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Research Article

Composition-Differentiation Operators on Derivative Hardy Spaces

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We first explore conditions under which every weighted composition-differentiation operator on the Hardy space $H^1(\mathbb{D})$ is completely continuous. We then discuss necessary and sufficient conditions for these operators to be Hilbert–Schmidt on the derivative Hardy space $S^2(\mathbb{D})$.

1. Introduction

Let $\mathscr{X}$ be a Banach space consisting of analytic functions on a domain Ω in the complex plane. For simplicity, we shall assume that the underlying domain is the unit disk.

$$\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}. \tag{1}$$

The boundary of the unit disk will be denoted by

$$\mathbb{T} = \partial\mathbb{D} = \{z \in \mathbb{C}: |z| = 1\}. \tag{2}$$

Let φ be an analytic self-mapping on the unit disk; this means that φ is an analytic function that maps the unit disk into the unit disk. The composition operator with symbol φ, denoted by C_φ: $\mathscr{X} \longrightarrow \mathscr{X}$, is defined as follows:

$$C_\varphi(f)(z) = (f \circ \varphi)(z), \quad f \in \mathscr{X}, z \in \mathbb{D}. \tag{3}$$

This operator was first introduced by Nordgren [1] in the framework of the classical Hardy space H^2. Assuming that the symbol function φ is an inner function (an analytic self-map of the unit disk whose radial limit equals 1 almost everywhere on $\mathbb{T}$), Nordgren studied the boundedness of C_φ, computed its norm, described the spectrum of the composition operator, and finally related properties of this operator to the existence of fixed points of the Poisson integral of φ. It turned out that this operator plays a prominent role in the operator theory of function spaces. There are a bulk of papers on this topic, and to mention just a few, we should cite the papers [2–10] and the references therein; see also the books written on this topic [11, 12]. Among other properties, several authors have obtained conditions that ensure compactness, Fredholmness, and Hilbert–Schmidtness of the composition operator. It is well known that the composition operator is bounded on the Hardy space H^p and on the Bergman space A^p where p is a positive number (see [12]).

For an analytic function ψ: $\mathbb{D} \longrightarrow \mathbb{C}$, the weighted composition operator $C_{\psi,\varphi}$: $\mathscr{X} \longrightarrow \mathscr{X}$ is defined by

$$C_{\psi,\varphi}(f) = \psi \cdot (f \circ \varphi). \tag{4}$$

Similarly, we can define the composition-differentiation operator D_φ: $\mathscr{X} \longrightarrow \mathscr{X}$ by

$$D_\varphi(f) = f' \circ \varphi. \tag{5}$$

Continuing in this way, we define the weighted composition-differentiation operator $D_{\psi,\varphi}$: $\mathscr{X} \longrightarrow \mathscr{X}$ as follows:

$$D_{\psi,\varphi}(f) = \psi \cdot (f' \circ \varphi). \tag{6}$$

We should mention that in most cases, the functional Banach space $\mathcal{X}$ equals either the Hardy space H^p or the Bergman space A^p for $p \geq 1$. According to [[13], Corollary 3.2], for a univalent self-map φ of the unit disk, the operator D_φ on the Hardy space H^2 is bounded if and only if

$$\sup_{z \in \mathbb{D}} \frac{1 - |z|}{(1 - |\varphi(z)|)^3} < \infty. \tag{7}$$

Moreover, D_φ is compact on H^2 if and only if

$$\lim_{|z| \longrightarrow 1} \frac{1 - |z|}{(1 - |\varphi(z)|)^3} = 0. \tag{8}$$

The weighted composition-differentiation operator was recently studied in [2, 3]. In [2], the authors found necessary and sufficient conditions for $D_{\psi,\varphi}$ to be Hilbert–Schmidt both on the Hardy space and on the Bergman space. We also consider the operator $D'_{\psi,\varphi}$: $\mathcal{X} \longrightarrow \mathcal{X}$ by

$$D'_{\psi,\varphi}(f) = \psi \cdot (f' \circ \varphi)\varphi', \tag{9}$$

and investigate its complete continuity as well as its Hilbert–Schmidtness on the derivative Hardy space S^2.

