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Research Article

Subgradient Extragradient Method for Finite Lipschitzian Demicontractions and Variational Inequality Problems in a Hilbert Space

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In this research, the modified subgradient extragradient method and K-mapping generated by a finite family of finite Lipschitzian demicontractions are introduced. Then, a strong convergence theorem for finding a common element of the common fixed point set of finite Lipschitzian demicontraction mappings and the common solution set of variational inequality problems is established. Furthermore, numerical examples are given to support the main theorem.

1. Introduction

Let $\mathscr{H}$ be a real Hilbert space and $\mathscr{C}$ be a nonempty closed convex subset of $\mathscr{H}$ with the inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$.

The fixed point problem for the mapping $\mathscr{S}: \mathscr{H} \longrightarrow \mathscr{H}$ is to find $u \in \mathscr{H}$ such that

$$u = \mathrm{S}u. \tag{1}$$

The term $F(\mathscr{S})$ is denoted by the set of fixed points of $\mathscr{S}$, that is, $F(\mathscr{S}) = \{x \in \mathscr{H}: \mathscr{S}x = x\}$. Fixed point problem has been widely studied and developed in the various literature studies, see [1].

Definition 1. Let $\mathscr{H}$ be a real Hilbert space.

(i) A mapping $\mathscr{S}: \mathscr{H} \longrightarrow \mathscr{H}$ is said to be nonexpansive if

$$\|\mathrm{S}u - \mathrm{S}v\| \leq \|u - v\|, \quad \forall u, v \in \mathscr{H}. \tag{2}$$

(ii) A mapping $\mathscr{S}: \mathscr{H} \longrightarrow \mathscr{H}$ is said to be quasinonexpansive if $\mathrm{Fix}(\mathscr{S}) \neq \varnothing$ and

$$\|\mathrm{S}u - v\| \leq \|u - v\|, \quad \forall u \in \mathscr{H} \text{ and } \mathrm{v} \in \mathrm{F}(\mathscr{S}). \tag{3}$$

(iii) A mapping $\mathscr{S}: \mathscr{H} \longrightarrow \mathscr{H}$ is called κ-strictly pseudocontractive if there exists a constant $\kappa \in [0, 1)$ such that

$$\|\mathrm{S}u - \mathrm{S}v\|^2 \leq \|u - v\|^2 + \kappa \|(I - \mathscr{S})u - (I - \mathscr{S})v\|^2, \quad \forall u, v \in \mathscr{H}. \tag{4}$$

If $F(\mathscr{S}) \neq \varnothing$, then a nonexpansive mapping is a quasi-nonexpansive mapping. Also, if $\kappa = 0$, then a strictly pseudocontractive reduces to a nonexpansive mapping.

In a real Hilbert space, the inequality (4) is equivalent to

$$\langle \mathrm{S}u - \mathrm{S}v, u - v \rangle \leq \|u - v\|^2 - \frac{1-\kappa}{2} \|(I - \mathscr{S})u - (I - \mathscr{S})v\|^2, \quad \forall u, v \in \mathscr{H}. \tag{5}$$

Definition 2 (see [2]). A mapping $\mathcal{S}$ is called demicontractive if $\mathrm{Fix}(\mathcal{S}) \neq \varnothing$ and there exists a constant $\kappa \in [0,1)$ such that

$$\|\mathrm{S}u - v\|^2 \leq \|u - v\|^2 + \kappa \|(I - \mathcal{S})u\|^2, \quad \forall u \in \mathcal{H} \text{ and } v \in F(\mathcal{S}). \tag{6}$$

The class of demicontractive mappings covers a variety of nonlinear mappings, including strictly pseudocontractive mappings, quasinonexpansive mappings, and nonexpansive mappings.

By using the same technique as in the proof of (5), we see that (6) is equivalent to the inequality shown below if $\mathcal{S}\colon \mathcal{H} \longrightarrow \mathcal{H}$ is a demicontractive mapping.

$$\langle \mathcal{S}u - v, u - v \rangle \leq \|u - v\|^2 - \frac{1-\kappa}{2} \|(I - \mathcal{S})u\|^2, \quad \forall u \in \mathcal{H} \text{ and } v \in F(\mathcal{S}). \tag{7}$$

Several mathematicians have taken an interest in studying the common fixed point of the finite family of nonlinear mappings and their characteristics during the past few decades; see [3–6].

In 2009, K-mapping for nonlinear mappings is introduced by Kangtunyakarn and Suantai [6] as follows:

$$\begin{aligned}
\mathcal{V}_1 &= \rho_1 \mathcal{S}_1 + (1-\rho_1) I,\\
\mathcal{V}_2 &= \rho_2 \mathcal{S}_2 \mathcal{V}_1 + (1-\rho_2)\mathcal{V}_1,\\
\mathcal{V}_3 &= \rho_3 \mathcal{S}_3 \mathcal{V}_2 + (1-\rho_3)\mathcal{V}_2,\\
&\vdots\\
\mathcal{V}_{N-1} &= \rho_{N-1} \mathcal{S}_{N-1} \mathcal{V}_{N-2} + (1-\rho_{N-1})\mathcal{V}_{N-2},\\
K = \mathcal{V}_N &= \rho_N \mathcal{S}_N \mathcal{V}_{N-1} + (1-\rho_N)\mathcal{V}_{N-1},
\end{aligned} \tag{8}$$

where $0 \leq \rho_i \leq 1$ for every $i = 1, 2, \ldots, N$. This mapping is called K-mapping generated by $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N$ and $\rho_1, \rho_2, \ldots, \rho_N$. Furthermore, $F(K) = \cap_{i=1}^N F(\mathcal{S}_i)$.

Let $\mathcal{D}\colon \mathcal{C} \longrightarrow \mathcal{H}$ is a nonlinear mapping. The variational inequality problem (VIP) is to finding an element $u \in \mathcal{C}$ such that

$$\langle v - u, \mathrm{D}u \rangle \geq 0, \quad \forall v \in \mathcal{C}. \tag{9}$$

The solution set of the problem (9) is denoted by $\mathrm{VI}(\mathcal{C}, \mathcal{D})$.

Stampacchia [7] introduced and investigated variational inequalities in 1964. In addition to offering a comprehensive, unifying framework for the study of optimization, equilibrium problems, and related problems, it also serves as a helpful computational framework for the resolution of various problems in a wide range of applications. For additional information, see [8–13]. Various approaches are investigated to solve variational inequality problems and the related optimization problems through iterative methods.

Several researchers have presented a variety of iterative algorithms designed for solving the variational inequality problem (VIP). The projected gradient method (GM), which can be defined as follows, is the most fundamental projection technique for solving the VIP.

$$x_{n+1} = P_{\mathcal{C}}(x_n - \rho \mathrm{D} x_n), \quad \forall n \geq 1, \tag{10}$$

where $P_{\mathcal{C}}$ denotes the metric projection mapping, $\mathcal{D}$ is the α-strongly monotone, and $L-$ is Lipschitz continuous with $\rho \in (0, 2\alpha/L^2)$.

In 1976, Korpelevich [14] and Antipin [15] proposed the extragradient method (EGM) in a finite-dimensional Euclidean space as follows:

$$\begin{cases}
x_1 \in \mathcal{C},\\
y_n = P_{\mathcal{C}}(x_n - \rho \mathcal{D} x_n),\\
x_{n+1} = P_{\mathcal{C}}(x_n - \rho \mathcal{D} y_n), \quad \forall n \geq 1,
\end{cases} \tag{11}$$

where $\rho \in (0, 1/L)$ and $\mathcal{D}\colon \mathbb{R}^n \longrightarrow \mathbb{R}^2$ are monotones and L is Lipschitz continuous. If $\mathrm{VI}(\mathcal{C}, \mathcal{D})$ is nonempty, the sequence generated by (11) converges to a solution of VIP.

According to [16–18] and related references, the EGM has undergone modifications and enhancements in the past few years.

Later, in 2012, Censor et al. [19] defined the subgradient extragradient method (SEGM) by modifying the EGM and replacing the second projection with a projection onto a half-space which is presented as follows:

$$\begin{cases}
u_1 \in \mathcal{H},\\
v_n = P_{\mathcal{C}}(x_n - \rho \mathcal{D} x_n),\\
\mathcal{S}_n = \{z \in \mathcal{H}\colon \langle u_n - \rho \mathcal{D} u_n - v_n, z - v_n \rangle \leq 0\},\\
u_{n+1} = P_{\mathcal{S}_n}(u_n - \rho \mathcal{D} v_n), \quad \forall n \geq 1.
\end{cases} \tag{12}$$

Weak convergence theorem is obtained for SEGM (4) under some control conditions.

Recently, in 2021, Kheawborisut and Kangtunyakarn [20] introduced the modified subgradient extragradient method (MSEGM) as follows:

$$\begin{cases}
z, u_1 \in \mathcal{H},\\
v_n = P_{\mathcal{C}}\left(I - \rho \sum_{i=1}^N a_i \mathcal{D}_i\right) u_n,\\
\mathcal{R}_n = \left\{ y \in \mathcal{H}\colon \left\langle \left(I - \rho \sum_{i=1}^N a_i \mathcal{D}_i\right) u_n - v_n, v_n - y \right\rangle \geq 0 \right\},\\
u_{n+1} = \sigma_n z + \rho_n P_{\mathcal{R}_n}\left(u_n - \rho \sum_{i=1}^N a_i \mathcal{D}_i v_n\right) + \mu_n G u_n, \quad \forall n \geq 1,
\end{cases} \tag{13}$$

where $\mathcal{D}_i\colon \mathcal{H} \longrightarrow \mathcal{H}$ be α_i-inverse strongly monotone mappings, $0 < a_i < 1, \forall i = 1, 2, \ldots, N$, $\sum_{i=1}^N a_i = 1$, $\{\sigma_n\}, \{\rho_n\}, \{\mu_n\} \subset [0,1]$ with $\sigma_n + \rho_n + \mu_n = 1$, and G is a nonexpansive mapping. Afterwards, under certain control settings, a strong convergence theorem is obtained.

If $\mathscr{D} = \mathscr{D}_i, \forall i = 1, 2, \ldots, N$ and $\alpha_n = \gamma_n = 0, \forall n \in \mathbb{N}$, then the modified subgradient extragradient method (MSEGM) reduces to the subgradient extragradient method (SEGM).

Motivated by the recent research, the S-subgradient extragradient method (SSEGM) is introduced as follows:

$$\begin{cases} u, \mathscr{X}_1 \in \mathscr{H}, \\ \mathscr{Z}_n = P_{\mathscr{C}}\left(I - \rho \sum_{i=1}^{N} a_i \mathscr{D}_i\right)\mathscr{X}_n, \\ \mathscr{R}_n = \left\{ z \in \mathscr{H} \colon \left\langle \left(I - \rho \sum_{i=1}^{N} a_i \mathscr{D}_i\right)\mathscr{X}_n - \mathscr{Z}_n, \mathscr{Z}_n - z \right\rangle \geq 0 \right\}, \\ \mathscr{Y}_n = \alpha_n u + \beta_n P_{\mathscr{R}_n}\left(\mathscr{X}_n - \rho \sum_{i=1}^{N} a_i \mathscr{D}_i \mathscr{Z}_n\right) + \gamma_n G \mathscr{X}_n, \\ \mathscr{X}_{n+1} = \sigma_n \mathscr{Y}_n + (1 - \sigma_n) S \mathscr{X}_n, \quad \forall n \geq 1, \end{cases} \tag{14}$$

where S is a nonexpansive mapping. If $\sigma_n = 1$, then the S-subgradient extragradient method (SSEGM) reduces to the modified subgradient extragradient method (MSEGM).

