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Research Article

A Solution Matrix by IEVP under the Central Principle Submatrix Constraints

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The $n \times n$ real matrix P is called centrosymmetric matrix if $P = RPR$, where R is permutation matrix with ones on cross diagonal (bottom left to top right) and zeroes elsewhere. In this article, the solvability conditions for left and right inverse eigenvalue problem (which is special case of inverse eigenvalue problem) under the submatrix constraint for generalized centrosymmetric matrices are derived, and the general solution is also given. In addition, we provide a feasible algorithm for computing the general solution, which is proved by a numerical example.

1. Introduction

Linear algebra plays an important role in different fields of engineering and sciences. Especially matrix algebra has been used in many fields of applications since the 19$^{\text{th}}$ century [1, 2]. A symmetric matrix is a well-known type of matrix that also possesses numerous applications [3, 4]. One of the famous types of a symmetric matrix is the Centro-symmetric matrix (also known as cross-symmetric matrix) [5]. These matrices are symmetric about their center and play an important role in several applications such as antenna theory, vibration in structure, detection, estimation, electrically network engineering, communication sciences, predication theory, speech analysis, quantum physics, etc. [3, 4, 6–13]. The study of eigenvalues and eigenvectors plays an important role in enhancement in the fields of applications of Centro-symmetric matrices. Eigenvalues are a particular set of scalars associated with the system of linear equations whereas, eigenvectors are vectors that are changed by scalars when the linear transformation is applied [14, 15]. Symmetric matrices are an unusual structure that helps with main component extraction and determinant assessment by reducing the computational complexity [12]. For n ordered Centro-symmetric matrices, the issue of principal component extraction may be simplified into two subproblems of principal component assessment with orders $n/2$ for even and $(n-1)/2$ and $(n+1)/2$ for odd. The multiplicative complexity associated with the computation of the determinant and main components is reduced roughly by 75% [12]. An algorithm for the evaluation of eigenvalues and eigenvectors for centro-symmetric matrices with numerical solutions has been discussed [13]. Currently, many researchers have been working on special types of eigenvalues problems known as IEP (Inverse eigenvalues problem) [16]. IEP concerns the reconstruction of a structured matrix from prescribed spectral data, which has been used in control theory, vibration theory, structural design, molecular spectroscopy, etc. In the case of Centro-symmetric potentials, authors [17] explicitly solved IEP for classical Liouville functions in the term of Hermitian characteristics functions and found that eigenvalues were related to the constant of motion. Furthermore [17], obtained the same results for the Wigner distribution function by using the semi-classical quantum wave function. The results related to the eigenvector and eigenvalues for per-symmetric matrices occurred in communication and information theories have been already discussed [18]. In [19], the authors studied quadratic IEP for dumped structural updating model and developed a new approach without using eigenvector expansion techniques. In [20, 21], the natural frequencies of vibration of a beam of a given length in the free configuration were found by solving the Eigenvalue problem. On the bases of the solution of IEP, the method was developed for the design of the structure with low-order natural frequencies [22]. IEP is also applicable for finding the solution to problems related to molecular spectroscopy [23]. In [24], the authors modified various quadratically convergent methods for solving IEVP with consideration of both cases such as distinct and multiple eigenvalues. While studying the papers, it has been observed that many authors studied IEVP for Centro-symmetric matrices under the submatrix constraint [25–29]. But few authors studied left and right IEVP for Centro-symmetric matrix under submatrix constraint [30–32]. Therefore, we will consider the problem of the left and right IEVP of the Centro-symmetric matrix with central principal submatrix constraint. In this paper, we will consider that the extend matrix and submatrix both have the same structure ad also both are centro-symmetric matrices. This paper is divided into 4 sections. The first part is introductory, notations and some important definitions are in the second part. In the third part, we provide existence and expression for solution of the inverse eigenvalue problem for centrosymmetric matrices under central principal submatrix constraint and conclusion included in fourth part.

2. Notations and Preliminaries

In this article, we consider O and $R^{m\times n}$ be the sets of all $(n\times n)$ orthogonal matrices and $(m\times n)$ the real matrices respectively. P^{+}, $\rho(P)$ and P^{T} denotes the Moore-Penrose generalized inverse, rank, and transpose of matrix P respectively. An n^{th} order identity matrix, reverse identity matrix and zero matrices are represented by I_n, S_n and 0 respectively. For any two matrices $P=(p_{i,j})$ and $Q=(q_{i,j})\in\mathscr{R}^{m\times n}$, the Hadamard and inner product are represented by $P*Q=(p_{i,j}q_{i,j})$ and $\langle P,Q\rangle=\text{trace}\,(Q^{T}P)$, respectively. $\mathscr{R}^{m\times n}$ represented as Hilbert space and notation used for Frobenius norm of matrix P is $\|P\|=(\text{trace}\,(P^{T}P))^{1/2}$. The scalars λ and μ denotes the left and right eigenvalues of matrix and x and y be left and right eigenvectors corresponding these eigenvalues.

2.1. Basic Definitions for Centro-Symmetric Matrix. In this section, we present some important definitions for Centro-symmetric matrices with some appropriate examples. Definition 1 is defined for the construction of the Centro-symmetric matrix and Definition 2 for finding the central principal submatrix and trailing principal submatrix, Definition 3 for orthogonality, and Definition 4 for left and right eigenpairs. Similarly, Definition 5 for symmetric and anti-symmetric vectors.

Definition 1. The centro-symmetric matrix $P=(p_{i,j})\in\mathscr{R}^{n\times n}$ is defined by

$$p_{i,j}=p_{n+1-i,n+1-j},\tag{1}$$

where i, j are natural numbers. $\text{CS}\mathscr{R}^{n\times n}$ denote the set of all $(n\times n)$ Centro-symmetric matrices. For example, $P=\begin{bmatrix} a & b & c\\ d & e & d\\ c & b & a\end{bmatrix}$ is a Centro-symmetric matrix of order 3.

Several researchers have shown great interest in the study of inverse eigenvalue problems under submatrices constraint [25, 27, 28, 33]. Due to the unique structure of the Centro-symmetric matrix, it is not suitable for discussing Centro-symmetric matrices under principal submatrices constraints, for it destroys the symmetry in the structure of Centro-symmetric matrices. Therefore, we discuss the different concepts like the central principal submatrix, which is first defined in [25].

Definition 2. An m-square central principal matrix $P_c(m)$ of the matrix P, is given by

$$P_c(m)=(0,I_m,0)P(0,I_m,0)^{T},\tag{2}$$

where 0 is a zero matrix of order $(m\times((n-m)/2))$, and I is an identity matrix of order m.

For example, for the matrix P is of order 4, then $P_c(2)$ is

$$P=\begin{bmatrix} a_1 & a_2 & a_3 & a_4\\ b_1 & \mathbf{b_2} & \mathbf{b_3} & b_4\\ b_4 & \mathbf{b_3} & \mathbf{b_2} & b_1\\ a_4 & a_3 & a_2 & a_1\end{bmatrix},\tag{3}$$

Here, $P_c(2)=\begin{bmatrix} b_2 & b_3\\ b_3 & b_2\end{bmatrix}$ is the central principal submatrix of order 2.