In this paper, we first focus on the nonreflexive Hardy space H^1 and try to find conditions under which the weighted composition-differentiation operator $D_{\psi,\varphi}$ is completely continuous. We shall provide characterizations for the complete continuity of this operator in terms of the boundary behaviors of ψ and φ. More precisely, we prove that $D_{\psi,\varphi}$ is completely continuous if and only if $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$. Similar problem will be solved for the operator $D'_{\psi,\varphi}$ (Theorems 1 and 2).

The second topic to investigate is the Hilbert–Schmidt operators on a closed subspace of H^2; that is, the derivative Hardy space S^2 consisting of all analytic functions in the unit disk for which $f' \in H^2$ or

$$S^2 = \left\{f \in \mathrm{Hol}(\mathbb{D}) : f' \in H^2\right\}. \tag{10}$$

We will prove that D_φ is Hilbert–Schmidt on S^2 if and only if

$$\sup_{0<r<1} \frac{1}{2\pi} \int_0^{2\pi} \frac{\left|\varphi'\left(re^{i\theta}\right)\right|^2}{\left(1 - \left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3} \mathrm{d}\theta < \infty. \tag{11}$$

Similar problem for the operator D'_φ will also be discussed (Theorems 3 and 4).

2. Preliminaries

An analytic function f on the unit disk is said to belong to the Hardy space $H^p = H^p(\mathbb{D})$ if

$$\|f\|_{H^p}^p = \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \left|f\left(re^{i\theta}\right)\right|^p \mathrm{d}\theta < \infty. \tag{12}$$

For $1 \leq p < \infty$, the Hardy space H^p is a Banach space of analytic functions, and for $p = 2$, it is a Hilbert space with the following inner product:

$$\langle f, g \rangle = \frac{1}{2\pi} \int_0^{2\pi} f^*\left(e^{i\theta}\right)\overline{g^*\left(e^{i\theta}\right)} \mathrm{d}\theta, \tag{13}$$

where

$$f^*\left(e^{i\theta}\right) \coloneqq \lim_{r \longrightarrow 1^-} f\left(\mathrm{r}e^{i\theta}\right), \tag{14}$$

is the boundary function of f (for the existence of boundary function f^*, see [14], Chap. 17). It is easy to see that for $f \in H^2$ with Taylor series $f(z) = \sum_{n=0}^{\infty} a_n z^n$, the norm of f is given by

$$\|f\|_{H^2}^2 = \sum_{n=0}^{\infty} \left|a_n\right|^2. \tag{15}$$

The next space we study is the space S^p of all analytic functions in the unit disk for which $f' \in H^p$ or

$$S^p = \{f \in \mathrm{Hol}(\mathbb{D}): f' \in H^p\}. \tag{16}$$

The space S^p is called the derivative Hardy space. The norm in S^p is defined by the following relation:

$$\|f\|_{S^p}^p = \|f\|_{H^p}^p + \left\|f'\right\|_{H^p}^p. \tag{17}$$

It is clear that S^p is a closed subspace of H^p, and if $p = 2$, S^p is a Hilbert space of analytic functions. A computation shows that for $f(z) = \sum_{n=0}^{\infty} a_n z^n$, we have

$$\|f\|_{S^2}^2 = \sum_{n=0}^{\infty} \left(n^2 + 1\right)\left|a_n\right|^2. \tag{18}$$

This space equipped with the above norm was studied by Korenblum in 1972 [15]. We should mention that many authors use the expression

$$\|f\|_{S^p}^p = |f(0)|^p + \left\|f'\right\|_{H^p}^p, \tag{19}$$

for the norm of S^p (see for instance [16,17]). We shall use this norm in §3.

