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Research Article

Congruences Involving Special Sums of Triple Reciprocals

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Define the sums of triple reciprocals $Z(n) = \sum_{i+j+k=n} 1/ijk, i, j, k \geq 1$. Zhao discovered the following curious congruence for any odd prime p, $Z(p) \equiv -2B_{p-3} \pmod p$. Xia and Cai extended the above congruence to modulo p^2, $Z(p) \equiv 12B_{p-3}/(p-3) - 3B_{2p-4}/(p-2) \pmod{p^2}$, where $p > 5$ is a prime. In this paper, we consider the congruences about $Z([(p-1+N)/N])$ (where $[x]$ is the integral part of x, $N = 1, 2, 3, 4, 6$) modulo p^2. When $N = 1$, the results we obtain are the results of Zhao and Xia and Cai.

1. Introduction

Let

$$Z(n) = \sum_{\substack{i+j+k=n \\ i,j,k \in \mathscr{P}_p}} \frac{1}{ijk}, \tag{1}$$

where $\mathscr{P}_p$ denotes the set of positive integers which are prime to p.

In 2003, Zhao [1] first announced (see [2], Corollary 4.2) the following curious congruence involving multiple harmonic sums for any odd prime $p > 3$:

$$Z(p) \equiv -2B_{p-3} \pmod p, \tag{2}$$

which holds when $p = 3$ evidently. Here, Bernoulli numbers B_k are defined by the recursive relation:

$$B_0 = 1, \sum_{i=0}^{n} \binom{n+1}{i} B_i = 0, \quad n \geq 1. \tag{3}$$

The first few Bernoulli numbers are $B_0 = 1, B_1 = -1/2, B_2 = 1/6,$

$$\begin{aligned} B_{2n+1} &= 0, \quad (n \geq 1), \\ B_4 &= -\frac{1}{30}, \\ B_6 &= \frac{1}{42}, \\ B_8 &= -\frac{1}{30}, \\ B_{10} &= \frac{5}{66}, \ldots \end{aligned} \tag{4}$$

A simple proof of (2) was presented in [3]. This congruence has been generalized along several directions. First, Zhou and Cai [4] established the following harmonic congruence for prime $p > 3$ and integer $n \leq p-2$:

$$\sum_{l_1+l_2+\cdots+l_n=p} \frac{1}{l_1 l_2 \cdots l_n} \equiv \begin{cases} -(n-1)! B_{p-n} \pmod p, & \text{if } 2 \nmid n, \\ -\dfrac{n(n!)}{2(n+1)} p B_{p-n-1} \left(\bmod p^2\right), & \text{if } 2 \mid n. \end{cases} \tag{5}$$

Later, Xia and Cai [5] generalized (2) to

$$Z(p) \equiv \frac{12B_{p-3}}{p-3} - \frac{3B_{2p-4}}{p-2} \left(\bmod p^2\right), \tag{6}$$

where $p > 5$ is a prime.

In 2014, Wang and Cai [6] proved for every prime $p \geq 3$ and positive integer r,

$$Z(p^r) \equiv -2p^{r-1} B_{p-3} \left(\bmod p^r\right). \tag{7}$$

Let $n = 2$ or 4; for every positive integer $r \geq n/2$ and prime $p > n$, Zhao [7] generalized (7) to

$$\sum_{\substack{i_1+i_2+\cdots+i_n=p^r \\ i_1,i_2,\ldots,i_n \in \mathcal{P}_p}} \frac{1}{i_1 i_2 \cdots i_n} \equiv -\frac{n!}{n+1} p^r B_{p-n-1} \left(\bmod p^{r+1}\right). \tag{8}$$

In 2017, Cai et al. [8] replaced the odd prime p in the summation and modulus with a product of two odd prime powers. For other related papers, see [9–11].

In 2022, Chern [12] established the generalization of the conjecture in [8].

We have seen that the sum of the subscript variables in the literature is an integer multiple of the modulus. Next, we are going to consider the case where the sum of the subscript variables is not an integer multiple of the modulus. When positive integer $n < p$, we have

$$Z(n) = \sum_{\substack{i+j+k=n \\ i,j,k \geq 1}} \frac{1}{ijk}. \tag{9}$$

In this paper, we consider the congruences about $Z([(p-1+N)/N])$ (where $[x]$ is the integral part of x, $N = 1, 2, 3, 4, 6$) modulo p^2 and obtain the following theorems.

