X$. In (43), replacing y by y/b, we get

$$\left\|f(y) - \mathrm{Bg}\left(\frac{y}{b}\right)\right\| < \varphi\left(0, \frac{y}{b}\right)\Big], \quad \forall y \in X. \tag{44}$$

By the same way, from (38), we have the following inequality:

$$\left\|f(x + y) - \mathrm{Ah}\left(\frac{x}{a}\right) - \mathrm{Bg}\left(\frac{y}{b}\right)\right\| \leq \varphi\left(\frac{x}{a}, \frac{y}{b}\right), \quad \forall x, y \in X. \tag{45}$$

From (42), (44), and (45), it follows that

$$\begin{aligned} \|f(x + y) - f(x) - f(y)\| &\leq \left\|f(x + y) - \mathrm{Ah}\left(\frac{x}{a}\right) - \mathrm{Bg}\left(\frac{y}{b}\right)\right\| + \left\|f(x) - \mathrm{Ah}\left(\frac{x}{a}\right)\right\| + \left\|f(y) - \mathrm{Bg}\left(\frac{y}{b}\right)\right\| \\ &\leq \varphi\left(\frac{x}{a}, 0\right) + \varphi\left(0, \frac{y}{b}\right) + \varphi\left(\frac{x}{a}, \frac{y}{b}\right) \leq \hat{\varphi}(x, y), \quad \forall x \in X, \end{aligned} \tag{46}$$

where $\hat{\varphi}(x, y) = \varphi(x/a, y/b) + \varphi(x/a, 0) + \varphi(0, y/b)$.

It follows from (46) that

$$\begin{aligned} \left\|\left(\frac{1}{2}\right)^l f\left(2^l x\right) - \left(\frac{1}{2}\right)^m f\left(2^m x\right)\right\| &\leq \sum_{j=l}^{m-1} \left\|\left(\frac{1}{2}\right)^j f\left(2^j x\right) - \left(\frac{1}{2}\right)^{j+1} f\left(2^{j+1} x\right)\right\| \\ &\leq \sum_{j=l}^{m-1} \left(\frac{1}{2}\right)^j \left[\hat{\varphi}\left(2^j x, 2^j x\right)\right], \end{aligned} \tag{47}$$

for all nonnegative integers m and l with $m > l$ and all $x \in X$. It means that the sequence $\{(1/2)^n f(2^n x)\}$ is a Cauchy sequence for all $x \in X$. Since Y is complete, the sequence $\{(1/2)^n f(2^n x)\}$ converges. We define the mapping $F: X \longrightarrow Y$ by

$$F(x) = \lim_{n\longrightarrow\infty}\left\{\left(\frac{1}{2}\right)^n f(2^n x)\right\}, \tag{48}$$

for all $x \in X$. Moreover, letting $l = 0$ and passing to the limit $m \longrightarrow \infty$, we get

$$\begin{aligned}\|f(x) - F(x)\| &\leq \sum_{n=0}^{\infty}\left(\frac{1}{2}\right)^n\left\{\varphi\left(2^n\frac{x}{a}, 2^n\frac{x}{b}\right) + \varphi\left(2^n\frac{x}{a}, 0\right) + \varphi\left(0, 2^n\frac{x}{b}\right)\right\}\\ &= \tilde{\varphi}\left(\frac{x}{a}, \frac{x}{b}\right) + \tilde{\varphi}\left(0, \frac{x}{b}\right) + \tilde{\varphi}\left(\frac{x}{a}, 0\right), \quad \forall x \in X.\end{aligned} \tag{49}$$

Similarly, there exists a mapping $H: X \longrightarrow Y$ such that $H(x) = \lim_{n\longrightarrow\infty} 1/2^n h(2^n x)$ and

$$\|h(x) - H(x)\| < \frac{1}{|A|}\left\{\tilde{\phi}\left(x, \frac{a}{\mathrm{bx}}\right) + \tilde{\phi}\left(0, \frac{a}{\mathrm{bx}}\right) + 3\tilde{\phi}(x, 0) + \tilde{\phi}(2x, 0)\right\}, \tag{50}$$

for all $x \in X$.