Let $\mathcal{X}$ be a Banach space. An operator T: $\mathcal{X} \longrightarrow \mathcal{X}$ is compact if for every bounded sequence (x_n) in $\mathcal{X}$, (Tx_n) has a convergent subsequence. On the other hand, the operator T is called completely continuous if the weak convergence of $x_n \longrightarrow x$ in $\mathcal{X}$ implies

$$\left\|\mathrm{Tx}_n - \mathrm{Tx}\right\| \longrightarrow 0, \quad n \longrightarrow \infty. \tag{20}$$

It is well known that every compact operator is completely continuous. We remark that for $1 < p < \infty$, the Hardy space H^p is reflexive, meaning that it is isometrically isomorphic with its dual. It is known that on reflexive Banach spaces, an operator T is compact if and only if it is completely continuous. In this paper, we concentrate on the nonreflexive Banach space H^1 and the composition-differentiation operator $D_{\psi,\varphi}$ on H^1. We will find conditions on the functions ψ and φ to ensure that the operator $D_{\psi,\varphi}$ is completely continuous on H^1. We also consider

$D'_{\psi,\varphi}: \mathcal{X} \longrightarrow \mathcal{X}$ and investigate its complete continuity (see Theorems 1 and 2).

The second problem to be discussed is the conditions under which the abovementioned differentiation operators are Hilbert–Schmidt. We recall that if $\mathcal{H}$ is a separable Hilbert space and if T is an operator on $\mathcal{H}$, then T is said to be Hilbert–Schmidt provided that

$$\sum_{n=1}^{\infty} \left\| \mathrm{T}e_n \right\|^2 < \infty, \tag{21}$$

where $\{e_n\}$ is an orthonormal basis in $\mathcal{H}$. In the next section, we take $\mathcal{H} = S^2$ and try to find the condition for the operators $D_{\psi,\varphi}$ and $D'_{\psi,\varphi}$ to be Hilbert–Schmidt.

3. Main Results

In the following theorem, we shall characterize the complete continuity of the composition-differentiation operator in terms of ψ and φ.

Theorem 1. *Let $\psi \in H^1$ and φ be a self-map on $\mathbb{D}$. Assume that $D_{\psi,\varphi}$ is bounded on H^1. Then, $D_{\psi,\varphi}$ is completely continuous on H^1 if and only if $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.*

Proof. Let $D_{\psi,\varphi}$ be completely continuous, and let $\mathbb{T}$ denote the unit circle. Assume that $f \in L^\infty(\mathbb{T})$ and let $\widehat{f}(n)$ be its n-th Fourier coefficient. By the Riemann–Lebesgue lemma, we have

$$\int_{\mathbb{T}} f(z)\overline{z}^n \mathrm{d}m = \widehat{f}(n) \longrightarrow 0, \quad n \longrightarrow \infty, \tag{22}$$

where $\mathrm{d}m$ is the normalized arc-length measure on $\mathbb{T}$. This means that $\{z^n\}$ converges to zero weakly in $L^1(\mathbb{T})$ and hence weakly in H^1. Since $D_{\psi,\varphi}$ is completely continuous, it follows that

$$\left\| D_{\psi,\varphi}(z^n) \right\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{23}$$

On the other hand, for each $n \in \mathbb{N}$,

$$\begin{aligned} 0 \le \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} |\psi| \mathrm{d}m &\le \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} n|\psi| \mathrm{d}m \\ &= \int_{\{e^{i\theta}:|\varphi(e^{i\theta})|=1\}} n|\psi||\varphi|^{n-1} \mathrm{d}m \\ &\le \int_{\mathbb{T}} n|\psi||\varphi|^{n-1} \mathrm{d}m \\ &= \left\| D_{\psi,\varphi}(z^n) \right\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \tag{24}$$

Therefore, the integral on the left-hand side must be zero, from which it follows that $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$.

Conversely, let (f_n) be a weak null sequence in H^1. It follows that $f'_n \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. Using this fact together with the assumption that $\psi = 0$ almost everywhere in $\{e^{i\theta} : |\varphi(e^{i\theta})| = 1\}$, we conclude that

$$D_{\psi,\varphi}(f_n)\left(e^{i\theta}\right) = \psi\left(e^{i\theta}\right) f'_n\left(\varphi\left(e^{i\theta}\right)\right) \longrightarrow 0, \text{a.e. in } \mathbb{T}. \tag{25}$$

It now follows that $D_{\psi,\varphi}(f_n)$ converges to zero in measure in $L^1(\mathbb{T})$ (see [18], page 74). Moreover, the boundedness of $D_{\psi,\varphi}$ on H^1 implies that $D_{\psi,\varphi}(f_n) \longrightarrow 0$ in the weak topology of H^1 and hence in the weak topology of $L^1(\mathbb{T})$. Finally, we invoke the fact that weak convergence of a given sequence together with its convergence in measure implies its norm convergence (see [19], page 295), that is, $\|D_{\psi,\varphi}(f_n)\|_{H^1} \longrightarrow 0$ as $n \longrightarrow \infty$. □

