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Research Article

Structure and Rank of a Cyclic Code over a Class of Nonchain Rings

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The rings $Z_4 + vZ_4$ have been classified into chain rings and nonchain rings based on the values of $v^2 \in Z_4 + vZ_4$. In this paper, the structure of a cyclic code of arbitrary length over the rings $Z_4 + vZ_4$ for those values of v^2 for which these are nonchain rings has been established. A unique form of generators for a cyclic code over these rings has also been obtained. Furthermore, the rank and cardinality of a cyclic code over these rings have been established by finding a minimal spanning set for the code.

1. Introduction

From a mathematical point of view, one of the main aims of algebraic coding theory is to construct codes that can detect and correct the maximum number of errors during data transmission. To construct such codes, it is important to know the structure of a code.

The class of cyclic codes is one of the significant classes of codes, as these codes offer efficient encoding and decoding of the data using shift registers. These codes have good error-detecting and error-correcting capabilities. The theory of cyclic codes over finite fields is well established. The study of cyclic codes over rings started after the remarkable work done by Calderbank et al. [1], wherein a Gray map was introduced to show that some nonlinear binary codes can be viewed as binary images of linear codes over Z_4.

Recent research involves various approaches to determine the generators of cyclic codes over various finite commutative rings. A vast literature is available on cyclic codes over integer residue rings [2–4], Galois rings [5, 6], and finite chain rings [7, 8].

The generators of a cyclic code of arbitrary length over finite chain rings of the type $Z_2 + uZ_2, u^2 = 0$ and $Z_2 + uZ_2 + u^2Z_2, u^3 = 0$ have been obtained by Abualrub and Siap [9]. The same approach is used to find the generators of a cyclic code over the ring $Z_2[u]/\langle u^k \rangle$ by Ashker and Hamoudeh [10] and $Z_p[u]/\langle u^k \rangle$ by Abhay Kumar and Kewat [11].

The study of cyclic codes over nonchain rings can lead to better performance in terms of error-correcting capabilities and efficiency compared to codes over chain rings. The algebraic structure of cyclic codes over nonchain rings can be more complex than the structure of cyclic codes over chain rings, which can lead to improved code properties.

The structure of linear and cyclic codes of odd length over a finite nonchain ring $F_2[u, v]/\langle u^2, v^2, uv - vu \rangle$ has been determined by Yildiz and Karadeniz [12, 13]. A unique set of generators of a cyclic code over the ring $F_{2^m}[u, v]/\langle u^2, v^2, uv - vu \rangle$ has been obtained by Sobhani and Molakarimi [14]. The structure of a cyclic code over the ring $F_2[u_1, u_2, \ldots, u_k]/\langle u_i^2, u_j^2, u_i u_j - u_j u_i \rangle$ has been obtained by Dougherty et al. [15]. The structure of a cyclic code of arbitrary length over the ring $Z_p[u, v]/\langle u^2, v^2, uv - vu \rangle$ has been determined by Parmod Kumar Kewat et al. [16]. Linear and cyclic codes over the nonchain ring $Z_4 + uZ_4$, $u^2 = 0$ were first introduced by Yildiz et al. [17, 18]. They have found some good linear codes over Z_4 as the Gray images of cyclic codes over $Z_4 + uZ_4, u^2 = 0$. The structure of a cyclic code of arbitrary length over $Z_4 + uZ_4$, $u^2 = 0$ has been studied by Bandi and Bhaintwal [19]. Cyclic and some constacyclic codes of odd length over the nonchain ring $Z_4 + uZ_4$, $u^2 = 1$ have been studied by Ozen et al. [20].

In most of the studies, the structural properties of a cyclic code over nonchain rings have been established when the square of the indeterminate coefficient, i.e., u^2, is equal to zero. We make advancements to this study in the direction of the structure of a cyclic code of arbitrary length when u^2 takes nonzero values also.

The rings $Z_4 + \nu Z_4$ with $\nu^2 \in Z_4 + \nu Z_4$ have been classified into chain rings and nonchain rings by Adel Alahmadi et al. [21]. They have proved that $Z_4 + \nu Z_4$ is a chain ring for $\nu^2 \in \{2, 3, 1 + \nu, 1 + 2\nu, 1 + 3\nu, 2 + 2\nu, 3 + \nu, 3 + 3\nu\}$ and is a nonchain ring for $\nu^2 \in \{0, 1, \nu, 2\nu, 3\nu, 2 + \nu, 2 + 3\nu, 3 + 2\nu\}$. This motivated us to investigate the algebraic structure of a cyclic code of arbitrary length over the nonchain rings $Z_4 + \nu Z_4$, where ν^2 is other than 0 which is not under consideration until now.

We noticed that among the eight nonchain rings $Z_4 + \nu Z_4, \nu^2 \in \{0, 1, \nu, 2\nu, 3\nu, 2 + \nu, 2 + 3\nu, 3 + 2\nu\}$, some exhibit isomorphism with each other. Specifically, the ring $Z_4 + \nu Z_4$, $\nu^2 = 0$ and the ring $Z_4 + \nu Z_4$, $\nu^2 = 3 + 2\nu$ are confirmed to be isomorphic. Consequently, we can skip the examination of the structure of the ring $Z_4 + \nu Z_4$ with the condition $\nu^2 = 3 + 2\nu$. Additionally, it is worth noting that the ring $Z_4 + \nu Z_4, \nu^2 = \nu$ is isomorphic to the rings $Z_4 + \nu Z_4$, where $\nu^2 \in \{3\nu, 2 + \nu, 2 + 3\nu\}$. Similarly, the ring $Z_4 + \nu Z_4$, where $\nu^2 = 1$, is isomorphic to the ring $Z_4 + \nu Z_4$, where $\nu^2 = 2\nu$. Consequently, it is enough to focus on the structure of a cyclic code of arbitrary length over the rings $Z_4 + \nu Z_4$, where $\nu^2 \in \{1, \nu\}$.