Let $H_n(k)$ be defined by

$$\begin{aligned} H_0(k) &= 0, \\ H_n(k) &= \sum_{j=1}^{n} \frac{1}{j^k}, \end{aligned} \tag{10}$$

where $H_n(1)$ is the n-th harmonic number H_n.

Theorem 1. *Let n be a positive integer greater than 2. Then,*

$$Z(n) = \frac{3}{n} \left(H_{n-1}^2 - H_{n-1}(2)\right). \tag{11}$$

Theorem 2. *Let p be a prime greater than 5. Then,*

$$Z(p) \equiv \frac{12B_{p-3}}{p-3} - \frac{3B_{2p-4}}{p-2} \left(\bmod p^2\right), \tag{12}$$

and

$$Z\left(\frac{p+1}{2}\right) \equiv 24q_p^2(2) - p\left(24q_p^2(2) + 24q_p^3(2) + 14B_{p-3}\right) \left(\bmod p^2\right), \tag{13}$$

where $q_p(n) = (n^{p-1} - 1)/p$ is the Fermat quotient. More generally, we have

$$Z\left(\left[\frac{p+2}{3}\right]\right) \equiv \frac{18}{2p+3+(p/3)} \left\{ \begin{array}{l} \frac{9}{4} q_p^2(3) + \frac{B_{2p-3}(\{-p/3\})}{2p-3} - 2\frac{B_{p-2}(\{-p/3\})}{p-2} \\ -p\left(\frac{9}{4} q_p^3(3) - \frac{q_p(3)}{2} \left(\frac{p}{3}\right) B_{p-2}\left(\frac{1}{3}\right) + \frac{2}{9} B_{p-3}\left(\frac{1}{3}\right)\right) \end{array} \right\} \left(\bmod p^2\right), \tag{14}$$

where $B_n(x)$ are the Bernoulli polynomials defined by

$$B_n(x) = \sum_{i=0}^{n} \binom{n}{i} B_i x^{n-i}, \quad n \geq 0, \tag{15}$$

and $(\div)$ is the Legendre symbol and $\{x\}$ is the fractional part of x.

From Theorem 2, we obtain

$$Z(p) \equiv -2B_{p-3} (\bmod p), \tag{16}$$

and this is the result of (2), and

$$Z\left(\frac{p+1}{2}\right) \equiv 24q_p^2(2) (\bmod p). \tag{17}$$

From Theorem 2, by Lemmas 6 and 7 (see Auxiliary Results), we also obtain

$$Z\left(\left[\frac{p+2}{3}\right]\right) \equiv \frac{18}{3+(p/3)}\left(\frac{9}{4}q_p^2(3) - \frac{1}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)\right) (\bmod p). \tag{18}$$

Theorem 3. *Let p be a prime greater than 7. Then,*

$$Z\left(\left[\frac{p+3}{4}\right]\right) \equiv \frac{12}{p+2+(-1/p)}\left\{\begin{array}{c} 9q_p^2(2) - \left(\frac{-1}{p}\right)\left(8E_{p-3} - 4E_{2p-4}\right) \\ -p\left(9q_p^3(2) - 6q_p(2)\left(\frac{-1}{p}\right)E_{p-3} + \frac{14}{3}B_{p-3}\right)\end{array}\right\} \left(\bmod p^2\right), \tag{19}$$

where E_n are Euler numbers defined by

$$\frac{2e^t}{e^{2t}+1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}\left(|t| < \frac{\pi}{2}\right). \tag{20}$$

From Theorem 3, we obtain

$$Z\left(\left[\frac{p+3}{4}\right]\right) \equiv \frac{12}{2+(-1/p)}\left(9q_p^2(2) - \left(\frac{-1}{p}\right)4E_{p-3}\right) (\bmod p). \tag{21}$$

Theorem 4. *Let p be a prime greater than 11. Then,*

$$Z\left(\left[\frac{p+5}{6}\right]\right) \equiv \frac{18}{p+3+2(p/3)} \cdot \left\{\begin{array}{c} 4q_p^2(2) + 6q_p(2)q_p(3) + \frac{9}{4}q_p^2(3) + \frac{B_{2p-3}(\{-p/6\})}{2p-3} \\ -2\frac{B_{p-2}(\{-p/6\})}{p-2} - p\left(4q_p^3(2) + 3q_p(2)q_p^2(3) + 3q_p^2(2)q_p(3)\right. \\ \left. +\frac{9}{4}q_p^3(3) - \frac{4q_p(2)+3q_p(3)}{12}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right) + \frac{1}{9}B_{p-3}\left(\frac{1}{6}\right)\right)\end{array}\right\} \left(\bmod p^2\right). \tag{22}$$

