

# Research Article **On a Perturbed Risk Model with Time-Dependent Claim Sizes**

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We consider a risk model perturbed by a Brownian motion, where the individual claim sizes are dependent on the inter-claim times. We study the Gerber–Shiu functions when ruin is due to a claim or the jump-diffusion process. Integro-differential equations and Laplace transforms satisfied by the Gerber–Shiu functions are obtained. Then, it is shown that the expected discounted penalty functions satisfy defective renewal equations. Explicit expressions can be obtained for exponential claim sizes. Finally, a numerical example is provided to measure the impact of the various dependence parameters in the risk model on the ruin probabilities.

#### 1. Introduction

Consider insurer's surplus process at time t defined as  $\{U(t), t \ge 0\}$  with

$$U(t) = u + ct - \sum_{i=1}^{N(t)} X_i + \sigma B(t),$$
 (1)

where  $u \ge 0$  is the initial surplus and c > 0 is the premium rate. The number of claims process  $\{N(t), t \ge 0\}$  is assumed to be a Poisson process with independent and identically distributed (i.i.d.) exponential inter-claim time random variables  $\{W_i, i \ge 1\}$  distributed like a generic variable W. The probability density function (pdf) of W is defined by  $k(t) = \lambda e^{-\lambda t}$ for  $\lambda > 0$ . The individual claim amounts  $\{X_i, i \ge 1\}$  are assumed to be a sequence of strictly positive i.i.d. random variables with generic X. The standard Brownian motion  $\{B(t), t \ge 0\}$  is independent of  $\{W_i, i \ge 1\}$  and  $\{X_i, i \ge 1\}$ , and  $\sigma > 0$  is the diffusion volatility that accounts for the perturbation of the diffusion process.

Under the condition that the inter-arrival times between two successive claims and the claim amounts are independent, model (1) was first proposed by Dufresne and Gerber [1]. Since then, many researchers have made

contributions to this kind of risk model. However, it is extremely restrictive and sometimes unrealistic to assume the independence between inter-claim times and individual claim sizes. For example, more considerable damages are expected with a longer period between claims for a line of business covering damages due to earthquakes. To avoid this restriction, some papers considered the dependent risk models. As for the risk model (1) without diffusion, dependence structure based on the Farlie-Gumbel-Morgenstern (FGM) copula has been extensively studied, see, e.g., [2, 3]. Boudreault et al. [4] proposed an extension to the classical compound Poisson risk model assuming a dependence structure for (X, W), in which the distribution of the next claim amount is defined in terms of the time elapsed since the last claim. For an arbitrary dependence structure, the asymptotic ruin probability was studied by Albrecher and Teugels [5]. For the perturbed risk model (1), Zhang and Yang [6] used the FGM copula to define the dependence structure and derived the integro-differential equations and the Laplace transforms for the Gerber-Shiu functions. Adékambi and Takouda [7] generalized the results of Zhang and Yang [6] by studying the unified ruin-related measure, in which the claim inter-occurrences follow an Erlang (n) distribution. Recently, the authors of [8, 9] investigated (1) with a time delay in the arrival of the first two claims. Chadjiconstantinidis and Papaioannou [10] considered an extension to (1) the compound Poisson risk process perturbed by diffusion in which two types of dependent claims, main claims and by-claims, are incorporated.

In the present paper, we consider the perturbed model (1) with the dependence structure proposed by Boudreault et al. [4], in which the distribution of the next claim amount is defined in terms of the time elapsed since the last claim. More precisely, we assume that the bivariate random vectors  $(W_j, X_j)$  for  $j \in \mathbb{N}^+$  are mutually independent but that the random variables  $W_j$  and  $X_j$  are no longer independent. The density of  $X_k | W_k$  is defined as a special mixture of two arbitrary density functions  $f_1$  and  $f_2$  with respective means  $\mu_1$  and  $\mu_2$ , i.e.,

$$f_{X_k|W_k}(x) = e^{-\beta W_k} f_1(x) + \left(1 - e^{-\beta W_k}\right) f_2(x), \quad x \ge 0,$$
(2)

for  $k \in \mathbb{N}^+$ . The resulting marginal distribution of  $X_k$  is

$$f_{X_k}(x) = \frac{\lambda}{\lambda + \beta} f_1(x) + \frac{\beta}{\lambda + \beta} f_2(x).$$
(3)

To guarantee that ruin is not a certain event, we assume that the following net profit condition holds.

$$E[cW - X] > 0. \tag{4}$$

By (3), it is easy to calculate that the positive loading condition (4) is equivalent to

$$\frac{c}{\lambda} - \frac{\lambda \mu_1 + \beta \mu_2}{\beta + \lambda} > 0.$$
(5)

Associated with the risk model (1), let  $\tau = \inf_{t \ge 0} \{t, U(t) < 0\}$  be the time of ruin with  $\tau = \infty$  if ruin does not occur. The deficit at ruin and the surplus just prior to ruin are denoted by  $|U(\tau)|$  and  $U(\tau-)$ , respectively. The Gerber–Shiu discounted penalty function  $\phi(u)$  is defined as

$$\phi(u) = E \Big[ e^{-\delta \tau} \omega(U(\tau-), |U(\tau)|) I(\tau < \infty) |U(0) = u \Big],$$
(6)

where  $\delta \ge 0$  is interpreted as the force of interest or the Laplace argument, I(A) represents the indicator function of the event *A*, and w(x, y) is a non-negative bivariate function of *x*,  $y \ge 0$ .

We remark that the penalty function provides a unified framework of identifying ruin-related quantities since it is proposed by Gerber and Shiu [11]. Now the function may be instrumental in understanding the vulnerability of an insurance institution and has been generalized in the literature in various models, see [12–20] for more details. He et al. [21] also provided a comprehensive review of existing works for the Gerber–Shiu function from practical perspectives.