In this paper, inspired by [6, 20], the S-subgradient extragradient method and K-mapping generated by a finite family of finite Lipschitzian demicontraction mappings are proposed. Under some control conditions, strong convergence theorems are proved. Moreover, numerical examples are given to support the main theorem.

2. Preliminaries

The notations $'' \rightharpoonup ''$ and $'' \longrightarrow ''$ are denoted weak convergence and strong convergence, respectively. For each $u \in \mathscr{H}$, there exists a unique nearest point $P_{\mathscr{C}} u \in \mathscr{C}$ such that

$$\left\| u - P_{\mathscr{C}} u \right\| = \min_{v \in \mathscr{C}} \| u - v \|. \tag{15}$$

The mapping $P_{\mathscr{C}}$ is called the metric projection of $\mathscr{H}$ onto $\mathscr{C}$. Also, $P_{\mathscr{C}}$ is a firmly nonexpansive mapping from $\mathscr{H}$ onto $\mathscr{C}$, that is,

$$\left\| P_{\mathscr{C}} u - P_{\mathscr{C}} v \right\|^2 \leq \langle u - v, P_{\mathscr{C}} u - P_{\mathscr{C}} v \rangle, \forall u, v \in \mathscr{H}. \tag{16}$$

Moreover, for any $u \in \mathscr{H}$ and $q \in \mathscr{C}$, $q = P_{\mathscr{C}} u$ if and only if

$$\langle u - q, q - v \rangle \geq 0, \quad \forall v \in \mathscr{C}. \tag{17}$$

Definition 3. Let $\mathscr{S} \colon \mathscr{H} \longrightarrow \mathscr{H}$ be a mapping. Then,

(i) $\mathscr{S}$ is said to be μ-Lipschitz continuous if there is a positive real number $\mu > 0$ such that

$$\| \mathrm{S}u - \mathrm{S}v \| \leq \mu \| u - v \|, \quad \forall u, v \in \mathscr{H}. \tag{18}$$

(ii) $\mathscr{S}$ is called ξ-inverse strongly monotone if there is a positive real number ξ such that

$$\langle u - v, \mathrm{S}u - \mathrm{S}v \rangle \geq \xi \| \mathrm{S}u - \mathrm{S}v \|^2, \quad \forall u, v \in \mathscr{H}. \tag{19}$$

Lemma 4 (see [21]). *Let $\{p_n\}$ be a sequence of nonnegative real numbers satisfying*

$$p_{n+1} \leq (1 - \varepsilon_n) p_n + \rho_n, \quad \forall n \geq 0, \tag{20}$$

where $\{\varepsilon_n\}$ is a sequence in (0,1) and $\{\rho_n\}$ is a sequence such that

(1) $\sum_{n=1}^{\infty} \varepsilon_n = \infty$;
(2) $\limsup_{n \longrightarrow \infty} \rho_n / \varepsilon_n \leq 0$ or $\sum_{n=1}^{\infty} |\rho_n| < \infty$.

Then, $\lim_{n \longrightarrow \infty} p_n = 0$.

Lemma 5. *Let $\mathscr{H}$ be a real Hilbert space. Then, the following properties hold:*

(i) For all $u, v \in \mathscr{H}$ and $\alpha \in [0, 1]$,

$$\| \alpha u + (1 - \alpha) v \|^2 = \alpha \| u \|^2 + (1 - \alpha) \| v \|^2 - \alpha (1 - \alpha) \| u - v \|^2. \tag{21}$$

(ii) $\| u + v \|^2 \leq \| u \|^2 + 2 \langle v, u + v \rangle$, for each $u, v \in \mathscr{H}$.

Lemma 6 (see [22]). *For $j = 1, 2, \ldots, \widetilde{N}$, let $\mathscr{D}_i \colon \mathscr{C} \longrightarrow \mathscr{H}$ be an α_j-inverse strongly monotone with $0 < \rho < 2\alpha_j$ and $\cap_{j=1}^{N} VI(\mathscr{C}, \mathscr{D}_j) \neq \emptyset$. Therefore, these properties hold:*

(i) $VI(\mathscr{C}, \sum_{j=1}^{\widetilde{N}} b_j \mathscr{D}_j) = \cup_{j=1}^{\widetilde{N}} VI(\mathscr{C}, \mathscr{D}_j)$;
(ii) $I - \rho \sum_{j=1}^{N} b_j \mathscr{D}_j$ is a nonexpansive mapping.

Here, $b_j \in (0, 1)$, for $j = 1, 2, \ldots, \widetilde{N}$, and $\sum_{j=1}^{\widetilde{N}} b_j = 1$

Definition 7 (see [6]). Let $\mathscr{C}$ be a nonempty closed convex subset of a real Banach space. Let $\{\mathscr{S}_i\}_{i=1}^{N}$ be a finite family of κ_i-demicontractive mapping of $\mathscr{C}$ into itself and let $\rho_1, \rho_2, \ldots, \rho_N$ be real numbers with $0 \leq \rho_i \leq 1$ for every $i = 1, 2, \ldots, N$. Define a mapping $K \colon \mathscr{C} \longrightarrow \mathscr{C}$ as follows:

$$\begin{aligned} \mathscr{V}_1 &= \rho_1\mathscr{S}_1 + (1-\rho_1)I,\\ \mathscr{V}_2 &= \rho_2\mathscr{S}_2\mathscr{V}_1 + (1-\rho_2)\mathscr{V}_1,\\ \mathscr{V}_3 &= \rho_3\mathscr{S}_3\mathscr{V}_2 + (1-\rho_3)\mathscr{V}_2,\\ &\vdots\\ \mathscr{V}_{N-1} &= \rho_{N-1}\mathscr{S}_{N-1}\mathscr{V}_{N-2} + (1-\rho_{N-1})\mathscr{V}_{N-2},\\ K = \mathscr{V}_N &= \rho_N\mathscr{S}_N\mathscr{V}_{N-1} + (1-\rho_N)\mathscr{V}_{N-1}. \end{aligned} \tag{22}$$

This mapping K is said to be the K-mapping generated by $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_N$ and $\rho_1, \rho_2, \ldots, \rho_N$.

The following lemmas are needed to prove the main result.

Lemma 8. *Let $\mathscr{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$. Let $\{\mathscr{S}_i\}_{i=1}^N$ be a finite family of κ_i-demicontractive mapping of $\mathscr{C}$ into itself with $\kappa_i \le \gamma_1$, for all $i = 1, 2, \ldots, N$, and $\cap_{i=1}^N F(\mathscr{S}_i) \ne \emptyset$. Let $\rho_1, \rho_2, \ldots, \rho_N$ be real numbers with $0 < \rho_i < \gamma_2$, for all $i = 1, 2, \ldots, N$ and $\gamma_1 + \gamma_2 < 1$. Let K be the K-mapping generated by $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_N$ and $\rho_1, \rho_2, \ldots, \rho_N$. Then, there hold the following properties:*

(i) $F(K) = \cap_{i=1}^N F(\mathscr{S}_i)$;

(ii) K is a nonexpansive mapping.

Proof. To prove (i), it is clear that $\cap_{i=1}^N F(\mathscr{S}_i) \subseteq F(K)$.

Next, we prove that $F(K) \subseteq \cap_{i=1}^N F(\mathscr{S}_i)$. To show this, suppose $u \in F(K)$ and $v \in \cap_{i=1}^N F(\mathscr{S}_i)$.

By the definition of K-mapping, we obtain

$$\begin{aligned} &\|u-v\|\\ &= \|Ku-v\|^2\\ &= \left\|\rho_N\mathscr{S}_N\mathscr{V}_{N-1}u + (1-\rho_N)\mathscr{V}_{N-1}u - v\right\|^2\\ &= \left\|\rho_N(\mathscr{S}_N\mathscr{V}_{N-1}u - v) + (1-\rho_N)(\mathscr{V}_{N-1}u - v)\right\|^2\\ &= \rho_N^2\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - v\right\|^2 + (1-\rho_N)^2\left\|\mathscr{V}_{N-1}u - v\right\|^2\\ &\quad + 2\rho_N(1-\rho_N)\langle \mathscr{S}_N\mathscr{V}_{N-1}u - v, \mathscr{V}_{N-1}u - v\rangle\\ &= \rho_N^2\left(\left\|\mathscr{V}_{N-1}u - v\right\|^2 + \kappa_N\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - \mathscr{V}_{N-1}u\right\|^2\right) + (1-\rho_N)^2\left\|\mathscr{V}_{N-1}u - v\right\|^2\\ &\quad + 2\rho_N(1-\rho_N)\left(\left\|\mathscr{V}_{N-1}u - v\right\|^2 - \frac{1-\kappa_N}{2}\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - \mathscr{V}_{N-1}u\right\|^2\right)\\ &= \left(\rho_N^2 + (1-\rho_N)^2 + 2\rho_N(1-\rho_N)\right)\left\|\mathscr{V}_{N-1}u - v\right\|^2\\ &\quad + \left(\rho_N^2\kappa_N - \rho_N(1-\rho_N)(1-\kappa_N)\right)\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - \mathscr{V}_{N-1}u\right\|^2\\ &= (\rho_N + 1 - \rho_N)^2\left\|\mathscr{V}_{N-1}u - v\right\|^2\\ &\quad + \rho_N(\rho_N\kappa_N - (1-\rho_N)(1-\kappa_N))\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - \mathscr{V}_{N-1}u\right\|^2\\ &= \left\|\mathscr{V}_{N-1}u - v\right\|^2\\ &\quad + \rho_N(\rho_N\kappa_N - (1-\kappa_N) + \rho_N(1-\kappa_N))\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - \mathscr{V}_{N-1}u\right\|^2\\ &= \left\|\mathscr{V}_{N-1}u - v\right\|^2 + \rho_N(\kappa_N + \rho_N - 1)\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - \mathscr{V}_{N-1}u\right\|^2\\ &\le \left\|\mathscr{V}_{N-1}u - v\right\|^2 + \rho_N(\gamma_1 + \gamma_2 - 1)\left\|\mathscr{S}_N\mathscr{V}_{N-1}u - \mathscr{V}_{N-1}u\right\|^2\\ &\le \left\|\mathscr{V}_{N-1}u - v\right\|^2\\ &\ \ \vdots\\ &= \left\|\mathscr{V}_2u - v\right\|^2 \end{aligned}$$