We also obtain a mapping $G: X \longrightarrow Y$ such that $G(x) \coloneqq \lim_{n\longrightarrow\infty} 1/2^n g(2^n x)$, and

$$\|g(x) - G(x)\| < \frac{1}{|B|}\left\{\tilde{\phi}\left(x, \frac{b}{\mathrm{ax}}\right) + \tilde{\phi}\left(0, \frac{b}{\mathrm{ax}}\right) + 3\tilde{\phi}(x, 0) + \tilde{\phi}(2x, 0)\right\}, \quad \forall x \in X. \tag{51}$$

Next, we show that F is an additive mapping.

$$\begin{aligned}\|F(x) + F(y) - F(x + y)\| &= \lim_{n\longrightarrow\infty}\left(\frac{1}{2}\right)^n\|f(2^n x) + f(2^n y) - f(2^n(x + y))\|\\ &< \lim_{n\longrightarrow\infty}\left(\frac{1}{2}\right)^n\left\{\varphi\left(2^n\frac{x}{a}, 2^n\frac{y}{b}\right) + \varphi\left(2^n\frac{x}{a}, 0\right) + \varphi\left(0, 2^n\frac{y}{b}\right)\right\} = 0,\end{aligned} \tag{52}$$

for all $x, y \in X$. Thus, the mapping $F: X \longrightarrow Y$ is additive.

Now, we prove the uniqueness of F. Assume that $T: X \longrightarrow Y$ is another additive mapping satisfying (40). We obtain

$$\|F(x) - T(x)\| = \frac{1}{2^n}\|F(2^n x) - T(2^n x)\| \leq \left(\frac{1}{2}\right)^n\left[\|F(2^n x) - f(2^n x)\| + \|T(2^n x) - f(2^n x)\|\right] \leq 2\frac{1}{2^n}\left[\hat{\varphi}(2^n x, 2^n x)\right], \tag{53}$$

which tends to zero as $n \longrightarrow \infty$ for all $x \in X$. Then we can conclude that $F(x) = T(x)$ for all $x \in X$. In fact, by (42), we get $F(X) = \mathrm{AH}(x/a)$. Similarly, we obtain $F(x) = \mathrm{BG}(x/b)$. □

Corollary 7. *Let r and θ be positive real numbers with $r > 1$. Let $f, g, h: X \longrightarrow Y$ be mappings with $g(0) = h(0) = 0$ satisfying*

$$\|f(\mathrm{ax} + \mathrm{by}) - \mathrm{Ah}(x) - \mathrm{Bg}(y)\| \leq \theta(\|x\|^r + \|x\|^r), \tag{54}$$

for all $x, y \in X$. Then there exists a unique additive mapping $F\colon X \longrightarrow Y$ such that

$$\left\| f(x) - F(x) \right\| \le \frac{2\theta}{|a|^r + |b|^r} \frac{1}{2^r - 1} \|x\|^r;$$
$$\left\| h(x) - \frac{1}{A} F(\mathrm{ax}) \right\| \le \left(\theta + \frac{2|a|^r\theta}{|a|^r + |b|^r} \frac{1}{2^r - 1} \right) \|x\|^r;$$
$$\left\| g(x) - \frac{1}{B} F(\mathrm{bx}) \right\| \le \left(\theta + \frac{2|b|^r\theta}{|a|^r + |b|^r} \frac{1}{2^r - 1} \right) \|x\|^r, \quad \forall x \in X. \tag{55}$$

4. Conclusion

In this paper, we have investigated the Hyers–Ulam stability of general Pexider function inequalities in Banach spaces by using the fixed point method and the direct method.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors equally conceived the study, participated in its design and coordination, drafted the manuscript, participated in the sequence alignment, and read and approved the final manuscript. Y. Liu conceptualized the study, developed the research question, oversaw the overall progress of the study, managed the collaboration, wrote the initial draft of the manuscript, and integrated feedback from co-authors. G. Lyu assisted in developing the methodology and research design, conducted the literature review and contributed significantly to the theoretical framework, performed data collection and carried out preliminary data analysis, and reviewed and provided substantive edits to subsequent versions of the manuscript. Y. Jin designed and implemented the statistical models for data analysis, interpreted the results, wrote the results section, participated in drafting the discussion and conclusion sections, and secured funding for the project. J. Yang proposed the application of innovative techniques used in the study, contributed to writing the methods section, supported data validation, and critically reviewed the manuscript for intellectual content and clarity.

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