Theorem 2. *Let $\psi \in H^1$ and φ be a self-map on $\mathbb{D}$. Assume that $D'_{\psi,\varphi}$ is bounded on H^1. Then, $D'_{\psi,\varphi}$ is completely continuous on H^1 if and only if $\psi = 0$ almost everywhere in*

$$E := \left\{e^{i\theta} : \left|\varphi\left(e^{i\theta}\right)\right| = 1\right\} \cap \left\{e^{i\theta} : \left|\varphi'\left(e^{i\theta}\right)\right| \ge 1\right\}. \tag{26}$$

Proof. Let $D'_{\psi,\varphi}$ be completely continuous, and let $\mathbb{T}$ denote the unit circle. As in the proof of Theorem 1, we use the Riemann–Lebesgue lemma to conclude that $\{z^n\}$ converges to zero weakly in $L^1(\mathbb{T})$ and hence weakly in H^1. Since $D'_{\psi,\varphi}$ is completely continuous, it follows that

$$\left\| D'_{\psi,\varphi}(z^n) \right\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty. \tag{27}$$

On the other hand, on E, we have $|\varphi'| \ge 1$, so that for each $n \in \mathbb{N}$,

$$\begin{aligned} 0 \le \int_E |\psi| \mathrm{d}m \le \int_E n|\psi||\varphi'| \mathrm{d}m &= \int_E n|\psi||\varphi'||\varphi|^{n-1} \mathrm{d}m \\ &\le \int_{\mathbb{T}} n|\psi||\varphi'||\varphi|^{n-1} \mathrm{d}m \\ &= \left\| D'_{\psi,\varphi}(z^n) \right\|_{H^1} \longrightarrow 0, \quad n \longrightarrow \infty. \end{aligned} \tag{28}$$

Therefore, the integral on the left-hand side must be zero, from which it follows that $\psi = 0$ almost everywhere in E.

Conversely, let (f_n) be a sequence in H^1 that converges to zero weakly. It follows that $f'_n \longrightarrow 0$ uniformly on compact subsets of $\mathbb{D}$. Using this fact together with the assumption that $\psi = 0$ almost everywhere in E, we conclude that

$$D'_{\psi,\varphi}(f_n)\left(e^{i\theta}\right) = \psi\left(e^{i\theta}\right) f'_n\left(\varphi\left(e^{i\theta}\right)\right)\varphi'\left(e^{i\theta}\right) \longrightarrow 0, \text{a.e. in } \mathbb{T}. \tag{29}$$

This implies that $D'_{\psi,\varphi}(f_n)$ converges to zero in measure in $L^1(\mathbb{T})$ (see [18], page 74). Moreover, the boundedness of $D'_{\psi,\varphi}$ on H^1 implies that $D'_{\psi,\varphi}(f_n) \longrightarrow 0$ in the weak topology of H^1 and hence in the weak topology of $L^1(\mathbb{T})$. Again, we recall that weak convergence together with convergence in measure implies norm convergence (see [7], page 295); therefore, $\|D'_{\psi,\varphi}(f_n)\|_{H^1} \longrightarrow 0$ as $n \longrightarrow \infty$.

In the following theorems, we discuss the conditions under which the composition-differentiation operator on S^2 is Hilbert–Schmidt. We consider the following norm:

$$\|f\|_{S^2}^2 = |f(0)|^2 + \|f'\|_{H^2}^2. \tag{30}$$

It is easy to verify that with respect to this norm, the vectors $1 \cup \{z^n/n\}_{n\geq 1}$ form an orthonormal basis for S^2. □

Theorem 3. *Let φ be a self-map on $\mathbb{D}$. Then, D_φ is Hilbert–Schmidt on S^2 if and only if*

$$\sup_{0<r<1} \frac{1}{2\pi}\int_0^{2\pi} \frac{\left|\varphi'\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3} d\theta < \infty. \tag{31}$$