And for the matrix P is of order 5, then $P_c(3)$ is

$$P=\begin{bmatrix} a_1 & a_2 & a_3 & a_4 & a_5\\ b_1 & \mathbf{b_2} & \mathbf{b_3} & \mathbf{b_4} & b_5\\ c_1 & \mathbf{c_2} & \mathbf{c_3} & \mathbf{c_2} & c_1\\ b_5 & \mathbf{b_4} & \mathbf{b_3} & \mathbf{b_2} & b_1\\ a_5 & a_4 & a_3 & a_2 & a_1\end{bmatrix},\tag{4}$$

Here, $P_c(3)=\begin{bmatrix} b_2 & b_3 & b_4\\ c_2 & c_3 & c_2\\ b_4 & b_3 & b_2\end{bmatrix}$ is the central principal submatrix of order 3.

From above these two examples, it is clearly shown that the submatrix is lie in the center of the matrix, and matrices of order even (odd) have only submatrices of order even (odd). The central principal submatrix is also a centrosymmetric matrix.

2.1.1. Trailing Principal Submatrix. A m-square trailing principal submatrix $P_t(m)$ of real matrix P if

$$P_t(m)=\left(0_{m,n-m},I_m\right)P\left(0_{m,n-m},I_m\right)^{T},\tag{5}$$

where 0 is a zero matrix of order $(m\times(n-m))$, and I is an identity matrix of order m.

For example, firstly, consider the matrix P is of even order given in (3), then using (5), we get $P_t(2) = \begin{bmatrix} c_3 & c_4 \\ d_3 & d_4 \end{bmatrix}$ is trailing principal submatrix of order 2.

Secondly, consider the matrix P is of odd order given in (4), then using (5), we get $P_t(3) = \begin{bmatrix} c_3 & c_4 & c_5 \\ d_3 & d_4 & d_5 \\ e_3 & e_4 & e_5 \end{bmatrix}$ is trailing principal submatrix of order 3.

From above these two examples, it is clearly shown that the submatrix is situated in the left corner of the matrix, and matrices of order even (odd) have only submatrices of order even (odd). The trailing principal submatrix is not centro-symmetric matrix.

Definition 3. If matrix O of order n satisfies $O^T O = OO^T = I$, (where I is an identity matrix of order n), then the matrix is called an orthogonal matrix [34].

Definition 4. A pair involved eigenvalue and its corresponding eigenvector i.e., (λ, x), where λ is an eigenvalue and x is corresponding eigenvector is known as an eigenpair. For partial right eigenpairs $(\lambda_i, x_i), i = 1, 2, \ldots, h_1$, and left eigenpairs $(\mu_j, y_j), j = 1, 2, \ldots, h_2$, to construct a $(n \times n)$ matrix $P \in S$, such that

$$\begin{cases} Px_i = \lambda_i x_i, & i = 1, 2, \ldots, h_1, \\ y_j^T P = \mu_j y_j^T, & j = 1, 2, \ldots, h_2, \end{cases} \tag{6}$$

where $h_1 \leq n, h_2 \leq n, \lambda_i, \mu_j$ are eigenvalues, and x_i, y_j are their corresponding eigenvectors, and S is a subspace of $\mathscr{R}^{n\times n}$. In this paper, we discuss the problem and its optimal approximation for the centro-symmetric matrix under central principal submatrix constraint that is in the light of extended matrix which preserves the Centro-symmetric property. Now, we discuss the special structure of eigenvalues and their corresponding eigenvectors for a real matrix. If P matrix has real right eigenpairs (λ_i, x_i), $i = 1, 2, \ldots, h_1$, where $\lambda_i = \alpha_i + \sqrt{-1}\beta_i, x_i = \xi_i + \sqrt{-1}\eta_i$ $(\alpha_i, \beta_i, \xi_i, \eta_i$ are real numbers), then, let $\widehat{\Lambda}_i = \lambda_i, \widehat{X}_i = x_i$. If P matrix has real right eigenpairs (λ_i, x_i), $i = 1, 2, \ldots, h_1$, where $\lambda_i = \alpha_i + \sqrt{-1}\beta_i$, $x_i = \xi_i + \sqrt{-1}\eta_i$ $(\alpha_i, \beta_i, \xi_i, \eta_i$ are real numbers, $1 \leq i \leq h_1)$, then, let $\widehat{X}_i = (\xi_i, \eta_i)$, and $\widehat{\Lambda}_i = \begin{pmatrix} \alpha_i & \beta_i \\ -\beta_i & \alpha_i \end{pmatrix}$. Hence, the equation (6) becomes

$$P\widehat{X}_i = \widehat{X}_i \widehat{\Lambda}_i, \tag{7}$$

Similarly, if $(\mu_j, y_j), j = 1, 2, \ldots, h_2$ are left real eigenpairs of P, then, let $\widehat{\Gamma}_j = \mu_j, \widehat{Y}_j = y_j$. If $(\mu_j, y_j), j = 1, 2, \ldots, h_2$ are left complex eigenpairs of P, then, let $\widehat{Y}_j = (\xi_i, \eta_i)$, and $\widehat{\Gamma}_j = \begin{pmatrix} \alpha_i & -\beta_i \\ \beta_i & \alpha_i \end{pmatrix}$. Now, (5) become

$$\widehat{Y}_j^{\,T} P = \widehat{\Gamma}_j \widehat{Y}_j^{\,T}. \tag{8}$$

For right eigenpairs (λ_i, x_i), $i = 1, 2, \ldots, h_1$, and left eigenpairs $(\mu_j, y_j), j = 1, 2, \ldots, h_2$, write

$$\left.\begin{aligned} &\Lambda = \operatorname{diag}\left(\widehat{\Lambda_1}, \widehat{\Lambda_2}, \ldots, \widehat{\Lambda_{h_1}}\right) \in R^{m\times m}, X = \left(\widehat{X_1}, \widehat{X_2}, \ldots, \widehat{X_{h_1}}\right) \in R^{n\times m} \\ &\Gamma = \operatorname{diag}\left(\widehat{\Gamma_1}, \widehat{\Gamma_2}, \ldots, \widehat{\Gamma_{h_2}}\right) \in R^{l\times l}, Y = \left(\widehat{Y_1}, \widehat{Y_2}, \ldots, \widehat{Y_{h_2}}\right) \in R^{n\times l} \end{aligned}\right\}. \tag{9}$$

Hence, equations (8) and (9) become

$$\left.\begin{aligned} PX &= X\Lambda \\ Y^T P &= \Gamma Y^T \end{aligned}\right\}. \tag{10}$$

Definition 5. Let $x \in R^n$. x is a symmetric vector if $S_n x = x$. x is an anti-symmetric vector if $S_n x = -x$.

Problem 6. If $\Lambda \in R^{m\times m}$, $X \in R^{n\times m}$, $\Gamma \in R^{l\times l}$, and $Y \in R^{n\times l}$ expressed in (8), and $P_0 \in CSR^{k\times k}$ be submatrix with Centro-symmetric structure, find an extended matrix $P \in CSR^{n\times n}$ such that

$$\begin{cases} PX = X\Lambda, \\ Y^T P = \Gamma Y^T, \end{cases} \text{and } P_C(k) = P_0. \tag{11}$$

3. Preliminary Lemmas and General Solutions to Problem 6

In this section, we discuss the central submatrix having the same symmetric properties and structure as the Centro-symmetric matrix. Therefore, they have similar expressions. Furthermore, we discuss the properties of eigenvalues and eigenvectors of the centro-symmetric matrix, and we expressed the special form of eigenvectors of the centro-symmetric matrix. Hence, using a special form of centro-symmetric matrix and its central principal submatrix, we convert Problem 6 into two inverse eigenvalue problems of half-sized independent real matrices under principal submatrices constraint. Furthermore, we provide necessary and sufficient conditions for the existence of a solution to Problem 6 and give an expression for the general solution.