$$
\begin{aligned}
&= \left\|\rho_2\left(\mathcal{S}_2\mathcal{V}_1 u - v\right) + \left(1-\rho_2\right)\left(\mathcal{V}_1 u - v\right)\right\|^2 \\
&= \rho_2^2\left\|\mathcal{S}_2\mathcal{V}_1 x - y\right\|^2 + \left(1-\rho_2\right)^2\left\|\mathcal{V}_1 u - v\right\|^2 \\
&\quad + 2\rho_2\left(1-\rho_2\right)\left\langle \mathcal{S}_2\mathcal{V}_1 u - v, \mathcal{V}_1 u - v\right\rangle \\
&= \rho_2^2\left(\left\|\mathcal{V}_1 u - v\right\|^2 + \kappa_2\left\|\mathcal{S}_2\mathcal{V}_1 u - \mathcal{V}_1 u\right\|^2\right) + \left(1-\rho_2\right)^2\left\|\mathcal{V}_1 u - v\right\|^2 \\
&\quad + 2\rho_2\left(1-\rho_2\right)\left(\left\|\mathcal{V}_1 u - v\right\|^2 - \frac{1-\kappa_2}{2}\left\|\mathcal{S}_2\mathcal{V}_1 u - \mathcal{V}_1 u\right\|^2\right) \\
&= \left(\rho_2^2 + \left(1-\rho_2\right)^2 + 2\rho_2\left(1-\rho_2\right)\right)\left\|\mathcal{V}_1 u - v\right\|^2 \\
&\quad + \left(\rho_2^2\kappa_2 - \rho_2\left(1-\rho_2\right)\left(1-\kappa_2\right)\right)\left\|\mathcal{S}_2\mathcal{V}_1 u - \mathcal{V}_1 u\right\|^2 \\
&= \left(\rho_2 + 1 - \rho_2\right)^2\left\|\mathcal{V}_1 u - v\right\|^2 \\
&\quad + \rho_2\left(\rho_2\kappa_2 - \left(1-\rho_2\right)\left(1-\kappa_2\right)\right)\left\|\mathcal{S}_2\mathcal{V}_1 u - \mathcal{V}_1 u\right\|^2 \\
&= \left\|\mathcal{V}_1 u - v\right\|^2 + \rho_2\left(\kappa_2 + \rho_2 - 1\right)\left\|\mathcal{S}_2\mathcal{V}_1 u - \mathcal{V}_1 u\right\|^2 \\
&\leq \left\|\mathcal{V}_1 u - v\right\|^2 + \rho_2\left(\left(\gamma_1 + \gamma_2\right) - 1\right)\left\|\mathcal{S}_2\mathcal{V}_1 u - \mathcal{V}_1 u\right\|^2 \\
&\leq \left\|\mathcal{V}_1 u - v\right\|^2 \\
&= \left\|\rho_1\left(\mathcal{S}_1 u - v\right) + \left(1-\rho_1\right)\left(u - v\right)\right\|^2 \\
&= \rho_1^2\left\|\mathcal{S}_1 u - v\right\|^2 + \left(1-\rho_1\right)^2\|u - v\|^2 \\
&\quad + 2\rho_1\left(1-\rho_1\right)\left\langle \mathcal{S}_1 u - v, u - v\right\rangle \\
&= \rho_1^2\left(\|u - v\|^2 + \kappa_1\left\|\mathcal{S}_1 u - u\right\|^2\right) + \left(1-\rho_1\right)^2\|u - v\|^2 \\
&\quad + 2\rho_1\left(1-\rho_1\right)\left(\|u - v\|^2 - \frac{1-\kappa_1}{2}\left\|\mathcal{S}_1 u - u\right\|^2\right) \\
&= \left(\rho_1^2 + \left(1-\rho_1\right)^2 + 2\rho_1\left(1-\rho_1\right)\right)\|u - v\|^2 \\
&\quad + \left(\rho_1^2\kappa_1 - \rho_1\left(1-\rho_1\right)\left(1-\kappa_1\right)\right)\left\|\mathcal{S}_1 u - u\right\|^2 \\
&= \left(\rho_1 + 1 - \rho_1\right)^2\|u - v\|^2 \\
&\quad + \rho_1\left(\rho_1\kappa_1 - \left(1-\rho_1\right)\left(1-\kappa_1\right)\right)\left\|\mathcal{S}_1 u - u\right\|^2 \\
&= \|u - v\|^2 + \rho_1\left(\kappa_1 + \rho_1 - 1\right)\left\|\mathcal{S}_1 u - u\right\|^2 \\
&\leq \|u - v\|^2 + \rho_1\left(\left(\gamma_1 + \gamma_2\right) - 1\right)\left\|\mathcal{S}_1 u - u\right\|^2.
\end{aligned} \tag{23}
$$

By (23), it follows that

$$\rho_1\left(1 - \left(\gamma_1 + \gamma_2\right)\right)\left\|\mathcal{S}_1 u - u\right\|^2 \leq 0. \tag{24}$$

Then, we have

$$\left\|\mathcal{S}_1 u - u\right\| = 0. \tag{25}$$

Hence, $u = \mathcal{S}_1 u$, that is,

$$u \in F\left(\mathcal{S}_1\right). \tag{26}$$

By the definition of $\mathscr{V}_1$ and (26), we get

$$\begin{aligned}\mathscr{V}_1 u &= \rho_1 \mathscr{S}_1 u + (1-\rho_1)u \\ &= u,\end{aligned} \tag{27}$$

that is,

$$u \in F(\mathscr{V}_1). \tag{28}$$

From (23) and (28), we have

$$\begin{aligned}\|u-v\|^2 &\le \|\mathscr{V}_1 u - v\|^2 + \rho_2((\gamma_1+\gamma_2)-1)\|\mathscr{S}_2\mathscr{V}_1 u - \mathscr{V}_1 u\|^2 \\ &= \|u-v\|^2 + \rho_2((\gamma_1+\gamma_2)-1)\|\mathscr{S}_2 u - u\|^2,\end{aligned} \tag{29}$$

which follows that $u = \mathscr{S}_2 u$, that is,

$$u \in F(\mathscr{S}_2). \tag{30}$$

By the definition of $\mathscr{V}_2$, (23) and (28), this implies that

$$\begin{aligned}\mathscr{V}_2 u &= \rho_2 \mathscr{S}_2 \mathscr{V}_1 u + (1-\rho_2)\mathscr{V}_1 u \\ &= u,\end{aligned} \tag{31}$$

which yields that

$$u \in F(\mathscr{V}_2). \tag{32}$$

Using the same method, we get

$$\begin{aligned}&u \in F(\mathscr{S}_i), \\ &\mathrm{u} \in \mathrm{F}(\mathscr{V}_{\mathrm{i}}), \quad \forall \mathrm{i} = 1,2,\ldots,\mathrm{N}-1.\end{aligned} \tag{33}$$

Next, we claim that $u \in F(\mathscr{S}_N)$. Since

$$\begin{aligned}0 &= Ku - u \\ &= \rho_N \mathscr{S}_N \mathscr{V}_{N-1} u + (1-\rho_N)\mathscr{V}_{N-1} u - u \\ &= \rho_N(\mathscr{S}_N u - u),\end{aligned} \tag{34}$$

and $\rho_N \in (0,1]$, we have

$$u \in F(\mathscr{S}_N), \tag{35}$$

which implies that

$$u \in \bigcap_{i=1}^{N} F(\mathscr{S}_i). \tag{36}$$

Hence,

$$F(K) \subseteq \bigcap_{i=1}^{N} F(\mathscr{S}_i). \tag{37}$$

Therefore,

$$F(K) = \bigcap_{i=1}^{N} F(\mathscr{S}_i). \tag{38}$$

Finally, applying the same proof as in (23), K is a quasinonexpansive mapping. □

Lemma 9. *Let $\mathscr{C}$ be a nonempty closed convex subset of a real Hilbert space $\mathscr{H}$. For $i = 1,2,\ldots,N$, let $\mathscr{S}_i: \mathscr{H} \longrightarrow \mathscr{H}$ be a finite family of κ_i-demicontractive mappings of $\mathscr{C}$ into itself and L_i-Lipschitzian mappings with $\kappa_i \le \gamma_1$ and $\cap_{i=1}^{N} F(\mathscr{S}_i) \neq \emptyset$. For each $i = 1,2,\ldots,N$ and $n \in \mathbb{N}$, let $\rho_1,\rho_2,\ldots,\rho_N$ and $\rho_1^n,\rho_2^n,\ldots,\rho_N^n$ be real numbers with $0 < \rho_i, \rho_i^n < \gamma_2$ and $\gamma_1 + \gamma_2 < 1$ such that $\rho_i^n \longrightarrow \rho_i$ as $n \longrightarrow \infty$. For each $n \in \mathbb{N}$, let K and K_n be the K-mappings generated by $\mathscr{S}_1,\mathscr{S}_1,\ldots,\mathscr{S}_N$ and $\rho_1,\rho_2,\ldots,\rho_N$ and $\mathscr{S}_1,\mathscr{S}_2,\ldots,\mathscr{S}_N$ and $\rho_1^n,\rho_2^n,\ldots,\rho_N^n$, respectively. Therefore, for each bounded sequence $\{u_n\}$ in $\mathscr{C}$, we have*

$$\lim_{n\longrightarrow\infty}\|K_n u_n - K u_n\| = 0. \tag{39}$$

Proof. Let $\{u_n\}$ be a bounded sequence in $\mathscr{C}$ and let $\mathscr{V}_k$ and $\mathscr{V}_{n,k}$ be generated by $\mathscr{S}_1,\mathscr{S}_1,\ldots,\mathscr{S}_N$ and $\rho_1,\rho_2,\ldots,\rho_N$ and $\mathscr{S}_1,\mathscr{S}_1,\ldots,\mathscr{S}_N$ and $\rho_1^n,\rho_2^n,\ldots,\rho_N^n$, respectively. For each $n \in \mathbb{N}$, we have

$$\begin{aligned}\|\mathscr{V}_{n,1} u_n - \mathscr{V}_1 u_n\| &= \|\rho_1^n \mathscr{S}_1 u_n + (1-\rho_1^n)u_n - (\rho_1 \mathscr{S}_1 u_n + (1-\rho_1)u_n)\| \\ &= \|\rho_1^n \mathscr{S}_1 u_n - \rho_1^n u_n - \rho_1 \mathscr{S}_1 u_n + \rho_1 u_n\| \\ &= \|(\rho_1^n - \rho_1)\mathscr{S}_1 u_n - (\rho_1^n - \rho_1)u_n\| \\ &= |\rho_1^n - \rho_1|\|\mathscr{S}_1 u_n - u_n\|.\end{aligned} \tag{40}$$