Proof. Assume that condition (1) holds. We first note that for $n\geq 1$,

$$\begin{aligned}\left\|D_\varphi\left(\frac{z^n}{n}\right)\right\|_{S^2}^2 &= \|\varphi^{n-1}\|_{S^2}^2 \\ &= \left|\varphi^{n-1}(0)\right|^2 + \|(n-1)\varphi^{n-2}\varphi'\|_{H^2}^2.\end{aligned} \tag{32}$$

Since $|\varphi(0)|<1$, the series $\sum_{n=1}^{\infty}|\varphi^{n-1}(0)|^2$ converges, so that the series

$$\sum_{n=1}^{\infty}\left\|D_\varphi\left(\frac{z^n}{n}\right)\right\|_{S^2}^2, \tag{33}$$

converges if and only if

$$\sum_{n=1}^{\infty}\|(n-1)\varphi^{n-2}\varphi'\|_{H^2}^2 < \infty. \tag{34}$$

It is easy to see that for $0\leq x<1$, we have

$$\sum_{n=1}^{\infty} n^2 x^{n-1} = \frac{1+x}{(1-x)^3}. \tag{35}$$

This implies that

$$\begin{aligned}\sum_{n=1}^{\infty}\|(n-1)\varphi^{n-2}\varphi'\|_{H^2}^2 &= \sum_{n=1}^{\infty}\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}(n-1)^2\left|\varphi\left(re^{i\theta}\right)\right|^{2(n-2)}\left|\varphi'\left(re^{i\theta}\right)\right|^2 d\theta \\ &= \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\sum_{n=1}^{\infty} n^2\left|\varphi\left(re^{i\theta}\right)\right|^{2(n-1)}\left|\varphi'\left(re^{i\theta}\right)\right|^2 d\theta \\ &= \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{1+\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}\left|\varphi'\left(re^{i\theta}\right)\right|^2 d\theta \\ &\leq \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{2\left|\varphi'\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3} d\theta \\ &< \infty,\end{aligned} \tag{36}$$

which means that D_φ is Hilbert–Schmidt on S^2.

On the other hand, assume that D_φ is Hilbert–Schmidt on S^2. By the above computations and equality (2), we have

$$\begin{aligned}\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\varphi'\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3} d\theta &\leq \sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{1+\left|\varphi\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}\left|\varphi'\left(re^{i\theta}\right)\right|^2 d\theta \\ &= \sum_{n=1}^{\infty}\|(n-1)\varphi^{n-2}\varphi'\|_{H^2}^2 \\ &< \infty,\end{aligned} \tag{37}$$

which is the desired result. □

Corollary 4. *Let φ be a self-map on $\mathbb{D}$ and ψ be an analytic function on $\mathbb{D}$. Then, $D_{\psi,\varphi}$ is Hilbert–Schmidt on S^2 if and only if*

$$\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\frac{\left|\psi\left(re^{i\theta}\right)\right|^2\left|\varphi'\left(re^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(re^{i\theta}\right)\right|^2\right)^3}\mathrm{d}\theta<\infty. \tag{38}$$

Proof. The proof is similar to that of the preceding theorem. □

Theorem 5. *Let φ be a self-map on $\mathbb{D}$. Then, D'_φ is Hilbert–Schmidt on S^2 if*

$$\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\left(\frac{\left|\varphi'\left(\mathrm{re}^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(\mathrm{re}^{i\theta}\right)\right|^2\right)^3}+\frac{\left|\varphi''\left(\mathrm{re}^{i\theta}\right)\right|^2}{1-\left|\varphi\left(\mathrm{re}^{i\theta}\right)\right|^2}+\frac{\left|\varphi'\left(\mathrm{re}^{i\theta}\right)\varphi''\left(\mathrm{re}^{i\theta}\right)\right|}{\left(1-\left|\varphi\left(\mathrm{re}^{i\theta}\right)\right|^2\right)^2}\right)\mathrm{d}\theta<\infty. \tag{39}$$

Proof. First, note that

$$\begin{aligned}\left\|D'_\varphi\left(\frac{z^n}{n}\right)\right\|_{S^2}^2&=\left\|\varphi^{n-1}\varphi'\right\|_{S^2}^2\\&=\left|\varphi^{n-1}(0)\varphi'(0)\right|^2+\left\|(n-1)\varphi^{n-2}\varphi'+\varphi^{n-1}\varphi''\right\|_{H^2}^2.\end{aligned} \tag{40}$$