Now, e_i be i^{th} ($i \in$ natural numbers) column of I_n, and let $S_n = (e_n, e_{n-1}, \ldots, e_2, e_1)$, then $S_n = S_n^{\,T}, S_n S_n^{\,T} = I_n$. Let $k = [n/2]$, where $[n/2]$ is the greatest integer less than or equal to $n/2$, and let orthogonal matrices:

$$D_n = \begin{cases} \dfrac{1}{\sqrt{2}} \begin{bmatrix} I_k & I_k \\ S_k & -S_k \end{bmatrix}, & n = 2k, \\ \\ \dfrac{1}{\sqrt{2}} \begin{bmatrix} I_k & 0 & I_k \\ 0 & \sqrt{2} & 0 \\ S_k & 0 & -S_k \end{bmatrix}, & n = 2k+1. \end{cases} \tag{12}$$

Lemma 7 (see [35]). *A matrix P is a Centro-symmetric matrix of order n iff $S_n P S_n = P$.*

Lemma 8 (see [25]). *A matrix $P \in CSR^{n\times n}$ if and only if there exist $P_1[n-k]$, $P_2[k]$ such that*

$$P = D_n\begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix}D_n^T. \quad (13)$$

Proof. When $n = 2k$, consider $P = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$, If $P \in CSR^{n\times n}$, then by Lemma 7 $S_kPS_k = P$

$\begin{bmatrix} 0 & S_k \\ S_k & 0 \end{bmatrix}\begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}\begin{bmatrix} 0 & S_k \\ S_k & 0 \end{bmatrix} = \begin{bmatrix} P_{11} & P_{12} \\ P_{21} & P_{22} \end{bmatrix}$,which is equivalent to $P_{22} = S_kP_{11}S_k = P_{11}, P_{12} = S_kP_{21}S_k = P_{21}$.

Hence,

$$D_n^TPD_n = \frac{1}{\sqrt{2}}\begin{bmatrix} I_k & S_k \\ I_k & -S_k. \end{bmatrix}.\begin{bmatrix} P_{11} & S_kP_{21}S_k \\ P_{21} & S_kP_{11}S_k \end{bmatrix}.\frac{1}{\sqrt{2}}\begin{bmatrix} I_k & I_k \\ S_k & -S_k. \end{bmatrix}$$
$$= \begin{bmatrix} P_{11} + P_{21}S_k & 0 \\ 0 & P_{11} - P_{21}S_k. \end{bmatrix}. \quad (14)$$

Let $P_1 = P_{11} + P_{21}S_k, P_2 = P_{11} - P_{21}S_k$, then $P = D_n\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}D_n^T$.

Conversely, for every $P_1, P_2 \in R^{k\times k}$, we have.

$$\begin{bmatrix} 0 & S_k \\ S_k & 0 \end{bmatrix}D_n\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}D_n^T\begin{bmatrix} 0 & S_k \\ S_k & 0 \end{bmatrix} = D_n\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}D_n^T. \quad (15)$$

From above, it is showing that it follows Lemma 7. Hence, $= D_n\begin{bmatrix} P_1 & 0 \\ 0 & P_2 \end{bmatrix}D_n^T \in \mathrm{CSR}^{n\times n}$.

Now, we discuss the special properties of the central principal submatrix of order k of the centro-symmetric matrix in which the submatrix has the same structure and symmetric properties as those given in the centro-symmetric matrix. Here, we assume = $[k/2]$, and $r = [n/2]$, where r and t are the greatest integer less than or equal to $k/2$ and $n/2$ respectively. □

Lemma 9. *Assume that $P \in CSR^{n\times n}$ have formed as in equation (12). Then the k-square central principal submatrix of P is given as*

$$P_c(k) = D_k\begin{bmatrix} P_1[k-t] & 0 \\ 0 & P_2[t] \end{bmatrix}D_k^T. \quad (16)$$

Proof. If $n = 2r$, from (12) and properties that are, a matrix of even order only having central principal submatrices of even order, so we have $k = 2t$, and

$P_c(k) = \begin{bmatrix} M & NS_t \\ S_tN & S_tMS_t \end{bmatrix}$, where $M, N \in R^{t\times t}$

Thus, $D_k^TP_c(k)D_k = 1/2.\begin{bmatrix} I_t & S_t \\ I_t & -S_t \end{bmatrix}.\begin{bmatrix} M & NS_t \\ S_tN & S_tMS_t \end{bmatrix}.\begin{bmatrix} I_t & I_t \\ S_t & -S_t \end{bmatrix} = \begin{bmatrix} M+N & 0 \\ 0 & M-N \end{bmatrix}$and set $M + N = P_1[k-t], M - N = P_2[t]$. Hence, the k-square central principal submatrix of P may be expressed as

$$P_c(k) = D_k\begin{bmatrix} P_1[t] & 0 \\ 0 & P_2[t] \end{bmatrix}D_k^T. \quad (17)$$

If $n = 2r + 1$, and a matrix of odd order has central principal submatrices of odd order, we have $k = 2t + 1$, and

$$P_c(k) = \begin{bmatrix} M & u_t & NS_t \\ v_t^T & \alpha & v_t^TS_t \\ S_tN & S_tu_t & S_tNS_t \end{bmatrix}. \quad (18)$$

where $M, N \in R^{t\times t}$, $u_t = (0, I_t)u, v_t^T = (0, I_t)v$.

Hence,

$$D_k^TP_c(k)D_k = \frac{1}{2}.\begin{bmatrix} I_t & 0 & S_t \\ 0 & \sqrt{2} & 0 \\ I_t & 0 & -S_t \end{bmatrix}.\begin{bmatrix} M & u_t & NS_t \\ v_t^T & \alpha & v_t^TS_t \\ S_tN & S_tu_t & S_tNS_t \end{bmatrix}.\begin{bmatrix} I_t & 0 & I_t \\ 0 & \sqrt{2} & 0 \\ S_t & 0 & -S_t \end{bmatrix}$$
$$= \begin{bmatrix} M+N & \sqrt{2}u_t & 0 \\ \sqrt{2}v_t^T & \alpha & 0 \\ 0 & 0 & M-N \end{bmatrix}. \quad (19)$$

By setting, $\begin{bmatrix} M+N & \sqrt{2}u_t \\ \sqrt{2}v_t^T & \alpha \end{bmatrix} = P_1[t+1]$, $M - N = P_2[t]$, then the $P_c(k)$ may be written as

$$P_c(k) = D_k\begin{bmatrix} P_1[t+1] & 0 \\ 0 & P_2[t] \end{bmatrix}D_k^T. \quad (20)$$

By combining (18) and (20), we get k-square central principal submatrix of P that has the form as in (16). □

Lemma 10. *Let $P \in CSR^{n\times n}$ have formed as in (12). Then the k-square central principal submatrix of P is given as*

$$P_0(k) = D_k \begin{bmatrix} P_{10} & 0 \\ 0 & P_{20} \end{bmatrix} D_k^T, \tag{21}$$

where $P_{10} \in R^{(k-t)\times(k-t)}$ and $P_{20} \in R^{t\times t}$. The matrix $P_0(k)$ is central principal both are trailing principal submatrix of P_1 and P_2, respectively.