In this paper, a unique form of generators of a cyclic code of arbitrary length over nonchain rings of the type $Z_4 + \nu Z_4$, $\nu^2 \in \{1, \nu\}$ has been determined. Furthermore, the rank and cardinality of a cyclic code over these rings have been obtained.

2. Preliminaries

Let R be a ring with unity. A subset of R^n is called a code of length n over R. A linear code C of length n is a submodule of R^n over the ring R. An element of a linear code C is termed a codeword. If a codeword $(s_0, s_1, \ldots, s_{n-1})$ of C, $(s_{n-1}, s_0, \ldots, s_{n-2})$ is also a codeword of C, then C is called a cyclic code of length n over R. There is a one-to-one correspondence between the cyclic codes of length n over R and the ideals of the ring $R[z]/\langle z^n - 1\rangle$. The rank of a cyclic code, denoted by $\text{rank}(C)$, is the number of elements in the minimal (linear) spanning set of code C over R. A finite commutative ring R is a chain ring if all its ideals form a chain under the inclusion relation; otherwise, R is a nonchain ring.

Throughout this article, we will denote the nonchain rings $Z_4 + \nu Z_4, \nu^2 = \theta$ for $\theta \in \{1, \nu\}$ by R_θ. Define

$$k_\theta = \begin{cases} \nu & ; \theta = \nu, \\ 1 + \nu & ; \theta = 1. \end{cases} \tag{1}$$

Lemma 1. *The map ϕ_θ: $R_\theta \longrightarrow Z_4$ defined as $\phi_\theta(x) = x(\text{mod} k_\theta)$ is a ring homomorphism for $\theta \in \{1, \nu\}$.*

Proof. Case 1: When $\theta = 1$. For $a + \nu b \in R_\theta$, $\phi_\theta(a + \nu b) = a - b$. Let $x_1 = a + \nu b$ and $x_2 = c + \nu d$ be arbitrary elements of R_θ, where $a, b, c, d \in Z_4$.

We have, $\phi_\theta(x_1 + x_2) = \phi_\theta(a + \nu b + c + \nu d) = \phi_\theta(a + c + \nu(b + d)) = (a + c) - (b + d) = (a - b) + (c - d) = \phi_\theta(x_1) + \phi_\theta(x_2)$.

Again, $\phi_\theta(x_1 x_2) = \phi_\theta((a + \nu b)(c + \nu d)) = \phi_\theta(ac + bd + \nu(ad + bc)) = (ac + bd) - (ad + bc) = (a - b)(c - d) = \phi_\theta(x_1)\phi_\theta(x_2)$.

Case 2: When $\theta = \nu$. For $a + \nu b \in R_\theta$, $\phi_\theta(a + \nu b) = a$. Let x_1 and $x_2 \in R_\theta$, where $x_1 = a + \nu b, x_2 = c + \nu d$ with $a, b, c, d \in Z_4$.

Now, $\phi_\theta(x_1 + x_2) = \phi_\theta(a + \nu b + c + \nu d) = \phi_\theta(a + c + \nu(b + d)) = a + c = \phi_\theta(x_1) + \phi_\theta(x_2)$.

Also, $\phi_\theta(x_1 x_2) = \phi_\theta((a + \nu b)(c + \nu d)) = \phi_\theta(ac + \nu(bd + ad + bc)) = ac = \phi_\theta(x_1)\phi_\theta(x_2)$.

Thus, ϕ_θ is a ring homomorphism for $\theta \in \{1, \nu\}$. □

The following lemma by Abualrub and Siap [22] determines the structure of cyclic codes of arbitrary length over Z_4.

Lemma 2 (see [22]). *Let C be a cyclic code of arbitrary length n over Z_4. Then, $C = \langle g(z) + 2p(z), 2a(z)\rangle$, where $g(z), a(z)$, and $p(z)$ are binary polynomials such that $a(z)|g(z)|z^n - 1$ and either $p(z) = 0$ or $a(z)|p(z)(z^n - 1)/g(z)$ with deg $a(z) >$ deg $p(z)$.*

3. Structure of a Cyclic Code of Arbitrary Length over $R_\theta, \theta \in \{1, \nu\}$

In this section, we establish the structure of a cyclic code of arbitrary length n over the nonchain rings R_θ, $\theta \in \{1, \nu\}$.

Theorem 3. *Let C_θ be a cyclic code of arbitrary length n over the rings $R_\theta, \theta \in \{1, \nu\}$. Then, $C_\theta = \langle f_{\theta_1}(z), f_{\theta_2}(z), f_{\theta_3}(z), f_{\theta_4}(z)\rangle$, where $f_{\theta_1}(z) = f_{11}(z) + 2f_{12}(z) + k_\theta f_{13}(z) + 2k_\theta f_{14}(z)$, $f_{\theta_2}(z) = 2f_{22}(z) + k_\theta f_{23}(z) + 2k_\theta f_{24}(z)$, $f_{\theta_3}(z) = k_\theta f_{33}(z) + 2k_\theta f_{34}(z)$, $f_{\theta_4}(z) = 2k_\theta f_{44}(z)$ such that the polynomials $f_{ij}(z)$ are in $Z_2[z]/\langle z^n - 1\rangle$ for $1 \le i \le 4$, $i \le j \le 4$. Furthermore,*

$$f_{22}(z)|f_{11}(z)|z^n - 1, \tag{2}$$

$$\text{either } f_{12}(z) = 0 \text{ or } f_{22}(z)|f_{12}(z)\frac{z^n - 1}{f_{11}(z)} \text{ with deg } f_{22}(z) > \text{deg } f_{12}(z), \tag{3}$$

$$f_{44}(z)\big|f_{33}(z)\big|z^n-1, \tag{4}$$

$$\text{either } f_{34}(z)=0 \text{ or } f_{44}(z)|f_{34}(z)\frac{z^n-1}{f_{33}(z)} \text{ with deg } f_{44}(z)> \text{ deg } f_{34}(z). \tag{5}$$