By observing the sample paths of U(t), we know that ruin can be caused either by the oscillation of the Brownian motion or a downward jump. We decompose  $\phi(u)$  as follows.

$$\phi(u) = \phi_w(u) + \phi_d(u), \tag{7}$$

where

$$\phi_{w}(u) = E \Big[ e^{-\delta \tau} \omega (U(\tau), |U(\tau)|) I(\tau < \infty, U(\tau) < 0) |U(0) = u \Big],$$
(8)

is the Gerber-Shiu function when ruin is caused by a claim and

$$\phi_{d}(u) = E\left[e^{-\delta\tau}\omega(U(\tau-), |U(\tau)|)I(\tau<\infty, U(\tau)=0)|U(0)=u\right] = \omega(0, 0)E\left[e^{-\delta\tau}I(\tau<\infty, U(\tau)=0)|U(0)=u\right],$$
(9)

is the Gerber–Shiu function when ruin is caused by oscillation. Without loss of generality, we assume that  $\omega(0, 0) =$ 1 in what follows. Further, if  $\delta = 0$  in addition to  $\omega(x_1, x_2) =$ 1 for any  $x_1$  and  $x_2$ , (8) and (9) correspond to the infinitetime ruin probabilities  $\psi_w(u)$  and  $\psi_d(u)$ .

The objective of this paper is to study the unified Gerber-Shiu function for a compound Poisson risk model perturbed by a diffusion process with dependence structure. The additional diffusion term may be interpreted as the future uncertainty of aggregate claims or the fluctuation of investment of surplus. We obtain the integro-differential equations satisfied by the Gerber-Shiu penalty functions by using a trivariate potential measure based on the joint distribution of a drifted Brownian motion, its running supremum, and the claim size. By using the Laplace transform technique, we derive the defective renewal equations satisfied by the Gerber–Shiu penalty functions. We also provide a numerical example to illustrate the behavior of the ruin probability and analyze the effect of the dependence structure.

The rest of the paper is structured as follows. In Section 2, we analyze Lundberg's generalized equation and its roots. The integro-differential equations for the Gerber–Shiu functions are obtained in Section 3. In Section 4, the Laplace transforms and defective renewal equations for the Gerber–Shiu functions are derived. In Section 5, we obtain the explicit expressions for the Laplace transforms for exponential claim size distributions and numerical illustrations are provided. Section 6 draws the conclusions.

#### 2. Lundberg's Generalized Equation

One important step in the analysis of the ruin measures is the derivation of the so-called Lundberg's generalized equation and the identification of the number of roots to it.

Let  $T_0 = 0$  and  $T_k = \sum_{i=1}^k W_i$ ,  $k \in \mathbb{N}^+$ , be the arrival time of the *k* th claim. Denote by  $U_n$  the surplus immediately after the *n*-th claim; it is not hard to see that

$$U_{n} \stackrel{d}{=} u + \sum_{i=1}^{n} (cW_{i} - X_{i} + \sigma B(W_{i})), \qquad (10)$$

where  $\stackrel{d}{=}$  means equality in distribution.

To derive Lundberg's generalized equation, we seek a number  $s \in \mathbb{C}$  such that the process  $\{e^{-\delta T_n + sU_n}, n \in \mathbb{N}\}$  forms a martingale. This is the condition that

$$E\left[e^{-\delta W+s\left(cW+\sigma B_W-X\right)}\right]=1.$$
 (11)

Throughout the entire paper,  $\sim$  is added above a letter to represent the Laplace transform of the corresponding quantity. Using (3), the left-hand side of (11) can be written as

$$E\left(e^{-\delta W+s\left(cW+\sigma B_{W}-X\right)}\right) = E\left(e^{-\delta W+s\left(cW+\sigma B_{W}\right)}E\left(e^{sX}\mid W\right)\right)$$
$$= E\left(e^{-\delta W+s\left(cW+\sigma B_{W}\right)}\left(e^{-\beta W}\tilde{f}_{1}\left(s\right)+\left(1-e^{-\beta W}\right)\tilde{f}_{2}\left(s\right)\right)\right)$$
$$= \frac{\lambda\left(\lambda+\delta-\pi\left(s\right)\right)\tilde{f}_{1}\left(s\right)+\lambda\beta\tilde{f}_{2}\left(s\right)}{\left(\lambda+\beta+\delta-\pi\left(s\right)\right)\left(\lambda+\delta-\pi\left(s\right)\right)},$$
(12)

where  $\pi(s) = (\sigma^2/2)s^2 + cs$ . Substituting (12) into (11) yields

$$h_1(s) - h_2(s) = 0, \tag{13}$$

with

$$h_1(s) = (\lambda + \beta + \delta - \pi(s))(\lambda + \delta - \pi(s)),$$
(14)

$$h_{2}(s) = \lambda \left(\lambda + \delta - \pi(s)\right) \tilde{f}_{1}(s) + \lambda \beta \tilde{f}_{2}(s).$$

We call (13) Lundberg's generalized equation. In order to derive the defective renewal equations for  $\varphi_{\omega}(u)$  and  $\varphi_{d}(u)$ , it is necessary to identify the number of roots to (13). By the Rouche theorem and analogously to Propositions 1 and 2 of [10], we have the following results.

**Lemma 1.** For  $\delta > 0$ , Lundberg's generalized equation (13) has exactly 2 solutions, say  $\rho_1, \rho_2$ , such that  $\operatorname{Re}(\rho_i) > 0$  and for  $\delta = 0$  one root is null.

In the case of  $\delta > 0$ , it holds that  $\rho_1$  and  $\rho_2$  are distinct positive real numbers, see [6] for related discussions.