Given $P \in CSR^{n\times n}$, if $(\lambda_i, x_i), (\mu_j, y_j)$ (where $1 \le i \le h_1$, $1 \le j \le h_2$) are right and left real eigenpairs respectively, then we get, from Lemma 7,

$$\begin{aligned} PS_n x_i &= S_n P x_i = \lambda_i S_n x_i, \\ \text{and } y_j^T S_n P &= y_j^T P S_n = \mu_j S_n y_j^T, \end{aligned} \tag{22}$$

Therefore, $x_i \pm S_n x_i$ are eigenvectors associated with λ_i, where $x_i + S_n x_i$ is a symmetric vector, while $x_i - S_n x_i$ is an anti-symmetric vector. Similarly, $y_j^T + S_n y_j^T$ is a symmetric vector, and $y_j^T - S_n y_j^T$ is an anti-symmetric vector.

If $(\lambda_i, x_i), (\mu_j, y_j)$ (where $1 \le i \le h_1, 1 \le j \le h_2$) are right and left complex eigenpairs respectively, then we get, from Lemma 7,

$$\begin{aligned} PS_n \widehat{X}_i &= S_n P \widehat{X}_i = \widehat{\Lambda}_i S_n \widehat{X}_i, \\ \text{and } \widehat{Y}_j^T S_n P &= \widehat{Y}_j^T P S_n = \widehat{\Gamma}_j S_n \widehat{Y}_j^T. \end{aligned} \tag{23}$$

Thus $P(\widehat{X}_i \pm S_n \widehat{X}_i) = (\widehat{X}_i \pm S_n \widehat{X}_i)\widehat{\Lambda}_i$ and $(\widehat{Y}_j^T \pm \widehat{Y}_j^T S_n)P = \widehat{\Gamma}_j(\widehat{Y}_j^T \pm \widehat{Y}_j^T S_n)$, where the columns of $\widehat{X}_i + S_n \widehat{X}_i = (\xi_i + S_n \xi_i, \eta_i + S_n \eta_i)$ are symmetric vectors, and $\widehat{X}_i - S_n \widehat{X}_i = (\xi_i - S_n \xi_i, \eta_i - S_n \eta_i)$. Similarly, the columns of $\widehat{Y}_j^T + \widehat{Y}_j^T S_n = (\xi_i + S_n \xi_i, \eta_i + S_n \eta_i)$ are symmetric vectors, and $\widehat{Y}_j^T - \widehat{Y}_j^T S_n = (\xi_i - S_n \xi_i, \eta_i - S_n \eta_i)$.

From the above analysis, without loss of generality, we may suppose that X, Y, and Γ, Λ in Problem 6 can be written as follows:

$$\begin{aligned} X &= \begin{cases} \begin{bmatrix} \widetilde{M}_1 & N_1 \\ S_r \widetilde{M}_1 & -S_r N_1 \end{bmatrix}, & n = 2r, \\ \begin{bmatrix} \widetilde{M}_1 & N_1 \\ \sqrt{2}\, c^T & 0 \\ S_r \widetilde{M}_1 & -S_r N_1 \end{bmatrix}, & n = 2r+1, \end{cases} \\ Y &= \begin{cases} \begin{bmatrix} \widetilde{M}_2 & N_2 \\ S_r \widetilde{M}_2 & -S_r N_2 \end{bmatrix}, & n = 2r, \\ \begin{bmatrix} \widetilde{M}_2 & N_2 \\ \sqrt{2}\, c^T & 0 \\ S_r \widetilde{M}_2 & -S_r N_2 \end{bmatrix}, & n = 2r+1, \end{cases} \end{aligned} \tag{24}$$

$$\begin{aligned} \Lambda &= \operatorname{diag}(\Lambda_1, \Lambda_2), \\ \Gamma &= \operatorname{diag}(\Gamma_1, \Gamma_2), \end{aligned} \tag{25}$$

where, $\widetilde{M}_1 \in R^{r\times s_1}$, $N_1 \in R^{r\times(m-s_1)}$, $c \in R^{s_1}$, $\widetilde{M}_2 \in R^{r\times s_2}$, $N_2 \in R^{r\times(l-s_2)}$, $d \in R^{s_2}$, $\Lambda_1 \in R^{s_1\times s_1}$, $\Lambda_2 \in R^{(m-s_1)\times(m-s_1)}$, $\Gamma_1 \in R^{s_2\times s_2}$, $\Gamma_2 \in R^{(l-s_2)\times(l-s_2)}$, where $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2$ are block diagonals. Thus, $D_n^T X$ and $D_n^T Y$ has the following form:

If $n = 2r$, then

$$\begin{aligned} D_n^T X &= \frac{1}{\sqrt{2}} \begin{bmatrix} I_r & S_r \\ I_r & -S_r \end{bmatrix} \cdot \begin{bmatrix} \widetilde{M}_1 & N_1 \\ S_r \widetilde{M}_1 & -S_r N_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,\widetilde{M}_1 & 0 \\ 0 & \sqrt{2}\, N_1 \end{bmatrix}, \\ D_n^T Y &= \frac{1}{\sqrt{2}} \begin{bmatrix} I_r & S_r \\ I_r & -S_r \end{bmatrix} \cdot \begin{bmatrix} \widetilde{M}_2 & N_2 \\ S_r \widetilde{M}_2 & -S_r N_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,\widetilde{M}_2 & 0 \\ 0 & \sqrt{2}\, N_2 \end{bmatrix}. \end{aligned} \tag{26}$$

If $n = 2r + 1$, then

$$\begin{aligned} D_n^T X &= \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} I_r & 0 & S_r \\ 0 & \sqrt{2} & 0 \\ I_r & 0 & -S_r \end{bmatrix} \cdot \begin{bmatrix} \widetilde{M}_1 & N_1 \\ \sqrt{2}\, c^T & 0 \\ S_r \widetilde{M}_1 & -S_r N_1 \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,\widetilde{M}_1 & 0 \\ \sqrt{2}\, c^T & 0 \\ 0 & \sqrt{2}\, N_1 \end{bmatrix}, \\ D_n^T Y &= \frac{1}{\sqrt{2}} \cdot \begin{bmatrix} I_r & 0 & S_r \\ 0 & \sqrt{2} & 0 \\ I_r & 0 & -S_r \end{bmatrix} \cdot \begin{bmatrix} \widetilde{M}_2 & N_2 \\ \sqrt{2}\, c^T & 0 \\ S_r \widetilde{M}_2 & -S_r N_2 \end{bmatrix} = \begin{bmatrix} \sqrt{2}\,\widetilde{M}_2 & 0 \\ \sqrt{2}\, c^T & 0 \\ 0 & \sqrt{2}\, N_2 \end{bmatrix}. \end{aligned} \tag{27}$$

Now, for $n = 2r$, set $\widetilde{M}_1 = M_1$, $\widetilde{M}_2 = M_2$, and for $n = 2r+1$, set $M_1 = \begin{pmatrix} \widetilde{M}_1 \\ c^T \end{pmatrix}$, $M_2 = \begin{pmatrix} \widetilde{N}_1 \\ d^T \end{pmatrix}$, then for all arbitrary n, $D_n^T X$ and $D_n^T Y$ may be written in the following form:

$$D_n^T X = \begin{bmatrix} \sqrt{2} M_1 & 0 \\ 0 & \sqrt{2} M_2 \end{bmatrix}, \quad D_n^T Y = \begin{bmatrix} \sqrt{2} N_1 & 0 \\ 0 & \sqrt{2} N_2 \end{bmatrix}. \tag{28}$$

where $M_1 \in R^{(n-r)\times s_1}$, $M_2 \in R^{r\times(m-s_1)}$, $N_1 \in R^{(n-r)\times}\ s_2$, $N_2 \in R^{r\times(l-s_2)}$.