For $k \in \{2,3,\ldots,N\}$, we get

$$\begin{aligned}
&\|\mathcal{V}_{n,k}u_n - \mathcal{V}_k u_n\| \\
&\quad = \|\rho_k^n \mathcal{S}_k \mathcal{V}_{n,k-1}u_n + (1-\rho_k^n)\mathcal{V}_{n,k-1}u_n - (\rho_k \mathcal{S}_k \mathcal{V}_{k-1}u_n + (1-\rho_k)\mathcal{V}_{k-1}u_n)\| \\
&\quad = \|\rho_k^n \mathcal{S}_k \mathcal{V}_{n,k-1}u_n - \rho_k \mathcal{S}_k \mathcal{V}_{k-1}u_n + (1-\rho_k^n)\mathcal{V}_{n,k-1}u_n - (1-\rho_k)\mathcal{V}_{k-1}u_n\| \\
&\quad = \|\rho_k^n \mathcal{S}_k \mathcal{V}_{n,k-1}u_n - \rho_k^n \mathcal{S}_k \mathcal{V}_{k-1}u_n + \rho_k^n \mathcal{S}_k \mathcal{V}_{k-1}u_n - \rho_k \mathcal{S}_k \mathcal{V}_{k-1}u_n \\
&\qquad + (1-\rho_k^n)\mathcal{V}_{n,k-1}u_n - (1-\rho_k^n)\mathcal{V}_{k-1}u_n + (1-\rho_k^n)\mathcal{V}_{k-1}u_n \\
&\qquad - (1-\rho_k)\mathcal{V}_{k-1}u_n\| \\
&\quad = \|\rho_k^n (\mathcal{S}_k \mathcal{V}_{n,k-1}u_n - \mathcal{S}_k \mathcal{V}_{k-1}u_n) + (\rho_k^n - \rho_k)\mathcal{S}_k \mathcal{V}_{k-1}u_n \\
&\qquad + (1-\rho_k^n)(\mathcal{V}_{n,k-1}u_n - \mathcal{V}_{k-1}u_n) + (1-\rho_k^n - (1-\rho_k))\mathcal{V}_{k-1}u_n\| \\
&\quad \le \rho_k^n \|\mathcal{S}_k \mathcal{V}_{n,k-1}u_n - \mathcal{S}_k \mathcal{V}_{k-1}u_n\| + |\rho_k^n - \rho_k| \|\mathcal{S}_k \mathcal{V}_{k-1}u_n\| \\
&\qquad + (1-\rho_k^n)\|\mathcal{V}_{n,k-1}u_n - \mathcal{V}_{k-1}u_n\| + |\rho_k - \rho_k^n| \|\mathcal{V}_{k-1}u_n\| \\
&\quad \le \rho_k^n L_k \|\mathcal{V}_{n,k-1}u_n - \mathcal{V}_{k-1}u_n\| + |\rho_k^n - \rho_k| \|\mathcal{S}_k \mathcal{V}_{k-1}u_n\| \\
&\qquad + (1-\rho_k^n)\|\mathcal{V}_{n,k-1}u_n - \mathcal{V}_{k-1}u_n\| + |\rho_k - \rho_k^n| \|\mathcal{V}_{k-1}u_n\| \\
&\quad \le (L_k + 1)\|\mathcal{V}_{n,k-1}u_n - \mathcal{V}_{k-1}u_n\| + |\rho_k^n - \rho_k| (\|\mathcal{S}_k \mathcal{V}_{k-1}u_n\| + \|\mathcal{V}_{k-1}u_n\|).
\end{aligned} \tag{41}$$

From (40) and (41), we obtain

$$\begin{aligned}
&\|K_n u_n - K u_n\| \\
&\quad = \|\mathcal{V}_{n,N}u_n - \mathcal{V}_N u_n\| \\
&\quad \le (L_N + 1)\|\mathcal{V}_{n,N-1}u_n - \mathcal{V}_{N-1}u_n\| + |\rho_N^n - \rho_N| (\|\mathcal{S}_N \mathcal{V}_{N-1}u_n\| + \|\mathcal{V}_{N-1}u_n\|) \\
&\quad \le (L_N + 1)\big((L_{N-1} + 1)\|\mathcal{V}_{n,N-2}u_n - \mathcal{V}_{N-2}u_n\| + |\rho_{N-1}^n - \rho_{N-1}| \\
&\cdot (\|\mathcal{S}_{N-1}\mathcal{V}_{N-2}u_n\| + \|\mathcal{V}_{N-2}u_n\|)\big) + |\rho_N^n - \rho_N| (\|\mathcal{S}_N \mathcal{V}_{N-1}u_n\| + \|\mathcal{V}_{N-1}u_n\|) \\
&\quad = (L_N + 1)(L_{N-1} + 1)\|\mathcal{V}_{n,N-2}u_n - \mathcal{V}_{N-2}u_n\| \\
&\qquad + (L_N + 1)|\rho_{N-1}^n - \rho_{N-1}| (\|\mathcal{S}_{N-1}\mathcal{V}_{N-2}u_n\| + \|\mathcal{V}_{N-2}u_n\|) \\
&\qquad + |\rho_N^n - \rho_N| (\|\mathcal{S}_N \mathcal{V}_{N-1}u_n\| + \|\mathcal{V}_{N-1}u_n\|) \\
&\quad = \prod_{j=N-1}^{N} (L_j + 1)\|\mathcal{V}_{n,N-2}u_n - \mathcal{V}_{N-2}u_n\| \\
&\qquad + \sum_{j=N-1}^{N} (L_{j+1} + 1)^{N-j} |\rho_j^n - \rho_j| \left(\|\mathcal{S}_j \mathcal{V}_{j-1}u_n\| + \|\mathcal{V}_{j-1}u_n\|\right) \\
&\quad \vdots \\
&\quad \le \prod_{j=2}^{N} (L_j + 1)\|\mathcal{V}_{n,1}u_n - \mathcal{V}_1 u_n\| \\
&\qquad + \sum_{j=2}^{N} (L_{j+1} + 1)^{N-j} |\rho_j^n - \rho_j| \left(\|\mathcal{S}_j \mathcal{V}_{j-1}u_n\| + \|\mathcal{V}_{j-1}u_n\|\right) \\
&\quad = \prod_{j=2}^{N} (L_j + 1)|\rho_1^n - \rho_1| \|\mathcal{S}_1 u_n - u_n\| \\
&\qquad + \sum_{j=2}^{N} (L_{j+1} + 1)^{N-j} |\rho_j^n - \rho_j| \left(\|\mathcal{S}_j \mathcal{V}_{j-1}u_n\| + \|\mathcal{V}_{j-1}u_n\|\right).
\end{aligned} \tag{42}$$

By (42) and the condition $\rho_i^n \longrightarrow \rho_i$ as $n \longrightarrow \infty$ for all $i = 1, 2, \ldots, N$, hence we obtain $\lim_{n\longrightarrow\infty}\|K_n u_n - K u_n\| = 0$. □

Lemma 10 (see [23]). *Let $\{\Gamma_n\}$ be a sequence of real numbers that do not decrease at infinity in the sense that there exists a subsequence $\{\Gamma_{n_j}\}$ of $\{\Gamma_n\}$ such that $\Gamma_{n_j} < \Gamma_{n_j+1}$ for all $j \geq 0$. Also, we consider the sequence of integers $\{\tau_n\}_{n\geq n_0}$ defined by*

$$\tau(n) = \max\{k \leq n\colon \Gamma_k < \Gamma_{k+1}\}. \tag{43}$$

Then, $\{\tau_n\}_{n\geq n_0}$ is a nondecreasing sequence verifying

$$\lim_{n\longrightarrow\infty} \tau(n) = \infty, \quad \max\{\Gamma_{\tau(n)}, \Gamma_n\} \leq \Gamma_{\tau(n)+1}. \tag{44}$$

Lemma 11 (see [20]). *Let $\mathscr{H}$ be a real Hilbert space, for every $i = 1, 2, \ldots, N$, let $\mathscr{D}_i\colon \mathscr{H} \longrightarrow \mathscr{H}$ be α_i-inverse strongly monotone mappings with $\beta = \min\{\alpha_i\}$. Let $\{u_n\}$ and $\{v_n\}$ be sequences generated by*

$$\begin{aligned} v_n &= P_{\mathscr{C}}\left(I - \rho\sum_{i=1}^{N} b_i\mathscr{D}_i\right)u_n,\\ \mathscr{R}_n &= \left\{z \in \mathscr{H}\colon \left\langle \left(I - \rho\sum_{i=1}^{N} b_i\mathscr{D}_i\right)u_n - v_n, v_n - z\right\rangle\right\},\\ u^* &\in \bigcap_{i=1}^{N} VI(\mathscr{C}, \mathscr{D}_i), \quad \text{for all } i = 1, 2, \ldots, N. \end{aligned} \tag{45}$$

Then, the following inequality holds:

$$\begin{aligned} &\left\|P_{\mathscr{R}_n}\left(u_n - \rho\sum_{i=1}^{N} b_i\mathscr{D}_i v_n\right) - u^*\right\|^2\\ &\leq \|u_n - u^*\|^2 - \left(1 - \frac{\rho}{\beta}\right)\left\|P_{\mathscr{R}_n}\left(u_n - \rho\sum_{i=1}^{N} b_i\mathscr{D}_i v_n\right) - v_n\right\|^2\\ &\quad - \left(1 - \frac{\rho}{\beta}\right)\|u_n - v_n\|^2, \end{aligned} \tag{46}$$

where $0 < b_i < 1, \sum_{i=1}^{N} b_i = 0$ and $\rho \in (0, \beta)$ with $\beta = \min_{i=1,2,\ldots,N}\{\alpha_i\}$ for every $i = 1, 2, \ldots, N$.

3. Strong Convergence Theorem

Theorem 12. *Let $\mathscr{C}$ be a closed convex subset of a real Hilbert space $\mathscr{H}$. For $i = 1, 2, \ldots, \bar{N}$, let $\mathscr{D}_i\colon \mathscr{H} \longrightarrow \mathscr{H}$ be ε_i-inverse strongly monotone mappings. For $i = 1, 2, \ldots, \hat{N}$, let $\mathscr{E}_i\colon \mathscr{H} \longrightarrow \mathscr{H}$ be ν_i-inverse strongly monotone mappings with $0 < \rho < \min_{i=1,2,\ldots,\hat{N}} 2\nu_i$. For $i = 1, 2, \ldots, N$, let $\mathscr{S}_i\colon \mathscr{H} \longrightarrow \mathscr{H}$ be a finite family of κ_i-demicontractive mappings and L_i-Lipschitzian mappings with $L_i \leq 1$, $\kappa_i \leq \gamma_1$ and $\Omega\colon = \cap_{i=1}^{N} F(\mathscr{S}_i) \cap \bigcup_{i=1}^{\hat{N}} VI(\mathscr{C}, \mathscr{D}_i) \cap \bigcup_{i=1}^{\hat{N}} VI(\mathscr{C}, \mathscr{E}_i) \neq \varnothing$. For every $n \in \mathbb{N}$ and $i = 1, 2, \ldots, N$, let K_n be the K-mapping generated by $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_N$ and $\rho_1^n, \rho_2^n, \ldots, \rho_N^n$ where $\rho_i^n \in [\mu, \epsilon] \subset (0, \gamma_2)$, for some $\mu, \epsilon > 0$ and $\gamma_1 + \gamma_2 < 1$. Let the sequence $\{y_n\}$ and $\{\mathscr{Q}_n\}$ be generated by $u, \mathscr{Q}_1 \in \mathscr{H}$ and*

$$\begin{cases} \mathscr{X}_n = P_{\mathscr{C}}\left(I - \rho\sum_{i=1}^{\bar{N}} a_i\mathscr{D}_i\right)\mathscr{Q}_n,\\ \mathscr{R}_n = \left\{z \in \mathscr{H}\colon \left\langle \left(I - \rho\sum_{i=1}^{\bar{N}} a_i\mathscr{D}_i\right)\mathscr{Q}_n - \mathscr{X}_n, \mathscr{X}_n - z\right\rangle \geq 0\right\},\\ \mathscr{Y}_n = \varepsilon_n u + \xi_n P_{\mathscr{R}_n}\left(\mathscr{Q}_n - \rho\sum_{i=1}^{\bar{N}} a_i\mathscr{D}_i\right)\mathscr{X}_n\Bigg) + \psi_n K_n\mathscr{Q}_n,\\ \mathscr{Q}_{n+1} = \zeta_n\mathscr{Y}_n + (1 - \zeta_n)P_{\mathscr{C}}\left(I - \rho\sum_{i=1}^{\hat{N}} b_n^i\mathscr{E}_i\right)\mathscr{Q}_n, \quad \forall n \geq 1, \end{cases} \tag{47}$$

where $\sum_{i=1}^{N} a_i = 1$, $\{\varepsilon_n\},\{\xi_n\},\{\psi_n\},\{\zeta_n\} \subset (0,1)$ *with* $\varepsilon_n + \xi_n + \psi_n = 1$, $\forall n \geq 1$, $\rho \in (0,\eta)$ *with* $\eta = \min_{i=1,2,\ldots,\bar{N}}\{\varepsilon_i\}$ *and* $b_n^i \in (0,1), \forall i = 1,2,\ldots,\hat{N}$ *with* $\sum_{i=1}^{\hat{N}} b_n^i = 1$ *satisfying the following conditions:*