#### 3. Integro-Differential Equations

Let  $Z(t) = -ct - \sigma B(t)$ , which is a Brownian motion starting from zero with drift -c and variance  $\sigma^2$ . Denote by  $\overline{Z}(t) = \sup_{\{0 \le s \le t\}} Z(s)$  and define potential measure  $\mathscr{P}(u, dy, dx)$  as follows:

$$\mathscr{P}(u, \mathrm{d}y, \mathrm{d}x) = E\left[e^{-\delta W}I(\overline{Z}(W) < u, Z(W) \in dy, X \in \mathrm{d}x)\right], u, x > 0, u > y.$$
(15)

Similar to [6], we can prove that the measure  $\mathcal{P}(u, dy, dx)$  has a density given by

$$p(u, y, x) = \frac{\lambda \alpha_1 \alpha_2}{(\lambda + \delta + \beta) (\alpha_1 + \alpha_2)} \Big( e^{-\alpha_1 y} - e^{-(\alpha_1 + \alpha_2)u + \alpha_2 y} \Big) (f_1(x) - f_2(x)) + \frac{\lambda \beta_1 \beta_2}{(\lambda + \delta) (\beta_1 + \beta_2)} \Big( e^{-\beta_1 y} - e^{-(\beta_1 + \beta_2)u + \beta_2 y} \Big) f_2(x),$$
(16)

# for $0 \le y < u$ , and

$$p(u, y, x) = \frac{\lambda \alpha_1 \alpha_2}{(\lambda + \delta + \beta) (\alpha_1 + \alpha_2)} \Big( e^{\alpha_2 y} - e^{-(\alpha_1 + \alpha_2)u + \alpha_2 y} \Big) (f_1(x) - f_2(x)) + \frac{\lambda \beta_1 \beta_2}{(\lambda + \delta) (\beta_1 + \beta_2)} \Big( e^{\beta_2 y} - e^{-(\beta_1 + \beta_2)u + \beta_2 y} \Big) f_2(x),$$
(17)

for y < 0, where

$$\alpha_{1} = \frac{c}{\sigma^{2}} + \sqrt{\frac{2(\lambda + \delta + \beta)}{\sigma^{2}} + \frac{c^{2}}{\sigma^{4}}},$$

$$\alpha_{2} = -\frac{c}{\sigma^{2}} + \sqrt{\frac{2(\lambda + \delta + \beta)}{\sigma^{2}} + \frac{c^{2}}{\sigma^{4}}},$$

$$\beta_{1} = \frac{c}{\sigma^{2}} + \sqrt{\frac{2(\lambda + \delta)}{\sigma^{2}} + \frac{c^{2}}{\sigma^{4}}},$$

$$\beta_{2} = -\frac{c}{\sigma^{2}} + \sqrt{\frac{2(\lambda + \delta)}{\sigma^{2}} + \frac{c^{2}}{\sigma^{4}}}.$$
(18)

Now we consider  $\varphi_{\omega}(u).$  By conditioning on the time and amount of the first claim, one finds

$$\phi_{\omega}(u) = \int_{0}^{\infty} \int_{-\infty}^{u} \int_{0}^{u-y} e^{-\delta t} \Pr(\overline{Z}(t) < u, Z(t) \in dy)$$

$$\cdot \phi_{\omega}(u - y - x) f_{X,W}(x, t) dx dt$$

$$+ \int_{0}^{\infty} \int_{-\infty}^{u} \int_{u-y}^{\infty} e^{-\delta t} \Pr(\overline{Z}(t) < u, Z(t) \in dy)$$

$$\cdot \omega (u - y, x - (u - y)) f_{X,W}(x, t) dx dt$$

$$= \int_{-\infty}^{u} \int_{0}^{u-y} \phi_{\omega}(u - y - x) p(u, y, x) dx dy$$

$$+ \int_{-\infty}^{u} \int_{u-y}^{\infty} \omega (u - y, x - (u - y)) p(u, y, x) dx dy.$$
(19)

Let

$$\sigma_{\omega,1}(u) = \int_{0}^{u} \phi_{\omega}(u-x)f_{1}(x)dx + \omega_{1}(u),$$
  

$$\sigma_{\omega,2}(u) = \int_{0}^{u} \phi_{\omega}(u-x)f_{2}(x)dx + \omega_{2}(u),$$
  

$$\omega_{1}(u) = \int_{u}^{\infty} \omega(u, x-u)f_{1}(x)dx,$$
  

$$\omega_{2}(u) = \int_{u}^{\infty} \omega(u, x-u)f_{2}(x)dx.$$
(20)

Submitting (16) and (17) into (19), we obtain

$$\begin{split} \phi_{\omega}(u) &= \frac{\lambda \alpha_{1} \alpha_{2}}{(\lambda + \delta + \beta)(\alpha_{1} + \alpha_{2})} \int_{0}^{u} \left( e^{-\alpha_{1}y} - e^{-(\alpha_{1} + \alpha_{2})u + \alpha_{2}y} \right) \left( \sigma_{\omega,1}(u - y) - \sigma_{\omega,2}(u - y) \right) dy \\ &+ \frac{\lambda \beta_{1} \beta_{2}}{(\lambda + \delta)(\beta_{1} + \beta_{2})} \int_{0}^{u} \left( e^{-\beta_{1}y} - e^{-(\beta_{1} + \beta_{2})u + \beta_{2}y} \right) \sigma_{\omega,2}(u - y) dy \\ &+ \frac{\lambda \alpha_{1} \alpha_{2}}{(\lambda + \delta + \beta)(\alpha_{1} + \alpha_{2})} \int_{-\infty}^{0} \left( e^{\alpha_{2}y} - e^{-(\alpha_{1} + \alpha_{2})u + \alpha_{2}y} \right) \left( \sigma_{\omega,1}(u - y) - \sigma_{\omega,2}(u - y) \right) dy \\ &+ \frac{\lambda \beta_{1} \beta_{2}}{(\lambda + \delta)(\beta_{1} + \beta_{2})} \int_{-\infty}^{0} \left( e^{\beta_{2}y} - e^{-(\beta_{1} + \beta_{2})u + \beta_{2}y} \right) \sigma_{\omega,2}(u - y) dy. \end{split}$$
(21)