Proof. Let C_θ be a cyclic code of length n over R_θ, $\theta\in\{1,\nu\}$. Let ϕ_θ with a ring homomorphism as defined in Lemma 1 for $\theta\in\{1,\nu\}$. Clearly, $\phi_\theta(C_\theta)$ is a cyclic code of length n over Z_4. Using Lemma 2, we get $\phi_\theta(C_\theta)=\langle f_{11}(z)+2f_{12}(z), 2f_{22}(z)\rangle$, where $f_{22}(z)|f_{11}(z)|z^n-1$ and either $f_{12}(z)=0$ or $f_{22}(z)\ |f_{12}(z)(z^n-1)/f_{11}(z)$ with deg $f_{22}(z)>$ deg $f_{12}(z)$.

Let $\ker_\theta=\{x\in C_\theta \text{ such that } \phi_\theta(x)=0\}$. It is easy to see that $\ker_\theta$ is k_θ times a cyclic code of length n over Z_4. Therefore, using Lemma 2, we obtain $\ker_\theta=k_\theta\langle f_{33}(z)+2f_{34}(z), 2f_{44}(z)\rangle$, where $f_{44}(z)|f_{33}(z)|z^n-1$ and either $f_{34}(z)=0$ or $f_{44}(z)|f_{34}(z)(z^n-1)/f_{33}(z)$ with deg $f_{44}(z)>$ deg $f_{34}(z)$.

It follows that $C_\theta=\langle f_{\theta_1}(z), f_{\theta_2}(z), f_{\theta_3}(z), f_{\theta_4}(z)\rangle$, where $f_{\theta_1}(z)=f_{11}(z)+2f_{12}(z)+k_\theta f_{13}(z)+2k_\theta f_{14}(z)$, $f_{\theta_2}(z)=2f_{22}(z)+k_\theta f_{23}(z)+2k_\theta f_{24}(z)$, $f_{\theta_3}(z)=k_\theta f_{33}(z)+2k_\theta f_{34}(z)$, $f_{\theta_4}(z)=2k_\theta f_{44}(z)$ such that the polynomials $f_{ij}(z)$ are in $Z_2[z]/\langle z^n-1\rangle$ for $1\le i\le 4, i\le j\le 4$ and satisfy conditions (2)–(5).

Let C_θ be a cyclic code of length n over R_θ, $\theta\in\{1,\nu\}$ generated by the polynomials $f_{\theta_1}(z), f_{\theta_2}(z), f_{\theta_3}(z), f_{\theta_4}(z)$ as obtained in Theorem 3. Define Residue and Torsion of C_θ as

$$\begin{aligned}\operatorname{Res}(C_\theta)&=\left\{a(z)\in\frac{Z_4[z]}{\langle z^n-1\rangle}: a(z)+k_\theta b(z)\in C_\theta \text{ for some } b(z)\in\frac{Z_4[z]}{\langle z^n-1\rangle}\right\},\\ \operatorname{Tor}(C_\theta)&=\left\{a(z)\in\frac{Z_4[z]}{\langle z^n-1\rangle}: k_\theta a(z)\in C_\theta\right\}.\end{aligned} \tag{6}$$

Clearly, Res (C_θ) and Tor (C_θ) are the ideals of the ring $(Z_4[z]/\langle z^n-1\rangle)$. Furthermore, define

$C_{\theta_1}=\text{Res}(\text{Res } (C_\theta))=C_\theta \bmod (2,k_\theta)$

$C_{\theta_2}=\text{Tor}(\text{Res } (C_\theta))=\{a(z)\in Z_2[z]: 2a(z)\in C_\theta \bmod k_\theta\}$

$C_{\theta_3}=\text{Res}(\text{Tor } (C_\theta))=\{a(z)\in Z_2[z]: k_\theta a(z)\in C_\theta \bmod 2k_\theta\}$

$C_{\theta_4}=\text{Tor}(\text{Tor } (C_\theta))=\{a(z)\in Z_2[z]: 2k_\theta a(z)\in C_\theta\}$

It is easy to see that $C_{\theta_1}, C_{\theta_2}, C_{\theta_3}, C_{\theta_4}$ are ideals of the ring $Z_2[z]/\langle z^n-1\rangle$ generated by the unique minimal degree polynomials $f_{11}(z), f_{22}(z), f_{33}(z), f_{44}(z)$, respectively, as defined in Theorem 3. □