(i) $\lim_{n\longrightarrow\infty}\varepsilon_n = 0$ *and* $\sum_{n=1}^{\infty}\varepsilon_n = \infty$;

(ii) $0 < \tau \leq \xi_n, \psi_n, \zeta_n \leq \upsilon < 1$, *for some* $\tau, \upsilon > 0$.

Then, $\{\mathcal{Q}_n\}$ and $\{\mathcal{Z}_n\}$ converge strongly to $\nu = P_{\Omega}u$.

Proof. Let $c_n = P_{\mathcal{R}_n}(\mathcal{Q}_n - \rho\sum_{i=1}^{\bar{N}} a_i\mathcal{D}_i)\mathcal{Z}_n)$. First, we will show that $\{\mathcal{Q}_n\}$ is bounded. Consider

$$\begin{aligned}\mathcal{Y}_n &= \varepsilon_n u + \xi_n c_n + \psi_n K_n \mathcal{Q}_n \\ &= \varepsilon_n u + (1-\varepsilon_n)\left(\frac{\xi_n c_n + \psi_n K_n \mathcal{Q}_n}{1-\varepsilon_n}\right) \\ &= \varepsilon_n u + (1-\varepsilon_n) d_n,\end{aligned} \tag{48}$$

where $d_n = \xi_n c_n + \psi_n K_n \mathcal{Q}_n / 1 - \varepsilon_n$. Suppose that $g^* \in \Omega = \cap_{i=1}^{N} F(\mathcal{S}_i) \cap \bigcup_{i=1}^{\hat{N}} \mathrm{VI}(\mathcal{C},\mathcal{D}_i) \cap \bigcup_{i=1}^{\hat{N}} \mathrm{VI}(\mathcal{C},\mathcal{E}_i)$. By Lemma 5, we have

$$\begin{aligned}&\|d_n - g^*\|^2 \\ &= \left\|\frac{\xi_n c_n + \psi_n K_n \mathcal{Q}_n}{1-\varepsilon_n} - g^*\right\|^2 \\ &= \left\|\frac{\xi_n c_n + \psi_n K_n \mathcal{Q}_n - (1-\varepsilon_n) g^*}{1-\varepsilon_n}\right\|^2 \\ &= \frac{\xi_n}{1-\varepsilon_n}\|c_n - g^*\|^2 + \frac{\psi_n}{1-\varepsilon_n}\|K_n \mathcal{Q}_n - g^*\|^2 \\ &\quad - \frac{\xi_n \psi_n}{(1-\varepsilon_n)^2}\|c_n - K_n \mathcal{Q}_n\|^2.\end{aligned} \tag{49}$$

By the definition of $\mathcal{Q}_n$, (49), Lemmas 6 and 8, we obtain

$$\begin{aligned}&\|\mathcal{Q}_{n+1} - g^*\|^2 \\ &\leq \zeta_n\|\mathcal{Y}_n - g^*\|^2 + (1-\zeta_n)\left\|P_{\mathcal{C}}\left(I - \rho\sum_{i=1}^{\widehat{N}} b_n^i \mathcal{E}_i\right)\mathcal{Q}_n - g^*\right\|^2 \\ &\leq \zeta_n\|\mathcal{Y}_n - g^*\|^2 + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\ &= \zeta_n\|\varepsilon_n u + (1-\varepsilon_n) d_n - g^*\|^2 + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\ &= \zeta_n\|\varepsilon_n(u - g^*) + (1-\varepsilon_n)(d_n - g^*)\|^2 + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\ &= \zeta_n\left[\varepsilon_n\|u - g^*\|^2 + (1-\varepsilon_n)\|d_n - g^*\|^2 - \varepsilon_n(1-\varepsilon_n)\|f(\mathcal{Q}_n) - d_n\|^2\right] \\ &\quad + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\ &= \zeta_n\left(\varepsilon_n\|u - g^*\|^2 + (1-\varepsilon_n)\left[\frac{\xi_n}{1-\varepsilon_n}\|c_n - g^*\|^2 + \frac{\psi_n}{1-\varepsilon_n}\|K_n \mathcal{Q}_n - g^*\|^2\right.\right. \\ &\quad \left.\left. - \frac{\xi_n\psi_n}{(1-\varepsilon_n)^2}\|c_n - K_n \mathcal{Q}_n\|^2\right] - \varepsilon_n(1-\varepsilon_n)\|f(\mathcal{Q}_n) - d_n\|^2\right) \\ &\quad + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\ &= \zeta_n\left(\varepsilon_n\|u - g^*\|^2 + \xi_n\|c_n - g^*\|^2 + \psi_n\|K_n \mathcal{Q}_n - g^*\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n \mathcal{Q}_n\|^2\right. \\ &\quad \left. - \varepsilon_n(1-\varepsilon_n)\|f(\mathcal{Q}_n) - d_n\|^2\right) + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\ &\leq \zeta_n\left(\varepsilon_n\|u - g^*\|^2 + \xi_n\|c_n - g^*\|^2 + \psi_n\|\mathcal{Q}_n - g^*\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n \mathcal{Q}_n\|^2\right. \\ &\quad \left. - \varepsilon_n(1-\varepsilon_n)\|f(\mathcal{Q}_n) - d_n\|^2\right) + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2.\end{aligned} \tag{50}$$

From Lemma 11 and $\rho \in (0,\eta)$, we have

$$\|c_n - g^*\|^2 \le \|\mathcal{Q}_n - g^*\|^2. \quad (51)$$

From (50) and (51), we get

$$\begin{aligned}
&\|\mathcal{Q}_{n+1} - g^*\|^2 \\
&\le \zeta_n\Bigg(\varepsilon_n\|u - g^*\|^2 + \xi_n\|\mathcal{Q}_n - g^*\|^2 + \psi_n\|\mathcal{Q}_n - g^*\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n\mathcal{Q}_n\|^2 \\
&\quad - \varepsilon_n(1-\varepsilon_n)\|f(\mathcal{Q}_n) - d_n\|^2\Bigg) + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\
&\quad = \zeta_n\Bigg(\varepsilon_n\|u - g^*\|^2 + (1-\varepsilon_n)\|\mathcal{Q}_n - g^*\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n\mathcal{Q}_n\|^2 \\
&\quad - \varepsilon_n(1-\varepsilon_n)\|f(\mathcal{Q}_n) - d_n\|^2\Bigg) + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\
&\le \zeta_n\left(\varepsilon_n\|u - g^*\|^2 + (1-\varepsilon_n)\|\mathcal{Q}_n - g^*\|^2\right) + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\
&\vdots \\
&\le \max\left\{\|u - g^*\|^2, \|\mathcal{Q}_1 - g^*\|^2\right\}.
\end{aligned} \quad (52)$$

By induction, we obtain

$$\|\mathcal{Q}_n - g^*\|^2 \le \max\left\{\|u - g^*\|^2, \|\mathcal{Q}_1 - g^*\|^2\right\}. \quad (53)$$

This implies that $\{\mathcal{Q}_n\}$ is a bounded sequence. Next, from (50), observe that

$$\begin{aligned}
&\|\mathcal{Q}_{n+1} - g^*\|^2 \\
&\le \zeta_n\left(\varepsilon_n\|u - g^*\|^2 + \xi_n\|c_n - g^*\|^2 + \psi_n\|\mathcal{Q}_n - g^*\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n\mathcal{Q}_n\|^2\right) \\
&\quad + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\
&\le \zeta_n\Bigg(\varepsilon_n\|u - g^*\|^2 + \xi_n\Bigg[\|\mathcal{Q}_n - g^*\|^2 - \left(1 - \frac{\rho}{\eta}\right)\|c_n - \mathcal{Z}_n\|^2 \\
&\quad - \left(1 - \frac{\rho}{\eta}\right)\|\mathcal{Q}_n - \mathcal{Z}_n\|^2\Bigg] + \psi_n\|\mathcal{Q}_n - g^*\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n\mathcal{Q}_n\|^2\Bigg) \\
&\quad + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\
&= \zeta_n\Bigg(\varepsilon_n\|u - g^*\|^2 + (1-\varepsilon_n)\|\mathcal{Q}_n - g^*\|^2 - \xi_n\left(1 - \frac{\rho}{\eta}\right)\|c_n - \mathcal{Z}_n\|^2 \\
&\quad - \xi_n\left(1 - \frac{\rho}{\eta}\right)\|\mathcal{Q}_n - \mathcal{Z}_n\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n\mathcal{Q}_n\|^2\Bigg) + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\
&\le \zeta_n\Bigg(\varepsilon_n\|u - g^*\|^2 + \|\mathcal{Q}_n - g^*\|^2 - \xi_n\left(1 - \frac{\rho}{\eta}\right)\|c_n - \mathcal{Z}_n\|^2 \\
&\quad - \xi_n\left(1 - \frac{\rho}{\eta}\right)\|\mathcal{Q}_n - \mathcal{Z}_n\|^2 - \frac{\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n\mathcal{Q}_n\|^2\Bigg) + (1-\zeta_n)\|\mathcal{Q}_n - g^*\|^2 \\
&= \zeta_n\varepsilon_n\|u - g^*\|^2 + \|\mathcal{Q}_n - g^*\|^2 - \zeta_n\xi_n\left(1 - \frac{\rho}{\eta}\right)\|c_n - \mathcal{Z}_n\|^2 \\
&\quad - \zeta_n\xi_n\left(1 - \frac{\rho}{\eta}\right)\|\mathcal{Q}_n - \mathcal{Z}_n\|^2 - \frac{\zeta_n\xi_n\psi_n}{1-\varepsilon_n}\|c_n - K_n\mathcal{Q}_n\|^2.
\end{aligned} \quad (54)$$