Let s = u - y in (21), and we have

$$\phi_{\omega}(u) = \frac{\lambda \alpha_{1} \alpha_{2}}{(\lambda + \delta + \beta)(\alpha_{1} + \alpha_{2})}, \left(\int_{0}^{u} e^{-\alpha_{1}(u-s)}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds + \int_{u}^{\infty} e^{\alpha_{2}(u-s)}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds - \int_{0}^{\infty} e^{-\alpha_{1}u-\alpha_{2}s}(\sigma_{\omega,1}(s) - \sigma_{\omega,2}(s))ds\right),$$

$$+ \frac{\lambda \beta_{1} \beta_{2}}{(\lambda + \delta)(\beta_{1} + \beta_{2})}, \left(\int_{0}^{u} e^{-\beta_{1}(u-s)}\sigma_{\omega,2}(s)ds + \int_{u}^{\infty} e^{\beta_{2}(u-s)}\sigma_{\omega,2}(s)ds - \int_{0}^{\infty} e^{-\beta_{1}u-\beta_{2}s}\sigma_{\omega,2}(s)ds\right).$$
(22)

Let  $\mathscr{I}$  be the identity operator, and let  $\mathscr{D}$  be the differentiation operator. Then, we define the following differentiation operators:

$$\mathcal{A}_{1}(\mathcal{D}) = \mathcal{D}^{2} + \frac{2c}{\sigma^{2}} \mathcal{D} - \frac{2(\lambda + \delta + \beta)}{\sigma^{2}} \mathcal{I},$$

$$\mathcal{A}_{2}(\mathcal{D}) = \mathcal{D}^{2} + \frac{2c}{\sigma^{2}} \mathcal{D} - \frac{2(\lambda + \delta)}{\sigma^{2}} \mathcal{I}.$$
(23)

From the definitions of  $\alpha_i$  and  $\beta_i$ , we obtain immediately

$$\mathcal{A}_{1}(\mathcal{D}) = (\mathcal{D} + \alpha_{1}\mathcal{F})(\mathcal{D} - \alpha_{2}\mathcal{F}),$$
  
$$\mathcal{A}_{2}(\mathcal{D}) = (\mathcal{D} + \beta_{1}\mathcal{F})(\mathcal{D} - \beta_{2}\mathcal{F}).$$
 (24)

**Theorem 2.** The Gerber–Shiu function  $\varphi_{\omega}(u)$  defined in equation (8) when the ruin is caused by claims satisfies the following integro-differential equation:

$$\mathscr{A}_{1}(\mathscr{D})\mathscr{A}_{2}(\mathscr{D})\phi_{\omega}(u) + \frac{2\lambda}{\sigma^{2}}\mathscr{A}_{2}(\mathscr{D})\left(\sigma_{\omega,1}(u) - \sigma_{\omega,2}(u)\right) + \frac{2\lambda}{\sigma^{2}}\mathscr{A}_{1}(\mathscr{D})\sigma_{\omega,2}(u) = 0, \tag{25}$$

with the boundary conditions

$$\varphi_{\omega}(0) = 0, \varphi_{\omega}''(0) + \frac{2c}{\sigma^2} \varphi_{\omega}'(0) + \frac{2\lambda}{\sigma^2} \omega_1(0) = 0.$$
(26)

*Proof.* Applying the operator  $\mathscr{A}_1(\mathscr{D})\mathscr{A}_2(\mathscr{D})$  to both sides of (19) yields (25) after some rearrangements. The first boundary condition in (26) is obvious. By taking the first and second derivatives of (19) and then setting u = 0, respectively, the second boundary condition can be obtained by some comparisons.

In the same way as Theorem 2, we can give the integrodifferential equation for  $\varphi_d(u)$ . Let

$$\sigma_{d,1}(u) = \int_{0}^{u} \phi_{d}(u-x) f_{1}(x) dx,$$
  

$$\sigma_{d,2}(u) = \int_{0}^{u} \phi_{d}(u-x) f_{2}(x) dx,$$
(27)

and we have the following result.

(31)

**Theorem 3.** The Gerber–Shiu function  $\varphi_d(u)$  defined in equation (9) when the ruin is caused by oscillation satisfies the following integro-differential equation:

$$\mathscr{A}_{1}(\mathscr{D})\mathscr{A}_{2}(\mathscr{D})\varphi_{d}(u) + \frac{2\lambda}{\sigma^{2}}\mathscr{A}_{2}(\mathscr{D})\left(\sigma_{d,1}(u) - \sigma_{d,2}(u)\right) + \frac{2\lambda}{\sigma^{2}}\mathscr{A}_{1}(\mathscr{D})\sigma_{d,2}(u) = 0, \tag{28}$$

with the boundary conditions

$$\varphi_d(0) = 1, \varphi_d''(0) + \frac{2c}{\sigma^2} \varphi_d'(0) - \frac{2(\lambda + \delta)}{\sigma^2} = 0.$$
(29)

*Proof.* By conditioning on whether or not ruin occurs due to oscillation before the first claim, we have

 $E\left[e^{-\delta\tau_{u}}I\left(\tau_{u}< W\right)\right] = e^{-\beta_{1}u},$ 

$$\varphi_d(u) = \int_0^\infty \int_{-\infty}^u \int_0^{u-y} e^{-\delta t} \Pr\left(\overline{Z}(t) < u, Z(t) \in \mathrm{d}y\right) \varphi_d(u-y-x) f_{X,W}(x,t) \mathrm{d}x \mathrm{d}t + E\left[e^{-\delta \tau_u} I\left(\tau_u < W\right)\right], \tag{30}$$

where  $\tau_u = \inf\{t \ge 0: Z(t) = u\}$ . Using formula (2.01) of Borodin and Salminen [22], we have

where 
$$\beta_1 = c/\sigma^2 + \sqrt{2(\lambda + \delta)/\sigma^2 + c^2/\sigma^4}$$
.