Theorem 4. *Let $C_\theta=\langle f_{\theta_1}(z), f_{\theta_2}(z), f_{\theta_3}(z), f_{\theta_4}(z)\rangle$ be a cyclic code of arbitrary length n over the ring R_θ, $\theta\in\{1,\nu\}$, where $f_{\theta_i}(z)$, $1\le i\le 4$ are polynomials as defined in Theorem 3. Then, there exists a set of generators $\{g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z), g_{\theta_4}(z)\}$ of C_θ, where $g_{\theta_1}(z)=g_{11}(z)+2g_{12}(z)+k_\theta g_{13}(z)+2k_\theta g_{14}(z)$, $g_{\theta_2}(z)=2g_{22}(z)+k_\theta g_{23}(z)+2k_\theta g_{24}(z)$, $g_{\theta_3}(z)=k_\theta g_{33}(z)+2k_\theta g_{34}(z)$, and $g_{\theta_4}(z)=2k_\theta g_{44}(z)$ such that the polynomials $g_{ij}(z)$ are in $Z_2[z]/\langle z^n-1\rangle$ satisfy conditions (2)–(5) as defined in Theorem 3 and $g_{ii}(z)$ are unique minimal degree polynomial generators of C_{θ_i}, $1\le i\le 4$. Also, either $g_{ij}(z)=0$ or deg $g_{ij}(z)<$ deg $g_{jj}(z)$ for $1\le i\le 3, i<j\le 4$.*

Proof. Clearly, $f_{\theta_1}(z)=f_{11}(z)+2f_{12}(z)+k_\theta f_{13}(z)+2k_\theta f_{14}(z)$, $f_{\theta_2}(z)=2f_{22}(z)+k_\theta f_{23}(z)+2k_\theta f_{24}(z)$, $f_{\theta_3}(z)=k_\theta f_{33}(z)+2k_\theta f_{34}(z)$, and $f_{\theta_4}(z)=2k_\theta f_{44}(z)$ are the generators of C_θ such that either $f_{12}(z)=0$ or deg $f_{12}(z)<$ deg $f_{22}(z)$ and either $f_{34}(z)=0$ or deg $f_{34}(z)<$ deg $f_{44}(z)$. Furthermore, if either $f_{ij}(z)=0$ or deg $f_{ij}(z)<$ deg $f_{jj}(z)$ for all $1\le i\le 2, i<j\le 4$, then we get the required result. Otherwise, let us suppose that deg $f_{ij}(z)\ge$ deg $f_{jj}(z)$ for some $i=1$, 2 and $j=3$, 4. Assume that deg $f_{ij}(z)\ge$ deg $f_{jj}(z)$ for $i=1$ and $j=3$, 4, i.e., deg $f_{13}(z)\ge$ deg $f_{33}(z)$. By division algorithm, there exist some $q_{13}(z), g_{13}(z)\in Z_2[z]$ such that $f_{13}(z)=q_{13}(z)\ f_{33}(z)+g_{13}(z)$, where either $g_{13}(z)=0$ or deg $g_{13}(z)<$ deg $f_{33}(z)$. Consider $f_{\theta_1}(z)-q_{13}(z)f_{\theta_3}(z)=f_{11}(z)+2f_{12}(z)+k_\theta g_{13}(z)+2k_\theta(f_{14}(z)-q_{13}(z)f_{34}(z))$. Furthermore, deg $(f_{14}(z)-q_{13}(z)f_{34}(z))\ge$ deg $f_{44}(z)$. Again by division algorithm, there exist some $q_{14}(z), g_{14}(z)\in Z_2(z)$ such that $f_{14}(z)-q_{13}(z)f_{34}(z)=f_{44}(z)q_{14}(z)+g_{14}(z)$, where either $g_{14}(z)=0$ or deg $g_{14}(z)<$ deg $f_{44}(z)$. Now, consider the polynomial $g_{\theta_1}(z)=f_{\theta_1}(z)-q_{13}(z)f_{\theta_3}(z)-q_{14}(z)f_{\theta_4}(z)=f_{11}(z)+2f_{12}(z)+k_\theta g_{13}(z)+2k_\theta g_{14}(z)$. Clearly, $g_{\theta_1}(z)\in C_\theta$. Also, we have that either $g_{13}(z)=0$ or deg $g_{13}(z)<$ deg $f_{33}(z)$ and either $g_{14}(z)=0$ or deg $g_{14}(z)<$ deg $f_{44}(z)$. Since $g_{\theta_1}(z)$ is a linear combination of $f_{\theta_1}(z), f_{\theta_3}(z)$, and $f_{\theta_4}(z)$, we have $C_\theta=\langle f_{\theta_1}(z), f_{\theta_2}(z), f_{\theta_3}(z), f_{\theta_4}(z)\rangle=\langle g_{\theta_1}(z), f_{\theta_2}(z), f_{\theta_3}(z), f_{\theta_4}(z)\rangle$. Furthermore, using similar arguments, we can find polynomials $g_{\theta_2}(z), g_{\theta_3}(z)$ and $g_{\theta_4}(z)\in C_\theta$ satisfying the required properties such that $C_\theta=\langle g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z), g_{\theta_4}(z)\rangle$ and complete the proof of the theorem.

In the following theorem, a unique form of the generators of a cyclic code C_θ of arbitrary length n over $R_\theta, \theta \in \{1, v\}$ has been determined. □

Theorem 5. *Let $C_\theta = \langle g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z), g_{\theta_4}(z)\rangle$ be a cyclic code of arbitrary length n over the ring $R_\theta, \theta \in \{1, v\}$, where $g_{\theta_1}(z) = g_{11}(z) + 2g_{12}(z) + k_\theta g_{13}(z) + 2k_\theta g_{14}(z)$, $g_{\theta_2}(z) = 2g_{22}(z) + k_\theta g_{23}(z) + 2k_\theta g_{24}(z)$, $g_{\theta_3}(z) = k_\theta g_{33}(z) + 2k_\theta g_{34}(z)$, and $g_{\theta_4}(z) = 2k_\theta g_{44}(z)$ such that the polynomials $g_{ij}(z) \in Z_2[z]/\langle z^n - 1\rangle$ and satisfy conditions (2)–(5) as defined in Theorem 3 with either $g_{ij}(z) = 0$ or deg $g_{ij}(z) <$ deg $g_{jj}(z)$ for $1 \le i \le 3, i < j \le 4$ and $g_{ii}(z)$ are the unique minimal degree polynomial generators of $C_{\theta_i}, 1 \le i \le 4$. Then, the polynomials $g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z), g_{\theta_4}(z)$ are uniquely determined.*