Lemma 11. *Given $X \in R^{n\times m}$, $Y \in R^{n\times l}$, $\Lambda \in R^{m\times m}$, and $\Gamma \in R^{l\times l}$ as in (8), then there exists a matrix $P \in R^{n\times n}$ such that*

$$\begin{cases} PX = X\Lambda, \\ Y^T P = \Gamma Y^T. \end{cases} \tag{29}$$

If and only if $Y^T X\Lambda = \Gamma Y^T X$, $X\Lambda X^+ X = X\Lambda$, $\Gamma Y^T = Y^+ Y \Gamma Y^T$,

Moreover, its general solutions can be written as follows:

$$P = X\Lambda X^+ + \left(Y^T\right)^+ \Gamma Y^T \left(I - XX^+\right) + Q_1 G Q_2^T, \tag{30}$$

where $G \in R^{(n-r_1)\times(n-r_2)}$, $Q_1 \in R^{n\times(n-r_1)}$, $Q_1^T Q_1 = I_{n-r_1}$, $r_1 = \rho(Y)$, range space(Q_1) = null space(Y^T).

Lemma 12 (see [25]). *Assume that $X \in R^{m\times m}$, $Y \in R^{n\times l}$, $B \in R^{k\times l}$ be given. Denote*

$$U_1 \equiv \left\{P \in R^{m\times n} \mid f_1(P) = \|XPY - B\| = \min\right\}, \tag{31}$$

Then, every element of U_1 has following form:

$$P = X^+ BY^+ + G - X^+ XGYY^+, \forall G \in R^{m\times n}. \tag{32}$$

In particular, $f_1(P) = 0$ has matrices solutions in $R^{m\times n}$, iff $X^+ XBYY^+ = B$, and its general solution can be also expressed in the form of (32).

Theorem 13. *Given $P_0 \in CSR^{k\times k}$, partition P_0 as in (16). Let $X \in R^{m\times m}$, and $Y \in R^{n\times l}$ be given as in (17), $\Lambda \in R^{m\times m}$, and $\Gamma \in R^{l\times l}$ be given as in (18). Partition $D_n^T X$ and $D_n^T Y$ as in (19). Denote*

$$\begin{aligned} M_0 &= M_1\Lambda_1 M_1^+ + \left(N_1^T\right)^+ \Gamma_1 N_1^T \left(I - M_1 M_1^+\right), \\ N_0 &= M_2\Lambda_2 M_2^+ + \left(N_2^T\right)^+ \Gamma_2 N_2^T \left(I - M_2 M_2^+\right); \\ H_1 &= (0, I_{k-t})Q_3, \\ H_2 &= Q_4^T (0, I_{k-t})^T, \\ H_3 &= (0, I_t)Q_4, \\ H_4 &= Q_5^T (0, I_t)^T; \\ K_1 &= P_{10} - (0, I_{k-t})M_0 (0, I_{k-t})^T, \\ K_2 &= P_{20} - (0, I_t)N_0 (0, I_t)^T, \end{aligned} \tag{33}$$

where, $Q_3 \in R^{(n-r)\times(n-r-r_3)}$, $r_3 = \operatorname{rank}(N_1)$, $Q_4 \in R^{(n-r)\times(n-r-r_4)}$, $r_4 = \operatorname{rank}(M_1)$, $Q_5 \in R^{r\times(r-r_5)}$, $r_5 = \operatorname{rank}(N_2)$, $Q_6 \in R^{r\times(r-r_6)}$, $r_6 = \operatorname{rank}(M_2)$, range space$(Q_3)$ = null space(N_1^T), range space(Q_4) = null space(M_1^T), range space(Q_5) = null space(N_2^T), range space(Q_6) = null space(N_2^T);

$$\begin{aligned} Q_3^T Q_3 &= I_{n-r-r_3}, \\ Q_4^T Q_4 &= I_{n-r-r_4}, \\ Q_5^T Q_5 &= I_{r-r_5}, \\ Q_6^T Q_6 &= I_{r-r_6}, \end{aligned} \tag{34}$$

Then, Problem 6 is solvable if and only if

$$\begin{aligned} N_1^T M_1 \Lambda_1 &= \Gamma_1 N_1^T M_1, \\ M_1\Lambda_1 M_1^+ M_1 &= M_1\Lambda_1, \\ \Gamma_1 N_1^T &= N_1^+ N_1 \Gamma_1 N_1^T, \\ N_2^T M_2 \Lambda_2 &= \Gamma_2 N_2^T M_2, \\ M_2\Lambda_2 M_2^+ M_2 &= M_2\Lambda_2, \\ \Gamma_2 N_2^T &= N_2^+ N_2 \Gamma_2 N_2^T. \end{aligned} \tag{35}$$

$$\begin{aligned} H_1 H_1^+ K_1 H_2^+ H_2 &= K_1, \\ H_3 H_3^+ K_2 H_3^+ H_3 &= K_2. \end{aligned} \tag{36}$$

Furthermore, every matrix P in the solution set may be expressed as follows:

$$P = D_n \begin{bmatrix} M_0 + Q_3 G_1 Q_4^T & 0 \\ 0 & N_0 + Q_5 G_2 Q_6^T \end{bmatrix} D_n^T, \tag{37}$$

where, $G_1 = H_1^+ K_1 H_2^+ + G_3 - H_1^+ H_1 G_3 H_2 H_2^+$, $G_2 = H_3^+ K_2 H_4^+ + G_4 - H_3^+ H_3 G_4 H_4 H_4^+$, and $G_3 \in R^{(n-r-r_3)\times(n-r-r_4)}$ and $G_4 \in R^{(r-r_5)\times(r-r_6)}$ be any arbitrary (38).

Proof. From Lemmas 8, 9, and Problem 6 is equivalent to evaluating $P_1[n-r]$ and $P_2[r]$, such that

$$P = D_n \begin{pmatrix} P_1 & 0 \\ 0 & P_2 \end{pmatrix} D_n^T, \tag{38}$$

where P_1 and P_2 satisfy

$$\begin{cases} P_1 M_1 = M_1\Lambda_1, \\ Y_1^T P_1 = \Gamma_1 Y_1^T. \end{cases} \quad \begin{cases} P_2 M_2 = M_2\Lambda_2, \\ Y_2^T P_2 = \Gamma_2 Y_2^T. \end{cases} \tag{39}$$

$$\begin{aligned} P_{10} &= P_1[k-t] = (0, I_{k-t})P_1 (0, I_{k-t})^T, \\ P_{20} &= P_2[t] (0, I_t) P_2 (0, I_t)^T. \end{aligned} \tag{40}$$

By Lemma 11, we know that the equations in (38) hold if and only if

$$\begin{aligned} N_1^T M_1 \Lambda_1 &= \Gamma_1 N_1^T M_1, \\ M_1 \Lambda_1 M_1^+ M_1 &= M_1 \Lambda_1, \\ \Gamma_1 N_1^T &= N_1^+ N_1 \Gamma_1 N_1^T, \\ N_2^T M_2 \Lambda_2 &= \Gamma_2 N_2^T M_2, \\ M_2 \Lambda_2 M_2^+ M_2 &= M_2 \Lambda_2, \\ \Gamma_2 N_2^T &= N_2^+ N_2 \Gamma_2 N_2^T. \end{aligned} \quad (41)$$

which means that (35) hold. Furthermore, P_1 and P_2 can be expressed as

$$\begin{aligned} P_1 &= M_0 + Q_3 G_1 Q_4^T, \\ P_2 &= N_0 + Q_5 G_2 Q_6^T, \end{aligned} \quad (42)$$

where $G_1 \in R^{(n-r-r_3)\times(n-r-r_4)}$ and $G_2 \in R^{(r-r_5)\times(r-r_6)}$ be an arbitrary real matrix.