Therefore, (30) can be rewritten as

$$\varphi_{d}(u) = e^{-\beta_{1}u} + \int_{-\infty}^{u} \int_{0}^{u-y} \varphi_{d}(u-y-x)p(u,y,x)dxdy.$$
(32)

Submitting (16) and (17) into (32), (28) can be obtained by imitating the same steps as those of Theorem 2.  $\hfill \Box$ 

## 4. Laplace Transforms and Defective Renewal Equations

In this section, we first derive the Laplace transforms for the Gerber–Shiu function when ruin is caused by claims and by oscillations. Then, we prove that the Gerber–Shiu function satisfies the defective renewal equation. For simplicity, let

$$\pi_{1}(s) = \pi(s) - \pi(\rho_{1}),$$

$$\pi_{2}(s) = \pi(s) - \pi(\rho_{2}),$$

$$a(s) = \pi(s) - (2\lambda + 2\delta + \beta),$$

$$b(s) = \frac{\sigma^{2}}{2} (\pi(s) - (\lambda + \delta + \beta)).$$
(33)

**Theorem 4.** The Laplace transforms of  $\varphi_{\omega}(u)$  and  $\varphi_{d}(u)$  are given by

$$\widetilde{\varphi}_{\omega}(s) = \frac{l_1(s) - l_2(s)}{h_1(s) - h_2(s)},$$
(34)

$$\widetilde{\varphi}_{d}(s) = \frac{k_{1}(s) - k_{2}(s)}{h_{1}(s) - h_{2}(s)},$$
(35)

where  $h_1(s)$  and  $h_2(s)$  are determined by (14), and

$$l_{1}(s) = \frac{\sigma^{4}}{4} \phi_{\omega}^{'''}(0) + \sigma^{2} c \phi_{\omega}^{''}(0) + \left(c^{2} + \frac{\sigma^{2}}{2}a(s)\right) \phi_{\omega}^{\prime}(0) + \frac{\sigma^{2}}{2} \lambda \sigma_{\omega,1}^{\prime}(0) + \lambda c \omega_{1}(0) - \frac{\sigma^{2}}{2} \lambda \omega_{2}(0)s,$$

$$l_{2}(s) = \lambda (\pi(s) - (\lambda + \delta)) \widetilde{\omega}_{1}(s) - \beta \lambda \widetilde{\omega}_{2}(s),$$

$$k_{1}(s) = \frac{\sigma^{4}}{4} \phi_{\omega}^{'''}(0) + \sigma^{2} c \phi_{\omega}^{''}(0) + \left(c^{2} + \frac{\sigma^{2}}{2}a(s)\right) \phi_{d}^{\prime}(0) + \frac{\sigma^{2}}{2} \lambda \sigma_{d,1}^{\prime}(0) - (2\lambda + 2\delta + \beta)c,$$

$$k_{2}(s) = -\frac{\sigma^{4}}{4}s^{3} - \sigma^{2} cs^{2} - \left(c^{2} - \frac{\sigma^{2}}{2}(\lambda + \delta + \beta)\right)s.$$
(36)

Proof. After some careful calculations, we have

$$\int_{0}^{\infty} e^{-su} \mathscr{A}_{1}(\mathscr{D}) \mathscr{A}_{2}(\mathscr{D}) \phi_{\omega}(u) du = \mathscr{A}_{1}(s) \mathscr{A}_{2}(s) \widetilde{\phi}_{\omega}(s) + \frac{2\lambda}{\sigma^{2}} \left( \omega_{1}(0) + \omega_{2}(0) \right) s$$
$$- \left[ \phi_{\omega}^{''}(0) + \frac{4c}{\sigma^{2}} \phi_{\omega}^{''}(0) + \frac{4}{\sigma^{4}} \left( c^{2} + \frac{\sigma^{2}}{2} a(s) \right) \phi_{\omega}^{\prime}(0) \right],$$
(37)

$$\int_{0}^{\infty} e^{-su} \mathscr{A}_{2}(\mathscr{D}) \Big( \sigma_{\omega,1}(u) - \sigma_{\omega,2}(u) \Big) du = \mathscr{A}_{2}(s) \Big( \tilde{f}_{1}(s) - \tilde{f}_{2}(s) \Big) \widetilde{\phi}(s) + \mathscr{A}_{2}(s) \big( \tilde{\omega}_{1}(s) - \tilde{\omega}_{2}(s) \big) - \big( \omega_{1}(0) - \omega_{2}(0) \big) \Big( s + \frac{2c}{\sigma^{2}} \Big) - \big( \sigma_{\omega,1}'(0) - \sigma_{\omega,2}'(0) \big),$$
(38)

$$\int_{0}^{\infty} e^{-su} \mathscr{A}_{1}(\mathscr{D}) \sigma_{\omega,2}(u) \mathrm{d}u = \mathscr{A}_{1}(s) \tilde{f}_{2}(s) \tilde{\varphi}(s) + \mathscr{A}_{1}(s) \tilde{\omega}_{2}(s) - \omega_{2}(0) \left(s + \frac{2c}{\sigma^{2}}\right) - \sigma_{\omega,2}'(0).$$
(39)

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Taking the Laplace transform of (25) and using equations (37)–(39) with the boundary conditions, we get (34). In the same way as the proof of (34), we obtain (35).

Now we are ready to prove that the Gerber–Shiu function when ruin is caused by claims and oscillations satisfies the defective renewal equation. Let us recall the Dickson–Hipp operator  $\mathcal{T}_s$  defined by

$$\mathcal{T}_{s}q(x) = \int_{x}^{\infty} e^{-s(y-x)}q(y)\mathrm{d}y.$$
(40)

We refer the reader to [23] for more properties on the above operator.