This follows that

$$\zeta_n\xi_n\left(1-\frac{\rho}{\eta}\right)\|c_n-\mathcal{Z}_n\|^2+\zeta_n\xi_n\left(1-\frac{\rho}{\eta}\right)\|\mathcal{Q}_n-\mathcal{Z}_n\|^2+\frac{\zeta_n\xi_n\psi_n}{1-\varepsilon_n}\|c_n-K_n\mathcal{Q}_n\|^2 \leq \zeta_n\varepsilon_n\|u-g^*\|^2+\|\mathcal{Q}_n-g^*\|^2-\|\mathcal{Q}_{n+1}-g^*\|^2. \tag{55}$$

Take $S_n\colon=\zeta_n\xi_n(1-\rho/\eta)\|c_n-\mathcal{Z}_n\|^2+\zeta_n\xi_n(1-\rho/\eta)\|\mathcal{Q}_n-\mathcal{Z}_n\|^2+\zeta_n\xi_n\psi_n/1-\varepsilon_n\|c_n-K_n\mathcal{Q}_n\|^2$. Thus, we get

$$S_n\leq\zeta_n\varepsilon_n\|u-g^*\|^2+\|\mathcal{Q}_n-g^*\|^2-\|\mathcal{Q}_{n+1}-g^*\|^2. \tag{56}$$

Next, the following two possible cases are considered. □

Case 13. Put $\Gamma_n\colon=\|\mathcal{Q}_n-g^*\|^2$, for all $n\in\mathbb{N}$. Assume that there is no $n_0\geq 0$ such that, for any $n\geq n_0$, $\Gamma_{n+1}\leq\Gamma_n$. In this case, we have

$$\lim_{n\longrightarrow\infty}\Gamma_n \text{ exists}, \qquad \lim_{n\longrightarrow\infty}(\Gamma_n-\Gamma_{n+1})=0. \tag{57}$$

Since $\lim_{n\longrightarrow\infty}\varepsilon_n=0$, it yields from (56) that $\lim_{n\longrightarrow\infty}S_n=0$. Hence, we obtain

$$\lim_{n\longrightarrow\infty}\zeta_n\xi_n\left(1-\frac{\rho}{\eta}\right)\|c_n-\mathcal{Z}_n\|^2=\lim_{n\longrightarrow\infty}\zeta_n\xi_n\left(1-\frac{\rho}{\eta}\right)\|\mathcal{Q}_n-\mathcal{Z}_n\|^2 =\lim_{n\longrightarrow\infty}\frac{\zeta_n\xi_n\psi_n}{1-\varepsilon_n}\|c_n-K_n\mathcal{Q}_n\|^2 =0. \tag{58}$$

From condition (ii), we get

$$\lim_{n\longrightarrow\infty}\|c_n-\mathcal{Z}_n\|=\lim_{n\longrightarrow\infty}\|\mathcal{Q}_n-\mathcal{Z}_n\|=\lim_{n\longrightarrow\infty}\|c_n-K_n\mathcal{Q}_n\|=0. \tag{59}$$

Since

$$\|\mathcal{Q}_n-K_n\mathcal{Q}_n\|\leq\|\mathcal{Q}_n-\mathcal{Z}_n\|+\|\mathcal{Z}_n-c_n\|+\|c_n-K_n\mathcal{Q}_n\|, \tag{60}$$

then, from (59), we obtain

$$\lim_{n\longrightarrow\infty}\|\mathcal{Q}_n-K_n\mathcal{Q}_n\|=0. \tag{61}$$

Next, we choose a subsequence $\{\mathcal{Q}_{n_k}\}$ of $\{\mathcal{Q}_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle u-v,\mathcal{Q}_n-v\rangle=\lim_{k\longrightarrow\infty}\langle u-v,\mathcal{Q}_{n_k}-v\rangle, \text{where } v=P_\Omega u. \tag{62}$$

Since $\mathcal{Q}_n$ is bounded, this follows that $\mathcal{Q}_{n_k} \rightharpoonup \omega$ as $k \longrightarrow \infty$, where $\omega \in \mathcal{H}$. Assume that $\omega \notin \bigcup_{i=1}^{N} \mathrm{VI}(\mathcal{C}, \mathcal{D}_i)$. This follows by Lemma 6 that $\omega \notin F(P_{\mathcal{C}}(I - \rho\sum_{i=1}^{\bar{N}} a_i \mathcal{D}_i))$, that is, $\omega \neq P_{\mathcal{C}}(I - \rho\sum_{i=1}^{\bar{N}} a_i \mathcal{D}_i)\omega$. By nonexpansiveness of $P_{\mathcal{C}}(I - \rho\sum_{i=1}^{\bar{N}} a_i \mathcal{D}_i])$, (59), we get

$$\begin{aligned}
&\liminf_{n\longrightarrow\infty} \left\| \mathcal{Q}_{n_k} - \omega \right\| \\
&< \liminf_{n\longrightarrow\infty} \left\| \mathcal{Q}_{n_k} - P_{\mathcal{C}}\left(I - \rho \sum_{i=1}^{\overline{N}} a_i \mathcal{D}_i \right)\omega \right\| \\
&\le \liminf_{n\longrightarrow\infty} \left(\left\| \mathcal{Q}_{n_k} - y_{n_k} \right\| + \left\| y_{n_k} - P_{\mathcal{C}}\left(I - \rho \sum_{i=1}^{\overline{N}} a_i \mathcal{D}_i \right)\omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left(\left\| \mathcal{Q}_{n_k} - y_{n_k} \right\| + \left\| P_{\mathcal{C}}\left(I - \rho \sum_{i=1}^{\overline{N}} a_i \mathcal{D}_i \right)\mathcal{Q}_{n_k} - P_{\mathcal{C}}\left(I - \rho \sum_{i=1}^{\overline{N}} a_i \mathcal{D}_i \right)\omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left\| \mathcal{Q}_{n_k} - \omega \right\|.
\end{aligned} \tag{63}$$

This is a contradiction. Then, we obtain

$$\omega \in \bigcap_{i=1}^{\overline{N}} \mathrm{VI}(\mathcal{C}, \mathcal{D}_i). \tag{64}$$

Applying the same proof as in (63), we also have

$$\omega \in \bigcap_{i=1}^{\hat{N}} \mathrm{VI}(\mathcal{C}, \mathcal{E}_i). \tag{65}$$

For any $i = 1, 2, \ldots, N$ and $n \in \mathbb{N}$, $0 < \mu \le \rho_i^n \le \epsilon < \gamma_2 < 1$, without loss of generality, we have

$$\rho_i^{n_k} \longrightarrow \rho_i \in (0,1) \text{ as k} \longrightarrow \infty, \text{for every i} = 1, 2, \ldots, \text{N}. \tag{66}$$

Let K be the K-mapping generated by $\mathcal{S}_1, \mathcal{S}_2, \ldots, \mathcal{S}_N$ and $\rho_1, \rho_2, \ldots, \rho_N$. From Lemma 8, we obtain that K is nonexpansive and $F(K) = \bigcup_{i=1}^{N} F(\mathcal{S}_i)$. By Lemma 9, we get

$$\lim_{k\longrightarrow\infty} \left\| K_{n_k}\mathcal{Q}_{n_k} - K\mathcal{Q}_{n_k} \right\| = 0. \tag{67}$$

Suppose that $\omega \notin F(K)$. Therefore, we obtain $\omega \neq K\omega$. From (61) and (67), we have

$$\begin{aligned}
&\liminf_{n\longrightarrow\infty} \left\| \mathcal{Q}_{n_k} - \omega \right\| \\
&< \liminf_{n\longrightarrow\infty} \left\| \mathcal{Q}_{n_k} - K\omega \right\| \\
&\le \liminf_{n\longrightarrow\infty} \left(\left\| \mathcal{Q}_{n_k} - K_n\mathcal{Q}_{n_k} \right\| + \left\| K_n\mathcal{Q}_{n_k} - K\mathcal{Q}_{n_k} \right\| + \left\| K\mathcal{Q}_{n_k} - K\omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left\| K\mathcal{Q}_{n_k} - K\omega \right\| \\
&= \liminf_{n\longrightarrow\infty} \left\| \rho_N\left(\mathcal{S}_N\mathcal{V}_{N-1}\mathcal{Q}_{n_k} - \mathcal{S}_N\mathcal{V}_{N-1}\omega \right) + (1-\rho_N)\left(\mathcal{V}_{N-1}\mathcal{Q}_{n_k} - \mathcal{V}_{N-1}\omega \right) \right\| \\
&\le \liminf_{n\longrightarrow\infty} \left(\rho_N \left\| \mathcal{S}_N\mathcal{V}_{N-1}\mathcal{Q}_{n_k} - \mathcal{S}_N\mathcal{V}_{N-1}\omega \right\| + (1-\rho_N)\left\| \mathcal{V}_{N-1}\mathcal{Q}_{n_k} - \mathcal{V}_{N-1}\omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left(\rho_N L_N \left\| \mathcal{V}_{N-1}\mathcal{Q}_{n_k} - \mathcal{V}_{N-1}\omega \right\| + (1-\rho_N)\left\| \mathcal{V}_{N-1}\mathcal{Q}_{n_k} - \mathcal{V}_{N-1}\omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left\| \mathcal{V}_{N-1}\mathcal{Q}_{n_k} - \mathcal{V}_{N-1}\omega \right\| \\
&\le \liminf_{n\longrightarrow\infty} \left(\rho_{N-1} \left\| \mathcal{S}_{N-1}\mathcal{V}_{N-2}\mathcal{Q}_{n_k} - \mathcal{S}_{N-1}\mathcal{V}_{N-2}\omega \right\| + (1-\rho_{N-1})\left\| \mathcal{V}_{N-2}\mathcal{Q}_{n_k} - \mathcal{V}_{N-2}\omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left(\rho_{N-1}L_{N-1} \left\| \mathcal{V}_{N-2}\mathcal{Q}_{n_k} - \mathcal{V}_{N-2}\omega \right\| + (1-\rho_{N-1})\left\| \mathcal{V}_{N-2}\mathcal{Q}_{n_k} - \mathcal{V}_{N-2}\omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left\| \mathcal{V}_{N-2}\mathcal{Q}_{n_k} - \mathcal{V}_{N-2}\omega \right\| \\
&\vdots \\
&\le \liminf_{n\longrightarrow\infty} \left\| \mathcal{V}_1\mathcal{Q}_{n_k} - \mathcal{V}_1\omega \right\| \\
&\le \liminf_{n\longrightarrow\infty} \left(\rho_1 \left\| \mathcal{S}_1\mathcal{Q}_{n_k} - \mathcal{S}_1\omega \right\| + (1-\rho_1)\left\| \mathcal{Q}_{n_k} - \omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left(\rho_1 L_1 \left\| \mathcal{Q}_{n_k} - \omega \right\| + (1-\rho_1)\left\| \mathcal{Q}_{n_k} - \omega \right\| \right) \\
&\le \liminf_{n\longrightarrow\infty} \left\| \mathcal{Q}_{n_k} - \omega \right\|.
\end{aligned} \tag{68}$$