From Theorem 4, by Lemmas 6 and 7 (see Auxiliary Results), we obtain

$$Z\left(\left[\frac{p+5}{6}\right]\right) \equiv \frac{18}{3+2(p/3)}\left(4q_p^2(2) + 6q_p(2)q_p(3) + \frac{9}{4}q_p^2(3) - \frac{(p/3)}{2}B_{p-2}\left(\frac{1}{6}\right)\right) (\bmod p). \tag{23}$$

2. Auxiliary Results

Lemma 5 (see [13, 14]). *Let $p \geq 5$ be a prime and $k \leq p-4$ be a positive integer. Then,*

$$H_{p-1}(k) \equiv \begin{cases} \binom{k+1}{2}\dfrac{B_{p-2-k}}{p-2-k}p^2 \left(\bmod p^3\right), & \text{if } k \text{ is odd},\\ k\left(\dfrac{B_{2p-2-k}}{2p-2-k} - 2\dfrac{B_{p-1-k}}{p-1-k}\right)p \left(\bmod p^3\right), & \text{if } k \text{ is even}.\end{cases}$$
$$H_{\frac{p-1}{2}}(k) \equiv \begin{cases} \left(2^k-2\right)\left(2\dfrac{B_{p-k}}{p-k} - \dfrac{B_{2p-1-k}}{2p-1-k}\right)\left(\bmod p^2\right), & \text{if } k>1 \text{ is odd},\\ \dfrac{k\left(2^{k+1}-1\right)}{2(k+1)}pB_{p-1-k}\left(\bmod p^2\right), & \text{if } k \text{ is even},\\ -2q_p(2) + pq_p^2(2)\left(\bmod p^2\right), & \text{if } k=1.\end{cases} \tag{24}$$

Lemma 6 (see [15]). *Let n be a positive integer; then,*

$$\begin{aligned} B_{2n}\left(\frac{1}{3}\right) &= B_{2n}\left(\frac{2}{3}\right) = \frac{3-3^{2n}}{2\cdot 3^{2n}}B_{2n},\\ B_{2n}\left(\frac{1}{6}\right) &= B_{2n}\left(\frac{5}{6}\right) = \frac{\left(2-2^{2n}\right)\left(3-3^{2n}\right)}{2\cdot 6^{2n}}B_{2n}. \end{aligned} \tag{25}$$

Lemma 7 (see [15]). *Let p be an odd prime, $x \in Z_p$, and k, b be positive integers with $p-1 \nmid b$. Then,*

$$\begin{aligned} \frac{B_{k(p-1)+b}(x)}{k(p-1)+b} &\equiv \frac{B_b(x)}{b} \ (\bmod p),\\ B_b(1-x) &= (-1)^b B_b(x), \quad (b\geq 2). \end{aligned} \tag{26}$$

Lemma 8 (see [16]). *Let p be a prime greater than 5. Then,*

$$\begin{aligned} H_{[p/3]} &\equiv -\frac{3}{2}q_p(3) + p\left(\frac{3}{4}q_p^2(3) - \frac{1}{30}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)\left(\bmod p^2\right),\\ H_{[p/3]}(2) &\equiv -\frac{B_{2p-3}(\{-p/3\})}{2p-3} + 2\frac{B_{p-2}(\{-p/3\})}{p-2} + \frac{2p}{9}B_{p-3}\left(\left\{\frac{-p}{3}\right\}\right)\left(\bmod p^2\right). \end{aligned} \tag{27}$$

Lemma 9 (see [16]). *Let p be a prime greater than 5. Then,*

$$\begin{aligned} H_{[p/4]} &\equiv -3q_p(2) + p\left(\frac{3}{2}q_p^2(2) - (-1)^{(p-1)/2}E_{p-3}\right)\left(\bmod p^2\right),\\ H_{[p/4]}(2) &\equiv (-1)^{(p-1)/2}\left(8E_{p-3} - 4E_{2p-4}\right) + \frac{14}{3}pB_{p-3}\left(\bmod p^2\right). \end{aligned} \tag{28}$$