Since  $h_1(s) - \pi_1(s)\pi_2(s)$  is a polynomial function of  $\pi(s)$  with degree 1, then Lemma 1 and the Lagrange interpolation formula lead to

$$h_1(s) - \pi_1(s)\pi_2(s) = \frac{\pi_2(s)}{\pi_2(\rho_1)} h_2(\rho_1) + \frac{\pi_1(s)}{\pi_1(\rho_2)} h_2(\rho_2).$$
(41)

Note that

$$\frac{\pi_1(s)}{\pi_1(\rho_2)} + \frac{\pi_2(s)}{\pi_2(\rho_1)} = 1,$$
(42)

and we have

$$h_{1}(s) - h_{2}(s) = \pi_{1}(s)\pi_{2}(s) + \frac{\pi_{2}(s)}{\pi_{2}(\rho_{1})}(h_{2}(\rho_{1}) - h_{2}(s)) + \frac{\pi_{1}(s)}{\pi_{1}(\rho_{2})}(h_{2}(\rho_{2}) - h_{2}(s))$$

$$= \pi_{1}(s)\pi_{2}(s)\left(1 - \frac{h_{2}(s) - h_{2}(\rho_{1})}{\pi_{2}(\rho_{1})\pi_{1}(s)} - \frac{h_{2}(s) - h_{2}(\rho_{2})}{\pi_{1}(\rho_{2})\pi_{2}(s)}\right)$$

$$= \pi_{1}(s)\pi_{2}(s)\left(1 - \frac{(h_{2}(s) - h_{2}(\rho_{1})/s - \rho_{1})}{(\sigma^{2}/2)\pi_{2}(\rho_{1})(s + \rho_{1} + (2c/\sigma^{2}))} - \frac{(h_{2}(s) - h_{2}(\rho_{2})/s - \rho_{2})}{(\sigma^{2}/2)\pi_{1}(\rho_{2})(s + \rho_{2} + (2c/\sigma^{2}))}\right).$$
(43)

By Lemma 4 of [6], for i = 1, 2, we have

$$\frac{h_{2}(s) - h_{2}(\rho_{i})}{s - \rho_{i}} = \frac{\left[\lambda\left(\lambda + \delta - cs - \left(\sigma^{2}s^{2}/2\right)\right)\tilde{f}_{1}(s) + \lambda\beta\tilde{f}_{2}(s)\right] - \left[\lambda\left(\lambda + \delta - c\rho_{i} - \left(\sigma^{2}s^{2}/2\right)\right)\tilde{f}_{1}(\rho_{i}) + \lambda\beta\tilde{f}_{2}(\rho_{i})\right]}{s - \rho_{i}} \\
= \frac{\left(\sigma^{2}/2\right)\lambda\left(s^{2}\mathcal{T}_{s}f_{1}(0) - \rho_{i}^{2}\mathcal{T}_{\rho_{i}}f_{1}(0)\right)}{\rho_{i} - s} + \frac{\lambda c\left(s\mathcal{T}_{s}f_{1}(0) - \rho_{i}\mathcal{T}_{\rho_{i}}f_{1}(0)\right)}{\rho_{i} - s} \\
- \frac{\lambda\left(\lambda + \delta\right)\left(\mathcal{T}_{s}f_{1}(0) - \mathcal{T}_{\rho_{i}}f_{1}(0)\right)}{\rho_{i} - s} - \frac{\lambda\beta\left(\mathcal{T}_{s}f_{2}(0) - \mathcal{T}_{\rho_{i}}f_{2}(0)\right)}{\rho_{i} - s} \\
= \mathcal{T}_{s}\eta_{w,i}(0) - \left(\frac{\sigma^{2}}{2}\left(s + \rho_{i}\right) + c\right)\lambda\mathcal{T}_{s}f_{1}(0),$$
(44)

where

$$\eta_{w,i}(u) = \left(\frac{\sigma^2}{2}\rho_i^2 + c\rho_i - \lambda - \delta\right)\lambda \mathcal{F}_{\rho_i}f_1(u) - \lambda\beta \mathcal{F}_{\rho_i}f_2(u).$$
(45)

Hence, substituting (44) into (43) yields

$$h_{1}(s) - h_{2}(s) = \pi_{1}(s)\pi_{2}(s) \left(1 - \frac{\mathcal{T}_{s}\eta_{w,1}(0)}{\tau_{1}(s)} - \frac{\mathcal{T}_{s}\eta_{w,2}(0)}{\tau_{2}(s)}\right),$$
(46)

where

$$\tau_{1}(s) = \frac{\sigma^{2}}{2}\pi_{2}(\rho_{1})\left(s + \rho_{1} + \frac{2c}{\sigma^{2}}\right),$$

$$\tau_{2}(s) = \frac{\sigma^{2}}{2}\pi_{1}(\rho_{2})\left(s + \rho_{2} + \frac{2c}{\sigma^{2}}\right).$$
(47)

For i = 1, 2, define  $\eta_{w,i}^*(u) = \int_u^\infty \omega(u, x - u)\eta_{w,i}(x)dx$ . An analogous procedure can be employed to find alternative expressions for the numerators of (34) and (35) as follows.

$$l_{1}(s) - l_{2}(s) = \pi_{1}(s)\pi_{2}(s) \left(\frac{\mathcal{T}_{s}\eta_{w,1}^{*}(0)}{\tau_{1}(s)} + \frac{\mathcal{T}_{s}\eta_{w,2}^{*}(0)}{\tau_{2}(s)}\right),$$
(48)

$$k_1(s) - k_2(s) = \pi_1(s)\pi_2(s) \left(\frac{b(\rho_1)}{\tau_1(s)} + \frac{b(\rho_2)}{\tau_2(s)}\right).$$
(49)