This is a contradiction. Therefore, it follows that $\omega \in F(K)$. Applying Lemma 8, it implies that

$$\omega \in \bigcap_{i=1}^{N} F(\mathcal{S}_i). \tag{69}$$

From (64), (65), and (69), it follows that

$$\omega \in \Omega. \tag{70}$$

Since $\mathcal{Q}_{n_k} \rightharpoonup \omega$ and (70), we can conclude that

$$\begin{aligned}\limsup_{n\longrightarrow\infty}\langle u - v, \mathcal{Q}_n - v\rangle &= \lim_{k\longrightarrow\infty}\left\langle u - v, \mathcal{Q}_{n_k} - v\right\rangle \\ &= \langle u - v, \omega - v\rangle \\ &\leq 0.\end{aligned} \tag{71}$$

Let $v = P_\Omega u$, from Lemma 8, (49) and (51), we have

$$\begin{aligned}&\|d_n - v\|^2 \\ &= \frac{\xi_n}{1-\varepsilon_n}\|c_n - v\|^2 + \frac{\psi_n}{1-\varepsilon_n}\|K_n\mathcal{Q}_n - v\|^2 - \frac{\xi_n\psi_n}{(1-\varepsilon_n)^2}\|c_n - K_n\mathcal{Q}_n\|^2 \\ &\leq \frac{\xi_n}{1-\varepsilon_n}\|c_n - v\|^2 + \frac{\psi_n}{1-\varepsilon_n}\|K_n\mathcal{Q}_n - v\|^2 \\ &\leq \frac{\xi_n}{1-\varepsilon_n}\|\mathcal{Q}_n - v\|^2 + \frac{\psi_n}{1-\varepsilon_n}\|\mathcal{Q}_n - v\|^2 \\ &= \|\mathcal{Q}_n - v\|^2.\end{aligned} \tag{72}$$

Applying Lemma 5, the definition of $\mathcal{Q}_n$, (72), and $v = P_\Omega u$, thus we get

$$\begin{aligned}&\|\mathcal{Q}_{n+1} - v\|^2 \\ &= \left\|\zeta_n(\mathcal{Y}_n - v) + (1-\zeta_n)\left(P_{\mathcal{C}}\left(I - \rho\sum_{i=1}^{\widehat{N}} b_n^i\mathcal{E}_i\right)\mathcal{Q}_n - v\right)\right\|^2 \\ &= \Bigg\|\zeta_n(\varepsilon_n(u - v) + (1-\varepsilon_n)(d_n - v)) \\ &\quad + (1-\zeta_n)\left(P_{\mathcal{C}}\left(I - \rho\sum_{i=1}^{\widehat{N}} b_n^i\mathcal{E}_i\right)\mathcal{Q}_n - v\right)\Bigg\|^2 \\ &= \Bigg\|\zeta_n\varepsilon_n(u - v) + \zeta_n(1-\varepsilon_n)(d_n - v) \\ &\quad + (1-\zeta_n)\left(P_{\mathcal{C}}\left(I - \rho\sum_{i=1}^{\widehat{N}} b_n^i\mathcal{E}_i\right)\mathcal{Q}_n - v\right)\Bigg\|^2 \\ &\leq \left\|\zeta_n(1-\varepsilon_n)(d_n - v) + (1-\zeta_n)\left(P_{\mathcal{C}}\left(I - \rho\sum_{i=1}^{\widehat{N}} b_n^i\mathcal{E}_i\right)\mathcal{Q}_n - v\right)\right\|^2 \\ &\quad + 2\zeta_n\varepsilon_n\langle u - v, \mathcal{Q}_{n+1} - v\rangle \\ &\leq \left(\zeta_n(1-\varepsilon_n)\|d_n - v\| + (1-\zeta_n)\left\|P_{\mathcal{C}}\left(I - \rho\sum_{i=1}^{\widehat{N}} b_n^i\mathcal{E}_i\right)\mathcal{Q}_n - v\right\|\right)^2 \\ &\quad + 2\zeta_n\varepsilon_n\langle u - v, \mathcal{Q}_{n+1} - v\rangle \\ &\leq \left(\zeta_n(1-\varepsilon_n)\|\mathcal{Q}_n - v\| + (1-\zeta_n)\|\mathcal{Q}_n - v\|\right)^2 + 2\zeta_n\varepsilon_n\langle u - v, \mathcal{Q}_{n+1} - v\rangle \\ &= (1-\zeta_n\varepsilon_n)^2\|\mathcal{Q}_n - v\|^2 + 2\zeta_n\varepsilon_n\langle u - v, \mathcal{Q}_{n+1} - v\rangle \\ &\leq (1-\zeta_n\varepsilon_n)\|\mathcal{Q}_n - v\|^2 + 2\zeta_n\varepsilon_n\langle u - v, \mathcal{Q}_{n+1} - v\rangle.\end{aligned} \tag{73}$$

From (71), the conditions (i), (ii), and Lemma 4, we can conclude that $\{\mathcal{Q}_n\}$ converges strongly to $v = P_\Omega u$. From (59), we also obtain that $\{\mathcal{Z}_n\}$ converges strongly to $v = P_\Omega u$.

Case 14. Assume that there exists a subsequence $\{\Gamma_{n_i}\} \subset \{\Gamma_n\}$ such that $\Gamma n_i \le \Gamma_{n_i+1}$ for all $i \in \mathbb{N}$. In this case, we can define $\tau\colon \mathbb{N} \longrightarrow \mathbb{N}$ by $\tau(n) = \max\{k \le n\colon \Gamma_k < \Gamma_{k+1}\}$. Then, we obtain $\tau(n) \longrightarrow \infty$ as $n \longrightarrow \infty$ and $\Gamma_{\tau(n)} < \Gamma_{\tau_n+1}$. This implies by (72) that

$$\begin{aligned} &\zeta_{\tau(n)}\xi_{\tau(n)}\left(1-\frac{\rho}{\eta}\right)\left\|c_{\tau(n)} - y_{\tau(n)}\right\|^2 + \zeta_{\tau(n)}\xi_{\tau(n)}\left(1-\frac{\rho}{\eta}\right)\left\|x_{\tau(n)} - y_{\tau(n)}\right\|^2 \\ &\quad + \frac{\zeta_{\tau(n)}\xi_{\tau(n)}\psi_{\tau(n)}}{1-\varepsilon_{\tau(n)}}\left\|c_{\tau(n)} - K_{\tau(n)}x_{\tau(n)}\right\|^2 \\ &\le \zeta_{\tau(n)}\varepsilon_{\tau(n)}\left\|u - g^*\right\|^2 + \left\|x_{\tau(n)} - g^*\right\|^2 - \left\|x_{\tau(n)+1} - g^*\right\|^2. \end{aligned} \tag{74}$$

Using the same method as in Case 13, it yields that

$$\lim_{n\longrightarrow\infty}\left\|c_{\tau(n)} - y_{\tau(n)}\right\| = \lim_{n\longrightarrow\infty}\left\|x_{\tau(n)} - y_{\tau(n)}\right\| = \lim_{n\longrightarrow\infty}\left\|c_{\tau(n)} - K_{\tau(n)}x_{\tau(n)}\right\| = 0. \tag{75}$$

Since $\{x_{\tau(n)}\}$ is a bounded sequence, then there exists a subsequence $\{x_{\tau(n_k)}\}$ such that

$$\limsup_{n\longrightarrow\infty}\left\langle u - v, x_{\tau(n)} - v\right\rangle = \lim_{k\longrightarrow\infty}\left\langle u - v, x_{\tau(n_k)} - v\right\rangle, \text{where } \mathrm{v} = \mathrm{P}_\Omega \mathrm{u}. \tag{76}$$

Applying the same proof of Case 13 for $\{x_{\tau(n_k)}\}$, we get

$$\limsup_{n\longrightarrow\infty}\left\langle u - v, x_{\tau(n)} - v\right\rangle \le 0 \tag{77}$$

$$\left\|x_{\tau(n)+1} - v\right\|^2 \le \left(1 - \zeta_{\tau(n)}\varepsilon_{\tau(n)}\right)\left\|x_{\tau(n)} - v\right\|^2 + 2\zeta_{\tau(n)}\varepsilon_{\tau(n)}\left\langle u - v, x_{\tau(n)+1} - v\right\rangle. \tag{78}$$

By Lemma 4, we have

$$\begin{aligned} &\lim_{n\longrightarrow\infty}\left\|x_{\tau(n)} - v\right\| = 0, \\ &\lim_{n\longrightarrow\infty}\left\|\mathrm{x}_{\tau(n)+1} - \mathrm{v}\right\| = 0. \end{aligned} \tag{79}$$

Hence, by Lemma 10, we obtain

$$0 \le \left\|\mathcal{Q}_n - v\right\| \le \max\left\{\left\|x_{\tau(n)} - v\right\|, \left\|\mathcal{Q}_n - v\right\|\right\} \le \left\|x_{\tau(n)+1} - v\right\| \tag{80}$$

Therefore, we can conclude that $\{\mathcal{Q}_n\}$ converges strongly to $v = P_\Omega u$. From (59), we also have $\{\mathcal{Z}_n\}$ converging strongly to $v = P_\Omega u$. The proof is complete.

Remark 15. Since the S-subgradient extragradient method covers various type of iterations such as the modified subgradient extragradient method (MSEGM), the subgradient extragradient method (SEGM), and the extragradient method (EGM), Theorem 12 can be seen as a modification and extension of several research papers, see, for example, [14, 15, 19, 20].

4. Numerical Examples

Numerical examples are provided in this section to back up the main result.

Example 1. Let $\mathscr{C} = [0, 100]$ and $\mathbb{R}$ be the set of real numbers. Define the mappings $\mathscr{D}_1, \mathscr{D}_2, \mathscr{D}_3, \mathscr{E}_1, \mathscr{E}_2, \mathscr{E}_3$: $\mathscr{C} \longrightarrow \mathbb{R}$ by

$$\begin{aligned} &\mathscr{D}_1 u = 3u, \mathscr{D}_2 u = \frac{4}{3}u, \mathscr{D}_3 u = \frac{1}{2}u, \\ &\mathscr{E}_1 u = 5u, \mathscr{E}_2 u = \frac{2}{7}u, \mathscr{E}_3 u = \frac{4}{3}u, \quad \forall u \in \mathbb{R}. \end{aligned} \tag{81}$$

For $i = 1, 2, \ldots, N$, let $\mathscr{S}_i$: $\mathbb{R} \longrightarrow \mathbb{R}$ be defined by

$$\begin{aligned} \mathscr{S}_i u &= \frac{iu}{i + 10}, \quad \forall u \in \mathbb{R}, \\ \rho_i^n &= \frac{2n}{500n + i}, \quad \forall n \in \mathbb{N}. \end{aligned} \tag{82}$$

Put $\varepsilon_n = 1/6n$, $\xi_n = 3n - 2/6n$, $\psi_n = 3n + 2/6n$, and $\zeta_n = n + 1/3n + 2, b_n^1 = n + 1/3n + 5, b_n^2 = n + 4/3n + 5, b_n^3 = n/3n + 5, \forall n \in \mathbb{N}$. Choose $u = 5, \rho = 1/4, \rho = 1/5$ and $a_i = 1/3$, $i = 1, 2, 3$. Then, $\{\mathscr{Q}_n\}$ and $\{\mathscr{Z}_n\}$ converge strongly to 0.