Since $|\varphi(0)|<1$, the series

$$\sum_{n=1}^{\infty}\left|\varphi^{n-1}(0)\varphi'(0)\right|^2=|\phi(0)|^2\sum_{n=1}^{\infty}|\varphi(0)|^{2(n-1)}, \tag{41}$$

converges, so that the series

$$\sum_{n=1}^{\infty}\left\|D'_\varphi\left(\frac{z^n}{n}\right)\right\|_{S^2}^2, \tag{42}$$

converges if and only if

$$\sum_{n=1}^{\infty}\left\|n\varphi^{n-1}\varphi'+\varphi^n\varphi''\right\|_{H^2}^2<\infty. \tag{43}$$

On the other hand,

$$\begin{aligned}\sum_{n=1}^{\infty}\left|n\varphi^{n-1}\varphi'+\varphi^n\varphi''\right|^2&\le\sum_{n=1}^{\infty}n^2|\varphi|^{2(n-1)}|\varphi'|^2\\&\quad+\sum_{n=1}^{\infty}|\varphi|^{2n}|\varphi''|^2+\sum_{n=1}^{\infty}2n|\varphi|^{2n-1}|\varphi'\varphi''|\\&=\frac{1+|\varphi|^2}{\left(1-|\varphi|^2\right)^3}|\varphi'|^2+\frac{|\varphi|^2}{1-|\varphi|^2}|\varphi''|^2\\&\quad+\frac{2|\varphi|}{\left(1-|\varphi|^2\right)^2}|\varphi'\varphi''|,\end{aligned} \tag{44}$$

where for the last term, we used the identity

$$\sum_{n=1}^{\infty}2\mathrm{n}x^{2n-1}=\frac{2x}{\left(1-x^2\right)^2},\quad 0\le x<1. \tag{45}$$

Since $0\le|\varphi|<1$, it follows from (44) that D'_φ is Hilbert–Schmidt if (39) holds. □

Theorem 6. *Let φ be a self-map on $\mathbb{D}$ such that D'_φ is Hilbert–Schmidt on S^2. Then,*

$$\sup_{0<r<1}\frac{1}{2\pi}\int_0^{2\pi}\left[\frac{\left|\varphi'\left(\mathrm{re}^{i\theta}\right)\right|^2}{\left(1-\left|\varphi\left(\mathrm{re}^{i\theta}\right)\right|^2\right)^3}+\frac{\left|\varphi''\left(\mathrm{re}^{i\theta}\right)\right|^2}{1-\left|\varphi\left(\mathrm{re}^{i\theta}\right)\right|^2}+2\,\mathrm{Re}\left(\frac{\varphi}{\left(1-\varphi^2\right)^2}\varphi'\varphi''\right)\right]\mathrm{d}\theta<\infty. \tag{46}$$

Proof. Assume that D'_φ is Hilbert–Schmidt. We note that

$$\sum_{n=1}^{\infty}\left|n\varphi^{n-1}\varphi' + \varphi^{n}\varphi''\right|^{2} = \sum_{n=1}^{\infty} n^{2}|\varphi|^{2(n-1)}\left|\varphi'\right|^{2} + \sum_{n=1}^{\infty}|\varphi|^{2n}\left|\varphi''\right|^{2} + \sum_{n=1}^{\infty} 2n\,\mathrm{Re}\left(\varphi^{2n-1}\varphi'\varphi''\right) = \frac{1+|\varphi|^{2}}{\left(1-|\varphi|^{2}\right)^{3}}\left|\varphi'\right|^{2} + \frac{|\varphi|^{2}}{1-|\varphi|^{2}}\left|\varphi''\right|^{2} + 2\,\mathrm{Re}\left(\frac{\varphi}{\left(1-\varphi^{2}\right)^{2}}\varphi'\varphi''\right), \tag{47}$$

from which the result follows. □

Data Availability

Data sharing is not applicable to this article as no data sets were generated or analyzed during the current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors have contributed equally to this work.

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