Based on (46), (48), and (49), the Laplace transforms of  $\varphi_{\omega}(u)$  and  $\varphi_{d}(u)$  can be rewritten as

$$\widetilde{\varphi}_{\omega}(s) = \frac{\left(\mathcal{T}_{s}\eta_{w,1}^{*}(0)/\tau_{1}(s)\right) + \left(\mathcal{T}_{s}\eta_{w,2}^{*}(0)/\tau_{2}(s)\right)}{1 - \left(\mathcal{T}_{s}\eta_{w,1}(0)/\tau_{1}(s)\right) - \left(\mathcal{T}_{s}\eta_{w,2}(0)/\tau_{2}(s)\right)},\tag{50}$$

$$\widetilde{\varphi}_{d}(s) = \frac{(b(\rho_{1})/\tau_{1}(s)) + (b(\rho_{2})/\tau_{2}(s))}{1 - (\mathcal{T}_{s}\eta_{w,1}(0)/\tau_{1}(s)) - (\mathcal{T}_{s}\eta_{w,2}(0)/\tau_{2}(s))}.$$
 (51)

Finally, the inversion of the generating function in (50) and (51) gives the following results.  $\hfill \Box$ 

**Theorem 5.** The Gerber–Shiu functions  $\varphi_{\omega}(u)$  and  $\varphi_{d}(u)$  satisfy the following defective renewal equations:

$$\varphi_{\omega}(u) = \int_{0}^{u} \varphi_{\omega}(u-x)g(x)dx + H_{\omega}(u), \qquad (52)$$

$$\varphi_d(u) = \int_0^u \varphi_d(u-x)g(x)\mathrm{d}x + H_d(u), \qquad (53)$$

where the Laplace transforms of  $g, H_{\omega}, H_{d}$  are given by

$$\widetilde{g}(s) = \frac{\mathcal{T}_{s}\eta_{w,1}(0)}{\tau_{1}(s)} + \frac{\mathcal{T}_{s}\eta_{w,2}(0)}{\tau_{2}(s)},$$

$$\widetilde{H}_{\omega}(s) = \frac{\mathcal{T}_{s}\eta_{w,1}^{*}(0)}{\tau_{1}(s)} + \frac{\mathcal{T}_{s}\eta_{w,2}^{*}(0)}{\tau_{2}(s)},$$

$$\widetilde{H}_{d}(s) = \frac{b(\rho_{1})}{\tau_{1}(s)} + \frac{b(\rho_{2})}{\tau_{2}(s)}.$$
(54)

*Proof.* To demonstrate that the renewal equations are defective, it remains to show that  $\kappa(\delta) = \int_0^\infty g(x) dx < 1$ . By (46), we have

$$\tilde{g}(s) = 1 - \frac{h_1(s) - h_2(s)}{\pi_1(s)\pi_2(s)}.$$
(55)

Therefore.

$$\kappa(\delta) = \tilde{g}(0)$$
  
=  $1 - \frac{h_1(0) - h_2(0)}{\pi_1(0)\pi_2(0)}$  (56)

$$= 1 - \frac{(\lambda + \delta + \beta)\delta}{\left(\left(\sigma^2/2\right)\rho_1^2 + c\rho_1\right)\left(\left(\sigma^2/2\right)\rho_2^2 + c\rho_2\right)} < 1.$$

In the case of  $\delta = 0$ , setting  $s = \rho_2(\delta)$  in (11), we get

$$E\left[e^{-\delta W+\rho_2(\delta)(cW+\sigma B(W)-X)}\right] = 1.$$
(57)

Differentiating the above equation with respect to  $\delta$  and then setting  $\delta = 0$ , we obtain

$$\rho_2'(0) = \frac{EW}{E(cW - X)}.$$
 (58)

Since E(cW - X) > 0 due to (4), taking the limit  $\delta \longrightarrow 0$  in (56) gives

$$\lim_{\delta \to 0} \kappa(\delta) = 1 - \frac{(\lambda + \beta)E(cW - X)}{((\sigma^2/2)\rho_1^2(0) + c\rho_1(0))cEW} < 1.$$
(59)

The proof is completed.

#### 5. Numerical Illustration

In this section, we assume that  $f_1(x) = \lambda_1 e^{-\lambda_1 x}$ ,  $f_2(x) = \lambda_2 e^{-\lambda_2 x}$  with  $\lambda_1, \lambda_2 > 0$ . For i = 1, 2, it is readily seen from (45) that

$$\eta_{w,i}(u) = \xi_{i,1} e^{-\lambda_1 u} + \xi_{i,2} e^{-\lambda_2 u}, \tag{60}$$

where

$$\xi_{i,1} = \frac{\lambda \lambda_1 ((\sigma^2/2) \rho_i^2 + c\rho_i - \delta - \lambda)}{\rho_i + \lambda_1},$$

$$\xi_{i,2} = -\frac{\lambda \beta \lambda_2}{\rho_i + \lambda_2},$$
(61)

with

$$\mathcal{T}_s \eta_{w,i}(0) = \frac{\xi_{i,1}}{s+\lambda_1} + \frac{\xi_{i,2}}{s+\lambda_2}.$$
 (62)