Now, the definitions of $K_1, K_2, H_1, H_2, H_3, H_4$ as in (33), we substitute (42) into (40), then

$$\begin{aligned} H_1 G_1 H_2 &= K_1, \\ H_3 G_2 H_4 &= K_2, \end{aligned} \quad (43)$$

Lemma 12 implies that (43) holds if and only if

$$\begin{aligned} H_1 H_1^+ K_1 H_2^+ H_2 &= K_1, \\ H_3 H_3^+ K_2 H_3^+ H_3 &= K_2, \end{aligned} \quad (44)$$

which means (36) holds, and G_1, G_2 may be written as

$$\begin{aligned} G_1 &= H_1^+ K_1 H_2^+ + G_3 - H_1^+ H_1 G_3 H_2 H_2^+, \\ G_2 &= H_3^+ K_2 H_4^+ + G_4 - H_3^+ H_3 G_4 H_4 H_4^+, \end{aligned} \quad (45)$$

where, $G_3 \in R^{(n-r-r_3)\times(n-r-r_4)}$ and $G_4 \in R^{(r-r_5)\times(r-r_6)}$ be arbitrary.

Therefore, the solution to Problem 6 has the form of (37)

$$P = D_n \begin{bmatrix} M_0 + Q_3 G_1 Q_4^T & 0 \\ 0 & N_0 + Q_5 G_2 Q_6^T. \end{bmatrix} D_n^T. \quad (46)$$

The above matrix P is the solution of Problem 6, which satisfies equation (11) and centrosymmetric in nature. The conditions for solvability of Problem 6 given in equation (35), which are very useful conditions for finding the proof of Theorem 13 i.e., solution of problem-1. In above proof it is clearly shown that the left and right eigenpairs of matrix P and central principal submatrix of solution matrix P satisfies conditions of the Problem 6. In Problem 6, if $Y = 0$, then this problem becomes Problem 6 in [25]. (see Algorithm 1). □

Example 1. Let $n=8$ $m=4$, $l=3$, $k=4$, $s_1=2, s_2=2, t=2$ and given $\wedge = \mathrm{diag}(\Lambda_1, \Lambda_2)$, and $\Gamma = \mathrm{diag}(\Gamma_1, \Gamma_2)$, where $\Lambda_1 = \begin{pmatrix} 2.0528 & 0 \\ 0 & 1.0561 \end{pmatrix}$, $\Lambda_2 = \begin{pmatrix} 1.5482 & 0.7910 \\ -0.7910 & 1.5482 \end{pmatrix}$, $\Gamma_1 = \begin{pmatrix} 1.2841 & 0 \\ 0 & 1.8592 \end{pmatrix}$, and $\Gamma_2 = (0.4872)$.

$$\begin{aligned} X &= \begin{pmatrix} 0.9337 & 0.0414 & -2.2584 & -1.6702 \\ 0.3503 & -0.7342 & 2.2294 & 0.4716 \\ -0.0290 & -0.0308 & 0.3376 & -1.2128 \\ 0.1825 & 0.2323 & 1.0001 & 0.0662 \\ 0.1825 & 0.2323 & -1.0001 & -0.0662 \\ -0.0290 & -0.0308 & -0.3376 & 1.2128 \\ 0.3503 & -0.7342 & -2.2294 & -0.4716 \\ 0.9337 & 0.0414 & 2.2584 & 1.6702 \end{pmatrix}, \\ Y &= \begin{pmatrix} -0.6509 & 0.5946 & -1.6118 \\ 0.2571 & 0.3502 & -0.0245 \\ -0.9444 & 1.2503 & -1.9488 \\ -1.3218 & 0.9298 & 1.0205 \\ -1.3218 & 0.9298 & -1.0205 \\ -0.9444 & 1.2503 & 1.9488 \\ 0.2571 & 0.3502 & 0.0245 \\ -0.6509 & 0.5946 & 1.6118 \end{pmatrix}, \\ P_0 &= \begin{pmatrix} 0.4800 & -0.2300 & 0.7000 & -0.4500 \\ 0.5600 & -1.7200 & 0.4120 & 0.5600 \\ 0.5600 & 0.4120 & -1.7200 & 0.5600 \\ -0.4500 & 0.7000 & -0.2300 & 0.4800 \end{pmatrix}. \end{aligned} \quad (47)$$

(i) Input $X \in R^{n\times m}, Y \in R^{n\times l}$ as in (24), $\wedge \in R^{m\times m}, \Gamma \in R^{l\times l}$ as in (25), and $P_0 \in R^{k\times k}$.
(ii) Partition P_0 as in (21) and get P_{10}, P_{20}, and get $\Lambda_1, \Lambda_2, \Gamma_1, \Gamma_2$ as in (25), and derive M_1, M_2, N_1, N_2 as in (28).
(iii) Construct Q_3, Q_4, Q_5, Q_6 as in (22) and (23).
(iv) Obtain $H_1, H_2, H_3, H_4, K_1, K_2$ by using (33).
(v) If (35) and (36) hold, then go to the next step, otherwise reconstruct from step III.
(vi) Form (38), calculate G_1, G_2.
(vii) Follows (37), obtain matrix P.

Algorithm 1: Left-right inverse eigenvalue problem for generalized centro-symmetric matrices.

Equation (33) becomes.

$$\begin{aligned} H_1 &= \begin{pmatrix} -0.0523 & 0.6193 \\ -0.8972 & 0.1167 \end{pmatrix}, \\ H_2 &= \begin{pmatrix} -0.0511 & 0.7767 \\ -0.8024 & 0.2822 \end{pmatrix}, \\ H_3 &= \begin{pmatrix} -0.1584 & -0.0148 & 0.9680, \\ 0.3520 & -0.6665 & -0.0835. \end{pmatrix}, \\ H_4 &= \begin{pmatrix} 0.1944 & 0.0213 \\ 0.3921 & -0.7047 \end{pmatrix}, \\ K_1 &= \begin{pmatrix} -1.3356 & -0.1465, \\ 1.0849 & -1.8626. \end{pmatrix}, \\ K_2 &= \begin{pmatrix} 2.3319 & -0.3092 \\ 0.7170 & -1.9076 \end{pmatrix}. \end{aligned} \tag{48}$$

From equation (37), the general solution is.