Proof. Consider another set of generators $\{h_{\theta_1}(z), h_{\theta_2}(z), h_{\theta_3}(z), h_{\theta_4}(z)\}$ of C_θ, where $h_{\theta_1}(z) = h_{11}(z) + 2h_{12}(z) + k_\theta h_{13}(z) + 2k_\theta h_{14}(z)$, $h_{\theta_2}(z) = 2h_{22}(z) + k_\theta h_{23}(z) + 2k_\theta h_{24}(z)$, $h_{\theta_3}(z) = k_\theta h_{33}(z) + 2k_\theta h_{34}(z)$, and $h_{\theta_4}(z) = 2k_\theta h_{44}(z)$ such that the polynomials $h_{ij}(z)$ are in $Z_2[z]/\langle z^n - 1\rangle$ and satisfy conditions (2)–(5) as defined in Theorem 3 with either $h_{ij}(z) = 0$ or deg $h_{ij}(z) <$ deg $h_{jj}(z)$ for $1 \le i \le 3$, $i < j \le 4$ and $h_{ii}(z)$ are the unique minimal degree polynomial generators of $C_{\theta_i}, 1 \le i \le 4$.

Clearly, $g_{ii}(z) = h_{ii}(z)$, for $1 \le i \le 4$. Consider $g_{\theta_1}(z) - h_{\theta_1}(z) = 2(g_{12}(z) - h_{12}(z)) + k_\theta(g_{13}(z) - h_{13}(z)) + 2k_\theta (g_{14}(z) - h_{14}(z)) \in C_\theta$. This implies that $g_{12}(z) - h_{12}(z) \in C_{\theta_2} = \langle g_{22}(z)\rangle$. Also deg $(g_{12}(z) - h_{12}(z)) <$ deg $g_{22}(z)$ which is a contradiction because $g_{22}(z)$ is a minimal degree polynomial in C_{θ_2}. Hence, $g_{12}(z) = h_{12}(z)$. It follows that $g_{\theta_1}(z) - h_{\theta_1}(z) = k_\theta(g_{13}(z) - h_{13}(z)) + 2k_\theta(g_{14}(z) - h_{14}(z)) \in C_\theta$ which implies that $g_{13}(z) - h_{13}(z) \in C_{\theta_3} = \langle g_{33}(z)\rangle$. As deg $(g_{13}(z) - h_{13}(z)) <$ deg $g_{33}(z)$, we must have $g_{13}(z) = h_{13}(z)$.

Subsequently, $g_{\theta_1}(z) - h_{\theta_1}(z) = 2k_\theta(g_{14}(z) - h_{14}(z)) \in C_\theta$ implying that $g_{14}(z) - h_{14}(z) \in C_{\theta_4} = \langle g_{44}(z)\rangle$. This together with the fact that deg $(g_{14}(z) - h_{14}(z)) <$ deg $g_{44}(z)$ implies that $g_{14}(z) = h_{14}(z)$.

In a similar manner, we can prove that $g_{23}(z) = h_{23}(z)$, $g_{24}(z) = h_{24}(z)$, and $g_{34}(z) = h_{34}(z)$. Hence, the uniqueness of the polynomials $g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z)$, and $g_{\theta_4}(z)$ is established. □

The following theorem which gives some divisibility properties of polynomials $g_{ij}(z)$, $1 \le i \le 4, i \le j \le 4$ in $Z_2[z]/\langle z^n - 1\rangle$ can be proved through simple calculations. These properties will be required to prove the results of Section 4.

Theorem 6. *Let $C_\theta = \langle g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z), g_{\theta_4}(z)\rangle$ be a cyclic code of arbitrary length* n *over the ring $R_\theta, \theta \in \{1, v\}$, where the generators $g_{\theta_1}(z) = g_{11}(z) + 2g_{12}(z) + k_\theta g_{13}(z) + 2k_\theta g_{14}(z)$, $g_{\theta_2}(z) = 2g_{22}(z) + k_\theta g_{23}(z) + 2k_\theta g_{24}(z)$, $g_{\theta_3}(z) = k_\theta g_{33}(z) + 2k_\theta g_{34}(z)$, and $g_{\theta_4}(z) = 2k_\theta g_{44}(z)$ are in the unique form as in Theorem 5. Then, the following divisibility relations hold over the ring Z_2.*

(i) $g_{33}(z) | (z^n - 1)/g_{11}(z)(g_{13}(z) - (g_{12}(z)/g_{22}(z)) g_{23}(z))$

(ii) $g_{44}(z) | g_{23}(z)$

(iii) $g_{33}(z) | (g_{11}(z)/g_{22}(z)) g_{23}(z)$

(iv) $g_{44}(z) | (z^n - 1)/g_{22}(z)(g_{24}(z) - (g_{23}(z)/g_{33}(z)) g_{34}(z))$

(v) $g_{44}(z) | g_{13}(z) - (g_{11}(z)/g_{22}(z)) g_{24}(z) + (g_{11}(z)/g_{22}(z) g_{33}(z)) g_{23}(z) g_{34}(z)$

(vi) $g_{44}(z) | (z^n - 1)/g_{11}(z)(g_{14}(z) - (g_{12}(z)/g_{22}(z)) g_{24}(z) + (-g_{13}(z) + (g_{12}(z) g_{23}(z)/g_{22}(z)))/g_{33}(z)) g_{34}(z))$

(vii) $g_{33}(z) | g_{11}(z)$, $g_{44}(z) | g_{11}(z)$, $g_{44}(z) | g_{22}(z) + g_{23}(z)$ *for* $\theta = 1$

(viii) $g_{44}(z) | g_{12}(z) + g_{13}(z) - (g_{11}(z)/g_{33}(z)) g_{34}(z)$ *for* $\theta = 1$*, and* $g_{44}(z) | g_{13}(z)$ *for* $\theta = v$.