Multiplying both the denominators and the numerators in (35) by  $2/\sigma^2 \pi_1(\rho_2) \prod_{i=1}^2 (s + \lambda_i) \tau_i(s)$  gives

$$\widetilde{\varphi}_{\omega}(s) = \frac{\vartheta_1(s)\mathcal{T}_s\eta_{w,1}^*(0) - \vartheta_2(s)\mathcal{T}_s\eta_{w,2}^*(0)}{\vartheta(s)},\tag{63}$$

where

$$\begin{split} \vartheta_{i}(s) &= \left(s + \rho_{j} + \frac{2c}{\sigma^{2}}\right) \prod_{k=1}^{2} \left(s + \lambda_{k}\right), i, j = 1, 2, \quad i \neq j, \\ \vartheta(s) &= \frac{\sigma^{2}}{2} \pi_{2}\left(\rho_{1}\right) \prod_{i=1}^{2} \left(s + \lambda_{i}\right) \left(s + \rho_{i} + \frac{2c}{\sigma^{2}}\right) \\ &- \vartheta_{1}\left(s\right) \mathcal{T}_{s} \eta_{w,1}\left(0\right) + \vartheta_{2}\left(s\right) \mathcal{T}_{s} \eta_{w,2}\left(0\right). \end{split}$$
(64)

It is easy to see that  $\vartheta(s)$  is a polynomial of degree 4 with leading coefficient  $\sigma^2/2\pi_2(\rho_1)$ . On the other hand, Lemma 1 implied that  $\vartheta(s)$  has no zeros with nonnegative real part; then, it can be expressed as

$$\vartheta(s) = \frac{\sigma^2}{2} \pi_2(\rho_1) \prod_{i=1}^4 (s + R_i),$$
(65)



FIGURE 1: Ruin probabilities due to oscillation for different dependent parameters.



FIGURE 2: Ruin probabilities due to a claim for different dependent parameters.

with  $\operatorname{Re}(R_i) > 0$  for i = 1, 2, 3, 4. In what follows, we assume that  $R_i$  are distinct. By partial fractions, we have

$$\frac{\vartheta_i(s)}{\vartheta(s)} = \sum_{j=1}^4 \frac{\vartheta_{i,j}}{s+R_j}, \quad i = 1, 2,$$
(66)

where

$$\vartheta_{i,j} = \frac{\vartheta_i \left(-R_j\right)}{\left(\sigma^2/2\right) \pi_2\left(\rho_1\right) \prod_{n=1, n\neq j}^4 \left(R_n - R_j\right)}.$$
(67)

Submitting (66) into (63) yields

$$\widetilde{\varphi}_{\omega}(s) = \sum_{j=1}^{4} \left( \frac{\vartheta_{1,j} \mathcal{T}_{s} \eta_{\omega,1}^{*}(0)}{s + R_{j}} - \frac{\vartheta_{2,j} \mathcal{T}_{s} \eta_{\omega,1}^{*}(0)}{s + R_{j}} \right).$$
(68)

Denote by  $\gamma(s) = \vartheta_1(s)b(\rho_1) - \vartheta_2(s)b(\rho_2)$ . By using the same arguments, one gets the following expression of  $\tilde{\varphi}_d(s)$ .

$$\widetilde{\varphi}_d(s) = \sum_{j=1}^4 \frac{\gamma_j}{s+R_j},\tag{69}$$

where

$$\gamma_j = \frac{\gamma(-R_j)}{\left(\sigma^2/2\right)\pi_2\left(\rho_1\right)\prod_{n=1,n\neq j}^4 \left(R_n - R_j\right)}.$$
(70)

Upon inversion of the Laplace transforms in (68) and (69), we can obtain the explicit expressions for  $\varphi_{\omega}(u)$  and  $\varphi_{d}(u)$ , respectively. In the following example, we consider the ruin probabilities  $\psi_{\omega}(u)$  and  $\psi_{d}(u)$  by letting  $\delta = 0$  and  $\omega(x_{1}, x_{2}) \equiv 1$ .

*Example 1.* Suppose that  $f_1$  and  $f_2$  are exponentially distributed as above with  $\lambda_1 = 2, \lambda_2 = 0.5$ . Set  $c = 2, \lambda = 1.5$ ,  $(\sigma^2/2) = 1$ . And the values of  $\beta$  are 1.5, 1, 0.6, respectively. Then it is not difficult to check that the net profit condition (5) is fulfilled in such settings. After solving Lundberg's generalized (13) for these settings, we obtain the roots  $\rho_1$  and  $\rho_2$ . Furthermore, we can calculate exact values for  $\psi_d(u)$  by inverting (69). Figure 1 shows the behavior of  $\psi_d(u)$  with different  $\beta$ . As expected,  $\psi_d(u)$  decreases as the initial surplus u increases. Meanwhile,  $\psi_d(u)$  is increasing with respect to  $\beta$ .

In the same way, we can deal with  $\psi_{\omega}(u)$ . By inverting (68), we can give explicit expression for  $\psi_{\omega}(u)$ . Figure 2 shows the behavior of  $\psi_{\omega}(u)$  for different  $\beta$ . We notice that the ruin probabilities caused by claims increase first and then decrease as the initial capital increases.

#### 6. Conclusions

In this paper, we model insurance surplus by considering a perturbed risk model and time-dependent claims, in which the distribution of the next claim amount is defined in terms of the time elapsed since the last claim. By using some analytic techniques, the expected discounted penalty functions  $\varphi_{\omega}(u)$ and  $\varphi_d(u)$  when ruin is caused by claims and by oscillations are fully discussed. The integro-differential equations and the Laplace transforms for the Gerber–Shiu functions are obtained. We also prove that the Gerber–Shiu functions satisfy some defective renewal equations. For the situation when claim amounts follow exponential distribution, we give explicit expressions of the Gerber–Shiu functions. Numerical examples are provided to illustrate the ruin probabilities caused by claims and oscillations. It shows that the results obtained in the paper are readily programmable and confirm the expectancy. From practical perspectives, the model considered can be used to assess the vulnerability issues of insurance companies in a market full of uncertainties. Furthermore, the results derived may also be used to help an insurance company protect itself against possible bankruptcy by informing the minimum capital levels required to limit ruin probability below a certain threshold.

#### **Data Availability**

No data were used to support this study.

### **Conflicts of Interest**

The authors declare that they have no conflicts of interest.

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