$$P = \begin{pmatrix} -0.2008 & 1.7170 & -1.2681 & 0.4830 & 0.6523 & -0.5385 & -2.4266 & 3.2937 \\ 2.9098 & -1.3827 & 1.4386 & -1.6987 & 1.0063 & 0.2586 & 2.5994 & -2.2148, \\ 3.7523 & -4.3047 & \mathbf{0.4800} & \mathbf{-0.2300} & \mathbf{0.7000} & \mathbf{-0.4500} & 3.7934 & -2.8030 \\ 0.5204 & 1.0190 & \mathbf{0.5600} & \mathbf{-1.7200} & \mathbf{0.4120} & \mathbf{0.5600} & -0.8406 & 0.5811 \\ 0.5811 & -0.8406 & \mathbf{0.5600} & \mathbf{0.4120} & \mathbf{-1.7200} & \mathbf{0.5600} & 1.0190 & 0.5204, \\ -2.8030 & 3.7934 & \mathbf{-0.4500} & \mathbf{0.7000} & \mathbf{-0.2300} & \mathbf{0.4800} & -4.3047 & 3.7523 \\ -2.2148 & 2.5994 & 0.2586 & 1.0063 & -1.6987 & 1.4386 & -1.3827 & 2.9098 \\ 3.2937 & -2.4266 & -0.5385 & 0.6523 & 0.4830 & -1.2681 & 1.7170 & -0.2008. \end{pmatrix}, \tag{49}$$

where P is a general solution to Problem 6, which is a centro-symmetric matrix and P_0 is a central principle submatrix.

Example 2. Let $n = 10$, $m = 5$, $l = 4$, $k = 4$, and given $\wedge = \text{diag}(\Lambda_1, \Lambda_2)$, and $\Gamma = \text{diag}(\Gamma_1, \Gamma_2)$, where

$$
\begin{aligned}
\Lambda_1 &= \begin{pmatrix} 1.0528 & 0 \\ 0 & 1.9681 \end{pmatrix},\\
\Lambda_2 &= \begin{pmatrix} 1.5522 & 1.7910 & 0 \\ -1.7910 & 1.5522 & 0 \\ 0 & 0 & 0.2870 \end{pmatrix},\\
\Gamma_1 &= \begin{pmatrix} 1.8961 & 0 \\ 0 & 0.5842 \end{pmatrix},\\
\Gamma_2 &= \begin{pmatrix} 0.4872 & -1.5847 \\ 1.5847 & 0.4872 \end{pmatrix}.\\
X &= \begin{pmatrix}
0.0830 & -0.0376 & 0.0203 & 0.1472 & -0.0289 \\
0.0522 & 0.1746 & -0.0365 & -0.1159 & -0.1159 \\
0.1880 & 0.1518 & 0.1991 & -0.0735 & -0.0809 \\
0.0627 & 0.1644 & -0.0210 & -0.1246 & -0.0970 \\
0.1185 & -0.0052 & 0.1156 & 0.0299 & -0.0870 \\
0.1185 & -0.0052 & -0.1156 & -0.0299 & 0.0870 \\
0.0627 & 0.1644 & 0.0210 & 0.1246 & 0.0970 \\
0.1880 & 0.1518 & -0.1991 & 0.0735 & 0.0809 \\
0.0522 & 0.1746 & 0.0365 & 0.1159 & -0.0089 \\
0.0830 & -0.0376 & -0.0203 & -0.1472 & 0.0289
\end{pmatrix},\\
Y &= \begin{pmatrix}
0.1830 & -0.0892 & 0.0405 & 0.1027 \\
-0.0251 & -0.0784 & 0.2948 & 0.1557 \\
0.4740 & 0.0268 & 0.4944 & 0.3473 \\
-0.0342 & 0.4057 & 0.4064 & 0.3138 \\
0.1063 & 0.3794 & 0.4491 & 0.1184 \\
0.1063 & 0.3794 & -0.4491 & -0.1184 \\
-0.0342 & 0.4057 & -0.4064 & -0.3138 \\
0.4740 & 0.0268 & -0.4944 & -0.3473 \\
-0.0251 & -0.0784 & -0.2948 & -0.1557 \\
0.1830 & -0.0892 & -0.0405 & -0.1027
\end{pmatrix},\\
P_0 &= \begin{pmatrix}
-0.3400 & 0.5600 & -0.2300 & 0.7000 \\
0.5600 & 1.7200 & -0.9800 & 0.4600 \\
0.4600 & -0.9800 & 1.7200 & 0.5600 \\
0.7000 & -0.2300 & 0.5600 & -0.3400
\end{pmatrix}.
\end{aligned}
\tag{50}
$$

Equation (33) becomes

$$
\begin{aligned}
H_1 &= \begin{pmatrix} 0.5359 & -0.0997 & -0.1266 \\ -0.0120 & 0.6939 & 0.0385 \end{pmatrix}, \\
H_2 &= \begin{pmatrix} 0.2825 & -0.3142 \\ -0.5433 & -0.0026 \\ 0.6829 & 0.4810 \end{pmatrix}, \\
H_3 &= \begin{pmatrix} -0.2104 & -0.3845 & 0.0508 \\ 0.1438 & -0.4973 & 0.8249 \end{pmatrix}, \\
H_4 &= \begin{pmatrix} 0.5797 & 0.1518 \\ 0.1700 & 0.5688 \end{pmatrix} \\
K_1 &= \begin{pmatrix} -0.6316 & 0.4075 \\ 0.9661 & 0.2274 \end{pmatrix}, \\
K_2 &= \begin{pmatrix} -0.5528 & 0.7823 \\ 0.3541 & 2.0694 \end{pmatrix}
\end{aligned}
\tag{51}
$$

From equation (37), the general solution is.

$$
P = \begin{pmatrix}
3.3547 & 2.2843 & -1.1810 & -3.0852 & 0.8098 & -1.0362 & 1.7703 & 2.6204 & -1.8369 & -2.1216 \\
2.7299 & 2.0916 & -2.9843 & -0.2274 & -0.0075 & -0.3780 & 3.7054 & 2.0902 & -3.4446 & -3.4802 \\
-1.2378 & -1.3864 & -0.3213 & 0.1981 & -0.0949 & -0.3311 & 0.8983 & -1.0073 & -0.5882 & -2.1019 \\
0.9251 & 2.0187 & -2.3583 & \mathbf{-0.3400} & \mathbf{0.5600} & \mathbf{-0.2300} & \mathbf{0.7000} & 1.8033 & -1.1998 & -2.7183 \\
-0.6448 & -1.2582 & -1.3075 & \mathbf{0.5600} & \mathbf{1.7200} & \mathbf{-0.9800} & \mathbf{0.4600} & 0.4336 & -0.4679 & -0.3597 \\
-0.3597 & -0.4679 & 0.4336 & \mathbf{0.4600} & \mathbf{-0.9800} & \mathbf{1.7200} & \mathbf{0.5600} & -1.3075 & -1.2582 & -0.6448 \\
-2.7183 & -1.1998 & 1.8033 & \mathbf{0.7000} & \mathbf{-0.2300} & \mathbf{0.5600} & \mathbf{-0.3400} & -2.3583 & 2.0187 & 0.9251 \\
-2.1019 & -0.5882 & -1.0073 & 0.8983 & -0.3311 & -0.0949 & 0.1981 & -0.3213 & -1.3864 & -1.2378 \\
-3.4802 & -3.4446 & 2.0902 & 3.7054 & -0.3780 & -0.0075 & -0.2274 & -2.9843 & 2.0916 & 2.7299 \\
-2.1216 & -1.8369 & 2.6204 & 1.7703 & -1.0362 & 0.8098 & -3.0852 & -1.1810 & 2.2843 & 3.3547
\end{pmatrix},
\tag{52}
$$

where P is a general solution to Problem 6, which is a centrosymmetric matrix and P_0 is a central principle submatrix.