4. Rank and Cardinality of a Cyclic Code of Arbitrary Length over $R_\theta, \theta \in \{1, v\}$

In this section, the rank and cardinality of a cyclic code of arbitrary length over $R_\theta, \theta \in \{1, v\}$ have been obtained by determining a minimal spanning set of a cyclic code over R_θ.

Definition 7. A set S of elements of a cyclic code C over a finite commutative ring R is called a spanning set of C if each element of C can be written as a linear combination of elements of S with coefficients in R.

Definition 8. A spanning set S of a cyclic code C is called a minimal spanning set of C if no proper subset of S spans C.

Definition 9. The rank of a cyclic code C is the number of elements in the minimal spanning set of C.

Obviously, the minimal spanning set of a cyclic code C is not unique. However, the number of elements in any minimal spanning set of C remains the same. We prove this in the following theorem.

Theorem 10. *Let $S_1 = \{u_1, u_2, \ldots, u_m\}$ and $S_2 = \{v_1, v_2, \ldots, v_n\}$ be two minimal spanning sets of a cyclic code C over a finite commutative ring R. Then, m = n.*

Proof. Without loss of generality, we may assume that m is the least positive number such that no set with less than m elements spans C. Clearly, $m \le n$. Suppose, if possible, $m < n$. It is easy to see that v_i is not a zero divisor for some i, $1 \le i \le n$. Without loss of generality, we may assume that v_1 is not a zero divisor. Since S_1 spans C and $v_1 \in C$, we can find $\alpha_1, \alpha_2, \ldots, \alpha_m \in R$ such that $v_1 = \alpha_1 u_1 + \alpha_2 u_2 + \cdots + \alpha_m u_m$. Furthermore, it is easy to see that at least one α_j, $1 \le j \le m$ is not a zero divisor. Without loss of generality, we may assume that α_1 is not a zero divisor. It follows that

$$u_1 = \frac{1}{\alpha_1} v_1 - \frac{\alpha_2}{\alpha_1} u_2 - \cdots - \frac{\alpha_m}{\alpha_1} u_m, \tag{7}$$

and hence, the set $\{v_1, u_2, \ldots, u_m\}$ also spans C. Consequently, we have that $C = \langle v_1\rangle \oplus C_1$, where C_1 is the cyclic code spanned by the set $\{u_2, u_3, \ldots, u_m\}$. It is easy to see that $\{u_2, u_3, \ldots, u_m\}$ is a minimal spanning set for C_1. Clearly, $\{v_2, v_3, \ldots, v_n\}$ is also a minimal spanning set of C_1. Repeating the same arguments on the sets $\{u_2, u_3, \ldots, u_m\}$ and $\{v_2, v_3, \ldots, v_n\}$, we get that the set $\{v_2, u_3, \ldots, u_m\}$ also spans C_1 and therefore $\{v_1, v_2, u_3, \ldots, u_m\}$ spans C. Thus, $C = \langle v_1\rangle \oplus \langle v_2\rangle \oplus C_2$, where $\{u_3, u_4 \cdots, u_m\}$ and $\{v_3, v_4, \ldots, v_n\}$ are minimal spanning sets of C_2. Furthermore, repeating the above process a number of times, we obtain that the set $\{v_1, v_2, \ldots, v_m\}$ also spans C. This is a contradiction to the fact that S_2 is a minimal spanning set of C. Therefore, m must be equal to n.

In the following theorem, the rank of a cyclic code of arbitrary length over $R_\theta, \theta \in \{1, v\}$ has been obtained. □

Theorem 11. *Let $C_\theta = \langle g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z), g_{\theta_4}(z)\rangle$ be a cyclic code of arbitrary length* n *over the ring $R_\theta, \theta \in \{1, v\}$, where the generators $g_{\theta_1}(z) = g_{11}(z) + 2g_{12}(z) + k_\theta g_{13}(z) + 2k_\theta g_{14}(z)$, $g_{\theta_2}(z) = 2g_{22}(z) + k_\theta g_{23}(z) + 2k_\theta g_{24}(z)$, $g_{\theta_3}(z) = k_\theta g_{33}(z) + 2k_\theta g_{34}(z)$, and $g_{\theta_4}(z) = 2k_\theta g_{44}(z)$ are in the unique form as given in Theorem 5. Then, rank(C_θ) is $n + s_1 + s - s_2 - s_3 - s_4$, where $s_i = \deg g_{ii}(z)$ for $1 \le i \le 4$ and $s = \min\{s_2, s_3\}$.*

Proof. It can be easily seen that the set $A_\theta = \{g_{\theta_1}(z), zg_{\theta_1}(z), \ldots, z^{n-s_1-1}g_{\theta_1}(z), g_{\theta_2}(z), zg_{\theta_2}(z), \ldots, z^{n-s_2-1}g_{\theta_2}(z), g_{\theta_3}(z), zg_{\theta_3}(z), \ldots, z^{n-s_3-1}g_{\theta_3}(z), g_{\theta_4}(z), zg_{\theta_4}(z), \ldots, z^{n-s_4-1}g_{\theta_4}(z)\}$ is a spanning set of C_θ.