Lemma 10 (see [16]). *Let p be a prime greater than 5. Then,*

$$H_{\left[\frac{p}{6}\right]} \equiv -2q_p(2) - \frac{3}{2}q_p(3) + p\left(q_p^2(2) + \frac{3}{4}q_p^2(3) - \frac{1}{12}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)\left(\bmod p^2\right),$$
$$H_{\left[\frac{p}{6}\right]}(2) \equiv -\frac{B_{2p-3}(\{-p/6\})}{2p-3} + 2\frac{B_{p-2}(\{-p/6\})}{p-2} + \frac{p}{9}B_{p-3}\left(\left\{\frac{-p}{6}\right\}\right)\left(\bmod p^2\right). \tag{29}$$

3. Proofs

3.1. Proof of Theorem 1

Proof. By symmetry, we have

$$\begin{aligned} Z(n) &= \sum_{\substack{i+j+k=n\\ i,j,k\geq 1}} \frac{1}{ijk} \\ &= \frac{1}{n}\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}} \frac{i+j+k}{ijk} \\ &= \frac{3}{n}\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}} \frac{1}{ij} \\ &= \frac{3}{n}\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}} \frac{i+j}{ij(i+j)} \\ &= \frac{6}{n}\sum_{\substack{i+j+k=n\\ i,j,k\geq 1}} \frac{1}{i(i+j)}. \end{aligned} \tag{30}$$

Let $i+j=u$ in (30), and we get

$$\begin{aligned} Z(n) &= \frac{6}{n}\sum_{1\leq i<u\leq n-1}\frac{1}{iu} = \frac{3}{n}\left(\left(\sum_{i=1}^{n-1}\frac{1}{i}\right)^2 - \sum_{i=1}^{n-1}\frac{1}{i^2}\right) \\ &= \frac{3}{n}\left(H_{n-1}^2 - H_{n-1}(2)\right). \end{aligned} \tag{31}$$

Thus, we obtain Theorem 1.

Next, we consider the congruences about $Z([(p-1+N)/N])\,(N=1,2,3,4,6)$ modulo p^2. □

3.2. Proof of Theorem 2

Proof. For $N=1$, by Theorem 1 and Lemma 5, we have

$$\begin{aligned} Z(p) &= \frac{3}{p}\left(H_{p-1}^2 - H_{p-1}(2)\right) \\ &\equiv \frac{12B_{p-3}}{p-3} - \frac{3B_{2p-4}}{p-2}\left(\bmod p^2\right). \end{aligned} \tag{32}$$

This is the result of (6). For $N=2$, by Theorem 1 and Lemma 5, we have

$$\begin{aligned} Z\left(\frac{p+1}{2}\right) &= \frac{6}{p+1}\left(H_{(p-1)/2}^2 - H_{(p-1)/2}(2)\right) \\ &\equiv 6(1-p)\left(\left(-2q_p(2) + pq_p^2(2)\right)^2 - \frac{7}{3}pB_{p-3}\right) \\ &\equiv 24q_p^2(2) - p\left(24q_p^2(2) + 24q_p^3(2) + 14B_{p-3}\right)\left(\bmod p^2\right). \end{aligned} \tag{33}$$

For $N=3$, $[(p+2)/3]=(2p+3+(p/3))/6$, by Theorem 1 and Lemma 8, we have

$$\begin{aligned} Z\left(\left[\frac{p+2}{3}\right]\right) &= Z\left(\left[\frac{p}{3}\right]+1\right) = \frac{18}{2p+3+(p/3)}\left(H_{[p/3]}^{2}-H_{[p/3]}(2)\right) \\ &\equiv \frac{18}{2p+3+(p/3)} \\ &\quad \cdot \left\{\begin{array}{c} \left(-\frac{3}{2}q_p(3)+p\left(\frac{3}{4}q_p^2(3)-\frac{1}{30}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)\right)^2 \\ -\left(-\frac{B_{2p-3}(\{-p/3\})}{2p-3}+2\frac{B_{p-2}(\{-p/3\})}{p-2}+\frac{2p}{9}B_{p-3}\left(\left\{\frac{-p}{3}\right\}\right)\right) \end{array}\right\} \left(\bmod p^2\right). \end{aligned} \tag{34}$$