Example 3. Let $n = 10$, $m = 5$, $l = 4$, $k = 4$, and given $\wedge = \operatorname{diag}(\Lambda_1, \Lambda_2)$, and $\Gamma = \operatorname{diag}(\Gamma_1, \Gamma_2)$, where

$$
\begin{gathered}
\Lambda_1 = \begin{pmatrix} -2.9996 & 0 \\ 0 & 2.8021 \end{pmatrix}, \\
\Lambda_2 = \begin{pmatrix} -1.8519 & 0.6464 & 0 \\ -0.6464 & -1.8519 & 0 \\ 0 & 0 & 1.3045 \end{pmatrix}, \\
\Gamma_1 = \begin{pmatrix} 2.8021 & 0 \\ 0 & 1.4094 \end{pmatrix}, \\
\Gamma_2 = \begin{pmatrix} 0.7297 & -0.1497 \\ 0.1497 & 0.7297 \end{pmatrix}. \\
X = \begin{pmatrix}
-0.1940 & -0.2040 & -0.1525 & 0.0289 & -0.1827 \\
-0.2201 & 0.3856 & -0.2383 & 0.0178 & -0.5523 \\
-0.2403 & -0.0444 & 0.5239 & 0 & -0.2435 \\
0.4128 & 0.5009 & 0.2133 & -0.1773 & 0.3090 \\
-0.4310 & 0.2385 & -0.2205 & -0.1370 & 0.0825 \\
-0.4310 & 0.2385 & 0.2205 & 0.1370 & -0.0825 \\
0.4128 & 0.5009 & -0.2133 & 0.1773 & -0.3090 \\
-0.2403 & -0.0444 & -0.5239 & 0 & 0.2435 \\
-0.2201 & 0.3856 & 0.2383 & -0.0178 & 0.5523 \\
-0.1940 & -0.2040 & 0.1525 & -0.0289 & 0\,1827
\end{pmatrix}, \\
Y = \begin{pmatrix}
0.3003 & 0.1599 & -0.5677 & 0 \\
-0.4007 & 0.0851 & 0.2460 & -0.1018 \\
0.1443 & 0.5989 & -0.1283 & 0.0967 \\
-0.3396 & 0.1831 & 0.0401 & -0.1561 \\
-0.3362 & -0.2739 & -0.1395 & 0.1888 \\
-0.3362 & -0.2739 & 0.1395 & -0.1888 \\
-0.3396 & 0.1831 & -0.0401 & 0.1561 \\
0.1443 & 0.5989 & 0.1283 & -0.0967 \\
-0.4007 & 0.0851 & -0.2460 & 0.1018 \\
0.3003 & 0.1599 & 0.5677 & 0
\end{pmatrix}, \\
P_0 = \begin{pmatrix}
-0.1500 & -0.3800 & 1.7900 & 0.8600 \\
0.5700 & -0.6950 & -0.1050 & 0.7200 \\
0.7200 & -0.1050 & -0.6950 & 0.5700 \\
0.8600 & 1.7900 & -0.3800 & -0.1500
\end{pmatrix}.
\end{gathered}
\tag{53}
$$

Equation (33) becomes

$$
\begin{aligned}
H_1 &= \begin{pmatrix} -0.3763 & -0.6329 & 0.5953 \\ -0.6344 & 0.4327 & -0.0075 \end{pmatrix},\\
H_2 &= \begin{pmatrix} -0.4082 & -0.7480 \\ -0.7129 & -0.0459 \\ 0.2071 & -0.2842 \end{pmatrix},\\
H_3 &= \begin{pmatrix} 0.3320 & 0.2517 & -0.3472 \\ 0.3926 & -0.8805 & -0.0713 \end{pmatrix},\\
H_4 &= \begin{pmatrix} -0.1067 & -0.3163 \\ 0.1660 & 0.1660 \end{pmatrix},\\
K_1 &= \begin{pmatrix} 0.6873 & -0.2853 \\ 0.1792 & -0.1103 \end{pmatrix},\\
K_2 &= \begin{pmatrix} 0.1979 & -0.1474 \\ 0.6471 & -0.7488 \end{pmatrix}.
\end{aligned} \tag{54}
$$

From equation (37), the general solution is

$$
P = \begin{pmatrix}
-0.1212 & -1.7054 & -0.1169 & 0.3613 & -1.0276 & -1.5739 & -0.4393 & -1.0047 & 1.8765 & 0.3599 \\
-0.5414 & 0.6278 & -0.1536 & 0.7051 & -0.4663 & -0.2814 & -0.1218 & -0.9873 & 1.7574 & 0.1359 \\
-2.4427 & -7.9729 & 1.2883 & 0.7796 & -0.2780 & 2.8213 & 1.4695 & 1.0300 & 3.0321 & 1.8045 \\
-0.8333 & -2.6387 & -0.2655 & \mathbf{-0.1500} & \mathbf{-0.3800} & \mathbf{1.7900} & \mathbf{0.8600} & -0.2178 & 1.3916 & -0.0094 \\
-2.2203 & -5.0486 & -0.4678 & \mathbf{0.5700} & \mathbf{-0.6950} & \mathbf{-0.1050} & \mathbf{0.7200} & -0.9474 & 4.7792 & 1.3565 \\
1.3565 & 4.7792 & -0.9474 & \mathbf{0.7200} & \mathbf{-0.1050} & \mathbf{-0.6950} & \mathbf{0.5700} & -0.4678 & -5.0486 & -2.2203 \\
-0.0094 & 1.3916 & -0.2178 & \mathbf{0.8600} & \mathbf{1.7900} & \mathbf{-0.3800} & \mathbf{-0.1500} & 0.2655 & -2.6387 & -0.8333 \\
1.8045 & 3.0321 & 1.0300 & 1.4695 & 2.8213 & -0.2780 & 0.7796 & 1.2883 & -7.9729 & -2.4427 \\
0.1359 & 1.7574 & -0.9873 & -0.1218 & -1.2814 & -0.4663 & 0.7051 & -0.1536 & 0.6278 & -0.5414 \\
0.3599 & 1.8765 & -1.0047 & -0.4393 & -1.5739 & -1.0276 & 0.3613 & -0.1169 & -1.7054 & -0.1212
\end{pmatrix}. \tag{55}
$$

From above numerical example, the resultant matrix P is centrosymmetric matrix, which having P_0 as central principle submatrix.

4. Conclusion

In this paper, we expanded the system P from the center subsystem P_0 satisfying the matrix constraint, where P and P_0 are both centrosymmetric matrices. We consider the left and right inverse eigenvalue problem with a central principal submatrix constraint, which lies in center of the original matrix. Using special properties of left and right eigenpairs, we obtained the solvability conditions of the Problem 6 and its general solution. Furthermore, for feasibility of obtained general centrosymmetric matrix solutions we provided an algorithm with numerical example. In future, we will discuss the unique solution to the optimal approximation problem.

Data Availability

The data presented in this research article are available on request from the corresponding author.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors have contributed to all parts of the article. All authors have read and approved the final manuscript.

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