To prove that rank(C_θ) is $n + s_1 + \tilde{s} - s_2 - s_3 - s_4$, it is sufficient to show that the set $B_\theta = \{g_{\theta_1}(z), zg_{\theta_1}(z), \ldots, z^{n-s_1-1} g_{\theta_1}(z), g_{\theta_2}(z), z g_{\theta_2}(z), \ldots, z^{s_1-s_2-1} g_{\theta_2}(z), g_{\theta_3}(z), zg_{\theta_3}(z), \ldots, z^{s_1-s_3-1}g_{\theta_3}(z), g_{\theta_4}(z), zg_{\theta_4}(z), \ldots, z^{\tilde{s}-s_4-1}g_{\theta_4}(z)\}$ is a minimal spanning set of C_θ, where $\tilde{s} = \min\{s_2, s_3\}$.

To prove that the set B_θ spans C_θ, it is enough to show that $z^{\tilde{s}-s_4}g_{\theta_4}(z), z^{s_1-s_3}g_{\theta_3}(z), z^{s_1-s_2}g_{\theta_2}(z) \in \text{span}(B_\theta)$. First, let us suppose that $\tilde{s} = s_3$. As $g_{44}(z)|g_{33}(z)$ in $Z_2[z]/\langle z^n - 1\rangle$, there exists some $m(z) \in Z_2[z]$ with deg $m(z) = s_3 - s_4$ such that $g_{33}(z) = g_{44}(z)m(z) = g_{44}(z)(m_0 + zm_1 + \cdots + z^{s_3-s_4-1} m_{s_3-s_4-1} + z^{s_3-s_4}), m_i \in Z_2$. Multiplying both sides by $2k_\theta$, we get

$$2g_{\theta_3}(z) = \left(m_0 + zm_1 + \cdots + z^{s_3-s_4-1}m_{s_3-s_4-1}\right)g_{\theta_4}(z) + z^{s_3-s_4}g_{\theta_4}(z), \tag{8}$$

which implies that $z^{s_3-s_4}g_{\theta_4}(z) \in \text{span}(B_\theta)$. Next, suppose that $\tilde{s} = s_2$. Using the divisibilities $g_{44}(z)|g_{22}(z) + g_{23}(z)$ for $\theta = 1$ and $g_{44}(z)|g_{23}(z)$ for $\theta = v$ from Theorem 6, it can be proved that $z^{s_2-s_4}g_{\theta_4}(z) \in \text{span}(B_\theta)$ by working on the same lines as above. Thus, we have $z^{\tilde{s}-s_4}g_{\theta_4}(z) \in \text{span}(B_\theta)$, where $\tilde{s} = \min\{s_2, s_3\}$. By taking $g_{\theta_1}(z)$ as a divisor and applying the division algorithm for $g_{\theta_2}(z)$ and $g_{\theta_3}(z)$, respectively, we can show that $z^{s_1-s_2}g_{\theta_2}(z) \in \text{span}(B_\theta)$ and $z^{s_1-s_3}g_{\theta_3}(z) \in \text{span}(B_\theta)$. Thus, B_θ is a spanning set of C_θ.

Now to prove that the set B_θ is a minimal spanning set, it is enough to show that none of $z^{n-s_1-1}g_{\theta_1}(z), z^{s_1-s_2-1}g_{\theta_2}(z), z^{s_1-s_3-1}g_{\theta_3}(z)$ and $z^{\tilde{s}-s_4-1}g_{\theta_4}(z)$ can be written as a linear combination of other elements of B_θ. Suppose, if possible, that $z^{n-s_1-1}g_{\theta_1}(z)$ can be written as a linear combination of other elements of B_θ, i.e.,

$$\begin{aligned} z^{n-s_1-1}g_{\theta_1}(z) &= a(z)g_{\theta_1}(z) + b(z)g_{\theta_2}(z) \\ &\quad + c(z)g_{\theta_3}(z) + d(z)g_{\theta_4}(z), \end{aligned} \tag{9}$$

where $\deg a(z) < n - s_1 - 1$, $\deg b(z) < s_1 - s_2$, $\deg c(z) < s_1 - s_3$, and $\deg d(z) < \tilde{s} - s_4$. Multiplying equation (9) on both sides by $2k_\theta$ for $\theta = 1$, we get

$$2k_\theta z^{n-s_1-1}g_{11}(z) = 2k_\theta a(z)g_{11}(z), \theta = 1. \tag{10}$$

Multiplying equation (9) on both sides by $2(k_\theta - 1)$ for $\theta = v$, we get

$$2(k_\theta - 1)z^{n-s_1-1}g_{11}(z) = 2(k_\theta - 1)a(z)g_{11}(z), \theta = v. \tag{11}$$

Equations (10) and (11) are not possible as degrees of left-hand side and right-hand side in each of these equations do not match. Thus, $z^{n-s_1-1}g_{\theta_1}(z)$ cannot be written as a linear combination of other elements of B_θ. Using a similar argument, it can be shown that none of $z^{s_1-s_2-1}g_{\theta_2}(z), z^{s_1-s_3-1}g_{\theta_3}(z)$ and $z^{s-s_4-1}g_{\theta_4}(z)$ can be written as a linear combination of other elements of B_θ. Hence, B_θ is a minimal spanning set of C_θ. Furthermore, rank(C_θ) = number of elements in $B_\theta = (n - s_1) + (s_1 - s_2) + (s_1 - s_3) + (\tilde{s} - s_4) = n + s_1 + \tilde{s} - s_2 - s_3 - s_4$, where $\tilde{s} = \min\{s_2, s_3\}$.