Since $B_{p-2}(1/6)=5B_{p-2}(1/3)$ (see [16]) and by Lemma 6 in (34), we have

$$\begin{aligned} &Z\left(\left[\frac{p+2}{3}\right]\right) \\ &\equiv \frac{18}{2p+3+(p/3)}\left\{\begin{array}{c} \frac{9}{4}q_p^2(3)+\frac{B_{2p-3}(\{-p/3\})}{2p-3}-2\frac{B_{p-2}(\{-p/3\})}{p-2} \\ -p\left(\frac{9}{4}q_p^3(3)-\frac{q_p(3)}{2}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{3}\right)+\frac{2}{9}B_{p-3}\left(\frac{1}{3}\right)\right) \end{array}\right\}\left(\bmod p^2\right). \end{aligned} \tag{35}$$

Thus, we obtain Theorem 2. □

3.3. Proof of Theorem 3

Proof. For $N=4$, $[(p+3)/4]=(p+2+(-1/p))/4$, by Theorem 1 and Lemma 9, we have

$$\begin{aligned} Z\left(\left[\frac{p+3}{4}\right]\right) &= Z\left(\left[\frac{p}{4}\right]+1\right) = \frac{12}{p+2+(-1/p)}\left(H_{[p/4]}^{2}-H_{[p/4]}(2)\right) \\ &\equiv \frac{12}{p+2+(-1/p)}\left\{\begin{array}{c} \left(-3q_p(2)+p\left(\frac{3}{2}q_p^2(2)-(-1)^{(p-1)/2}E_{p-3}\right)\right)^2 \\ -(-1)^{(p-1)/2}\left(8E_{p-3}-4E_{2p-4}\right)-\frac{14}{3}pB_{p-3} \end{array}\right\} \\ &\equiv \frac{12}{p+2+(-1/p)}\left\{\begin{array}{c} 9q_p^2(2)-\left(\frac{-1}{p}\right)\left(8E_{p-3}-4E_{2p-4}\right) \\ -p\left(9q_p^3(2)-6q_p(2)\left(\frac{-1}{p}\right)E_{p-3}+\frac{14}{3}B_{p-3}\right) \end{array}\right\}\left(\bmod p^2\right). \end{aligned} \tag{36}$$

Thus, we obtain Theorem 3. □

3.4. Proof of Theorem 4

Proof. For $N = 6$, $[(p+5)/6] = (p+3+2(p/3))/6$, by Theorem 1 and Lemma 10, we have

$$
\begin{aligned}
Z\left(\left[\frac{p+5}{6}\right]\right) &= Z\left(\left[\frac{p}{6}\right]+1\right) = \frac{18}{p+3+2(p/3)}\left(H_{[p/6]}^{2} - H_{[p/6]}(2)\right)\\
&\equiv \frac{18}{p+3+2(p/3)}\left\{\begin{array}{c}\left(-2q_p(2) - \frac{3}{2}q_p(3) + p\left(q_p^2(2) + \frac{3}{4}q_p^2(3) - \frac{1}{12}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right)\right)\right)^2\\ -\left(-\frac{B_{2p-3}(\{-p/6\})}{2p-3} + 2\frac{B_{p-2}(\{-p/6\})}{p-2} + \frac{p}{9}B_{p-3}\left(\left\{\frac{-p}{6}\right\}\right)\right)\end{array}\right\}\\
&\equiv \frac{18}{p+3+2(p/3)}\\
&\cdot\left\{\begin{array}{c}4q_p^2(2) + 6q_p(2)q_p(3) + \frac{9}{4}q_p^2(3) + \frac{B_{2p-3}(\{-p/6\})}{2p-3}\\ -2\frac{B_{p-2}(\{-p/6\})}{p-2} - p\left(4q_p^3(2) + 3q_p(2)q_p^2(3) + 3q_p^2(2)q_p(3)\right.\\ \left.+\frac{9}{4}q_p^3(3) - \frac{4q_p(2)+3q_p(3)}{12}\left(\frac{p}{3}\right)B_{p-2}\left(\frac{1}{6}\right) + \frac{1}{9}B_{p-3}\left(\frac{1}{6}\right)\right)\end{array}\right\}\left(\bmod p^2\right).
\end{aligned}
\tag{37}
$$

Thus, we obtain Theorem 4. □

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The author declares that there are no conflicts of interest.

Acknowledgments

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