Solution. Clearly, all sequences $\{\varepsilon_n\}, \{\xi_n\}, \{\gamma_n\}, \{\zeta_n\}$ satisfy all conditions of Theorem 12. Moreover, $\mathscr{S}_i$ is 0-demicontractive mappings and $i/i + 10$-Lipschitzian mappings, for all $i = 1, 2, \ldots, N$. Choose $\gamma_1 = 1/3$ and $\gamma_2 = 1/2$, thus we get $\gamma_1 + \gamma_2 = 5/6 < 1$. Since K_n is a K-mapping generated by $\mathscr{S}_1, \mathscr{S}_2, \ldots, \mathscr{S}_N$ and $\rho_1^n, \rho_2^n, \rho_3^n, \ldots, \rho_N^n$, then

$$\begin{aligned} \mathscr{V}_1 \mathscr{Q}_n &= \frac{2n}{500n + 1} \mathscr{S}_1 \mathscr{Q}_n + \left(1 - \frac{2n}{500n + 1}\right) \mathscr{Q}_n, \\ \mathscr{V}_2 \mathscr{Q}_n &= \frac{2n}{500n + 2} \mathscr{S}_2 \mathscr{V}_1 \mathscr{Q}_n + \left(1 - \frac{2n}{500n + 2}\right) \mathscr{V}_1 \mathscr{Q}_n, \\ \mathscr{V}_3 \mathscr{Q}_n &= \frac{2n}{500n + 3} \mathscr{S}_3 \mathscr{V}_2 \mathscr{Q}_n + \left(1 - \frac{2n}{500n + 3}\right) \mathscr{V}_2 \mathscr{Q}_n, \\ &\vdots \\ \mathscr{V}_{N-1} \mathscr{Q}_n &= \frac{2n}{500n + N - 1} \mathscr{S}_{N-1} \mathscr{V}_{N-2} \mathscr{Q}_n + \left(1 - \frac{2n}{500n + N - 1}\right) \mathscr{V}_{N-2} \mathscr{Q}_n, \\ K_n \mathscr{Q}_n = \mathscr{V}_N \mathscr{Q}_n &= \frac{2n}{500n + N} \mathscr{S}_N \mathscr{V}_{N-1} \mathscr{Q}_n + \left(1 - \frac{2n}{500n + N}\right) \mathscr{V}_{N-1} \mathscr{Q}_n. \end{aligned} \tag{83}$$

Hence, we obtain

$$\{0\} = \bigcap_{i=1}^{N} F(\mathscr{S}_i) \bigcap \bigcap_{i=1}^{3} \mathrm{VI}(\mathscr{C}, \mathscr{D}_i) \bigcap \bigcap_{i=1}^{3} \mathrm{VI}(\mathscr{C}, \mathscr{E}_i). \tag{84}$$

By Theorem 12, the sequences $\{\mathscr{Q}_n\}$ and $\{\mathscr{Z}_n\}$ converge strongly to 0.

Table 1 and Figure 1 show the values of sequences $\{\mathscr{Q}_n\}$ and $\{\mathscr{Z}_n\}$ where $u = \mathscr{Q}_1 = 5$ and $n = N = 30$.

Example 2. Let $\mathbb{R}^2$ be the two-dimensional space of real numbers with an inner product $\langle \cdot, \cdot \rangle$: $\mathbb{R}^2 \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ defined by $\langle u, v \rangle = u \cdot v = u_1 v_1 + u_2 v_2$ and a usual norm $\|\cdot\|$: $\mathbb{R}^2 \longrightarrow \mathbb{R}$ given by $\|u\| = \sqrt{u_1^2 + u_2^2}$, for every $u = (u_1, u_2) \in \mathbb{R}^2$. Suppose $\mathscr{C} = \{(u_1, u_2) \in \mathscr{H}\colon 0 \le u_1, u_2 \le 100\}$. Define the mappings $\mathscr{D}_1, \mathscr{D}_2, \mathscr{E}_1, \mathscr{E}_2$: $\mathscr{C} \longrightarrow \mathbb{R}^2$ by

$$\begin{aligned} \mathscr{D}_1(u_1, u_2) &= \left(\frac{3u_1}{2}, \frac{3u_2}{2}\right), \\ \mathscr{D}_2(u_1, u_2) &= (5u_1, 5u_2), \\ \mathscr{E}_1(u_1, u_2) &= (4u_1, 4u_2), \\ \mathscr{E}_2(u_1, u_2) &= \left(\frac{5u_1}{8}, \frac{5u_2}{8}\right), \quad \forall (u_1, u_2) \in \mathbb{R}^2. \end{aligned} \tag{85}$$

For $i = 1, 2, \ldots, N$, let $\mathscr{S}_i$: $\mathbb{R}^2 \longrightarrow \mathbb{R}^2$ be defined by

$$\mathscr{S}_i(u_1, u_2) = \left(\frac{iu_1}{i + 100}, \frac{iu_2}{i + 100}\right), \quad \forall (u_1, u_2) \in \mathbb{R}^2. \tag{86}$$

Take $b_n^1 = n + 4/3n + 5, b_n^2 = 2n + 1/3n + 5, \forall n \in \mathbb{N}$. All sequences and other parameters are defined as in Example 1. Let $\rho = 1/4, \rho = 1/8$ and $a_i = 1/2, i = 1, 2$. Therefore, $\{\mathscr{Q}_n\}$ and $\{\mathscr{Z}_n\}$ converge strongly to (0,0).

Solution. Clearly, all sequences, parameters, and mappings satisfy all conditions of Theorem 12. Hence,

Table 1: The values of $\{\mathscr{Z}_n\}$ and $\{\mathscr{Q}_n\}$ with $n = N = 30$.

n	$\mathscr{Z}_n$	$\mathscr{Q}_n$
1	2.986111	5.000000
2	2.293376	3.840071
3	1.646740	2.757332
4	1.146351	1.919472
5	0.783242	1.311475
⋮	⋮	⋮
15	0.013890	0.023258
⋮	⋮	⋮
26	0.000123	0.000206
27	0.000082	0.000137
28	0.000055	0.000091
29	0.000035	0.000058
30	0.000022	0.000037

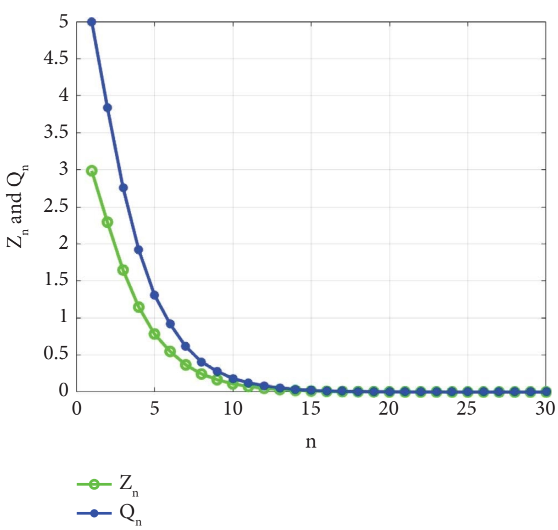

Figure 1: The convergence comparison of $\{\mathscr{Z}_n\}$ and $\{\mathscr{Q}_n\}$ with $u = \mathscr{Q}_1 = 5$.

Table 2: The values of $\{\mathscr{Z}_n\}$ and $\{\mathscr{Q}_n\}$ with $u = \mathscr{Q}_1 = (2, 2)$ and $n = N = 50$.

n	$\mathscr{Z}_n$	$\mathscr{Q}_n$
1	(0.375000, 0.375000)	(2.000000, 2.000000)
2	(0.263465, 0.263465)	(1.405144, 1.405144)
3	(0.181010, 0.181010)	(0.965388, 0.965388)
4	(0.125451, 0.125451)	(0.669071, 0.669071)
5	(0.100423, 0.100423)	(0.535589, 0.535589)
⋮	⋮	⋮
25	(0.004070, 0.004070)	(0.021705, 0.021705)
⋮	⋮	⋮
46	(0.001966, 0.001966)	(0.010486, 0.010486)
47	(0.001811, 0.001811)	(0.009660, 0.009660)
48	(0.001893, 0.001893)	(0.010095, 0.010095)
49	(0.001742, 0.001742)	(0.009290, 0.009290)
50	(0.001819, 0.001819)	(0.009704, 0.009704)

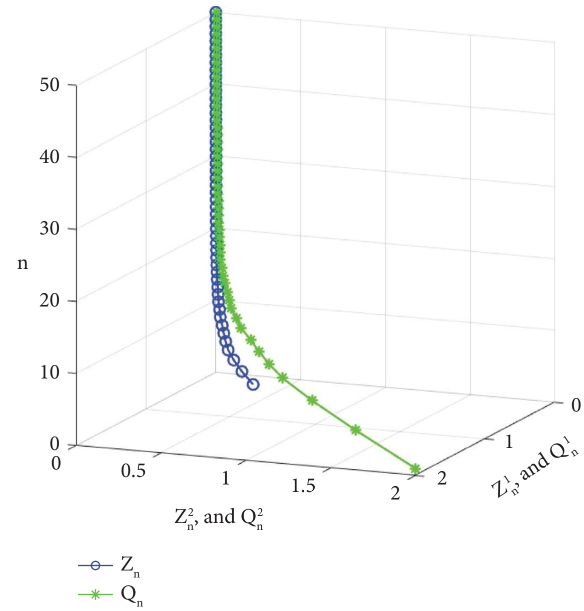

Figure 2: The convergence of $\{\mathscr{Z}_n\}$ and $\{\mathscr{Q}_n\}$ in a three-dimensional space with $u = \mathscr{Q}_1 = (2, 2)$ and $n = N = 50$.

$$\{(0,0)\} = \bigcap_{i=1}^{N} F(\mathscr{S}_i) \bigcap \bigcap_{i=1}^{2} VI(\mathscr{C}, \mathscr{D}_i) \cap \bigcap_{i=1}^{2} VI(\mathscr{C}, \mathscr{E}_i). \quad (87)$$

Applying Theorem 12, the sequences $\{\mathscr{Q}_n\}$ and $\{\mathscr{Z}_n\}$ converge strongly to (0,0).

Table 2 and Figure 2 show the values of sequences $\{\mathscr{Q}_n\}$ and $\{\mathscr{Z}_n\}$ where $u = \mathscr{Q}_1 = (2, 2)$ and $n = N = 50$.

5. Conclusion

This study proposes a new subgradient extragradient method for approximating a common fixed point of a finite family of demicontractive mappings and Lipschitzian mappings and a common solution of variational inequality problems. It can also be considered as an extension and modification of several currently used techniques for solving variational inequality problems as well as a fixed point problem with some associated mappings. As special cases of Theorem 12, previous publications such as [14, 15, 19, 20] can be considered. Also, numerical illustrations of the main theorem are given [24, 25].

Data Availability

No data were used to support this study.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

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