Corollary 12 follows immediately from the above theorem. □

Corollary 12. *Let $C_\theta = \langle g_{\theta_1}(z), g_{\theta_2}(z), g_{\theta_3}(z), g_{\theta_4}(z)\rangle$ be a cyclic code of arbitrary length n over the rings $R_\theta, \theta \in \{1, v\}$, where the generators $g_{\theta_1}(z) = g_{11}(z) + 2g_{12}(z) + k_\theta g_{13}(z) + 2k_\theta g_{14}(z)$, $g_{\theta_2}(z) = 2g_{22}(z) + k_\theta g_{23}(z) + 2k_\theta g_{24}(z)$, $g_{\theta_3}(z) = k_\theta g_{33}(z) + 2k_\theta g_{34}(z)$, and $g_{\theta_4}(z) = 2k_\theta g_{44}(z)$. Then, cardinality of C_θ is defined as follows:*

$$|C_\theta| = \begin{cases} 2^{4n+s_1+\tilde{s}-3s_2-2s_3-s_4}, & ; g_{23}(z) \neq 0, \\ 2^{4n+\tilde{s}-2s_2-2s_3-s_4}, & ; g_{23}(z) = 0, \end{cases} \tag{12}$$

where $s_i = \deg g_{ii}(z)$ for $1 \le i \le 4$ and $\tilde{s} = \min\{s_2, s_3\}$.

The following examples illustrate the above results.

Example 1. Let $C_\theta = \langle z^3 + z^2 + z + 1 + \nu(z+3), 2(z^2+1) + 2\nu, \nu(z^2+1), 2\nu(z+1)\rangle$ be a cyclic code of length 4 over the ring R_θ for $\theta = 1$. Here, $s_1 = 3, s_2 = 2, s_3 = 2, s_4 = 1$. Using Theorem 11, minimal spanning set of C_θ *is* $\{z^3 + z^2 + z + 1 + \nu(z+3), 2(z^2+1) + 2\nu, \nu(z^2+1), 2\nu(z+1)\}$. Hence, $\operatorname{rank}(C_\theta) = 4$ and $|C_\theta| = 2^9$.

Example 2. Let $C_\theta = \langle z^3 + z^2 + z + 3 + \nu(z^2+1) + 2\nu, 2(z+1), \nu(z^3 + z^2 + z + 1), 2\nu(z+1)\rangle$ be a cyclic code of length 4 over the ring R_θ for $\theta = \nu$. Here, $s_1 = 3, s_2 = 1, s_3 = 3, s_4 = 1$. Using Theorem 11, minimal spanning set of C_θ *is* $\{z^3 + z^2 + z + 3 + \nu(z^2+1) + 2\nu, 2\ (z+1), 2(z^2+z)\}$. Hence, $\operatorname{rank}(C_\theta) = 3$ and $|C_\theta| = 2^8$.

Example 3. Let $C_\theta = \langle z^4 + z^3 + z + 1 + 2(1+\nu)(z+1), 2(z^2+z+1), (1+\nu)(z^4 + z^3 + z + 1), 2(1+\nu)(z^2+z+1)\rangle$ be a cyclic code of length 6 over the ring R_θ for $\theta = 1$. *Here,* $s_1 = 4, s_2 = 2, s_3 = 4, s_4 = 2$. Using Theorem 11, minimal spanning set of C_θ *is* $\{z^4 + z^3 + z + 1 + 2(1+\nu)(z+1), z^5 + z^4 + z^2 + z + 2(1+\nu)(z^2+z)), 2(z^2+z+1), 2(z^3 + z^2 + z)\}$. Hence, $\mathit{rank}(C_\theta) = 4$ and $|C_\theta| = 2^{12}$.

Example 4. Let $C_\theta = \langle z^5 + z^4 + z^3 + z^2 + z + 1 + \nu(z^4 + z^2 + 1), 2(z+1) + \nu(z+1), \nu(z^5 + z^4 + z^3 + z^2 + z + 1), 2\nu\rangle$ be a cyclic code of length 6 over the ring R_θ *for* $\theta = \nu$. *Here,* $s_1 = 5, s_2 = 1, s_3 = 5, s_4 = 0$. Using Theorem 11, minimal spanning set of C_θ *is* $\{z^5 + z^4 + z^3 + z^2 + z + 1 + \nu(z^4 + z^2 + 1), 2(z+1) + \nu(z+1), 2z(z+1) + \nu z(z+1), 2z^2(z+1) + \nu z^2(z+1), 2z^3(z+1) + \nu z^3(z+1), 2\nu\}$. Hence, $\mathit{rank}(C_\theta) = 6$ and $|C_\theta| = 2^{17}$.

5. Conclusion and Future Scope

In this paper, the structure of a cyclic code of arbitrary length over the rings $Z_4 + \nu Z_4$ for those values of ν^2 for which these are nonchain rings has been established. A unique form of the generators of these codes has been obtained. Furthermore, formulae for rank and cardinality of a cyclic code over these rings have been established by finding their minimal spanning sets. This study can be used to find some new and good codes over Z_4. Also, the structural properties of cyclic codes of arbitrary length over the rings $R = Z_4 + \nu Z_4 + \cdots + \nu^{k-1} Z_4$ where $k \geq 3$ for $\nu^k \in R$ can be established.

Data Availability

Data sharing is not applicable to this article.

Disclosure

This paper has previously been archived in [23].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

Nikita Jain and Ranjeet Sehmi are contributing authors.

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