Hindawi
Journal of Mathematics
Volume 2024, Article ID 8844153, 9 pages
https://doi.org/10.1155/2024/8844153

Research Article

Some New Identities Related to Dedekind Sums Modulo a Prime

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Received 30 April 2023; Revised 10 June 2023; Accepted 29 March 2024; Published 15 April 2024

Academic Editor: Xiaogang Liu

The main purpose of this article is to use some identities of the classical Gauss sums, the properties of character sums, and Dedekind sums (modulo an odd prime) to study the computational problem of one-kind mean values related to Dedekind sums and give some interesting identities for them.

1. Introduction

To describe the results of this paper, we first need to introduce the Drichlet character and some famous sums in the analytic number theory. Let $q > 1$ be an integer and G be the group of reduced residue classes modulo q. Corresponding to each character f of G, we define an arithmetical function χ modulo q as follows:

$$\begin{aligned} &\chi(n) = f(\widehat{n}), \quad \text{if } (n,k) = 1, \\ &\chi(n) = 0, \quad \text{if } (n,k) > 1. \end{aligned} \tag{1}$$

In addition to periodicity, the Drichlet characters modulo q also have orthogonality which can be found in reference [1]. For any Dirichlet character χ modulo q, the classical Gauss sums $G(m,\chi;q)$ is defined as

$$G(m,\chi;q) = \sum_{a=1}^{q} \chi(a) e\left(\frac{\text{ma}}{q}\right), \tag{2}$$

where m is any integer, $e(y) = e^{2\pi i y}$, and $i^2 = -1$.

For convenience, we write $\tau(\chi) = G(1,\chi;q)$. About some properties of $G(m,\chi;q)$, many scholars have studied them and obtained a series of important results. Perhaps, the most important properties of $G(m,\chi;q)$ are the following two conclusions:

(i) If $(m,q) = 1$, then for any character modulo q, we have the identity (see [1, 2])

$$G(m,\chi;q) = \overline{\chi}(m) G(1,\chi;q) = \overline{\chi}(m)\tau(\chi). \tag{3}$$

(ii) If χ is any primitive character modulo q, then for any integer m, one has also $G(m,\chi;q) = \overline{\chi}(m)\tau(\chi)$ and the identity $|\tau(\chi)| = \sqrt{q}$.

In addition, for any prime p with $p \equiv 1 \bmod 4$ and any fourth-order primitive character χ_4 modulo p, Chen and Zhang [3] studied the properties of $\tau(\chi_4)$ and proved the identity

$$\tau^2(\chi_4) + \tau^2(\overline{\chi}_4) = 2\sqrt{p} \cdot \alpha(p), \tag{4}$$

where $\alpha(p) = 1/2\sum_{a=1}^{p-1}(a + \overline{a}/p)$, and $(*/p) = \chi_2$ denotes the Legendre's symbol modulo p.

Here, the constant $\alpha(p)$ in (4) has a special meaning. In fact, if prime $p \equiv 1 \bmod 4$, then we have the identity (for this, see Theorems 4–11 in [4]).

$$p = \alpha^2(p) + \beta^2(p) = \left(\frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a+\overline{a}}{p}\right)\right)^2 + \left(\frac{1}{2}\sum_{a=1}^{p-1}\left(\frac{a+r\overline{a}}{p}\right)\right)^2, \tag{5}$$

where r is any quadratic nonresidue modulo p. That is, $\chi_2(r) = -1$.

Chen [5] used the analytic methods to obtain another identity for the six-order primitive characters modulo p. That is, she proved the following conclusion. Let p be a prime with $p \equiv 1 \bmod 6$, then one has the identity

$$\tau^3(\chi_6)+\tau^3(\overline{\chi}_6)=\begin{cases} p^{(1/2)}\cdot\left(d^2-2p\right), & \text{if } p\equiv 1 \bmod 12;\\ -i\cdot p^{(1/2)}\cdot\left(d^2-2p\right), & \text{if } p\equiv 7 \bmod 12,\end{cases} \tag{6}$$

where $i^2=-1$, d is uniquely determined by $4p=d^2+27b^2$, and $d\equiv 1 \bmod 3$.

There are many other related results, and we will not list them all here.

Obviously, the identities (5) and (6) look very concise and beautiful, but whether they can be applied in theory or practice is what we care most. Recently, we have found that these identities can be used to calculating some mean value problems of the Dedekind sums. And for that, we need to introduce the definition of the Dedekind sums. For any integers $q\geq 2$ and h, the classical Dedekind sums $S(h,q)$ is defined as follows (see [6]):

$$S(h,q)=\sum_{a=1}^{q}\left(\left(\frac{a}{q}\right)\right)\left(\left(\frac{ah}{q}\right)\right), \tag{7}$$

where as usual,

$$((x))=\begin{cases} x-[x]-\frac{1}{2}, & \text{if } x \text{ is not an integer;}\\ 0, & \text{if } x \text{ is an integer.}\end{cases} \tag{8}$$

In fact, this sums describes the behaviour of the logarithm of the eta-function (see [7, 8]) under modular transformations. Because of the importance of $S(h,q)$ in the analytic number theory, many authors have studied the arithmetical properties of $S(h,q)$ and obtained many interesting results, some of them can be found in [9–15]. Some relevant and meaningful work can also be found in [16–18]. In order to avoid the tedious, we do not want to list them one by one. Maybe the most important properties of $S(h,q)$ are its reciprocity theorem (see [6, 9]). That is, for all positive integers h and q with $(h,q)=1$, we have the identity

$$S(h,q)+S(q,h)=\frac{h^2+q^2+1}{12\text{hq}}-\frac{1}{4}. \tag{9}$$

Rademacher and Grosswald [8] also obtained a three-term formula similar to (9).

The main purpose of this paper is to study the calculating problems of one kind mean values of $S(h,p)$. That is,

$$H(k,r;p)=\sum_{\substack{a_1=1\\ a_1^r+a_2^r+\cdots+a_k^r\equiv 0 \bmod p}}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_k=1}^{p-1}S\left(a_1a_2\cdots a_k,p\right), \tag{10}$$

where k and r are two positive integers.

This work is mainly because the high dimensional sums such as the h-dimensional Kloosterman sums $K(c_1, c_2,\ldots,c_h,b;q)$ and h-dimensional character sums play an important role in the research of number theory. For example, Li and Zhang [19] study the sums

$$S(m,h,\chi;p)=\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right), \tag{11}$$

where p is an odd prime, χ is any nonprincipal Dirichlet character mod p, h is any fixed positive integer, and m is any integer. Also, they obtained the following conclusions.

Theorem 1. *Let p be an odd prime, $h\geq 1$ is an integer with $(h+1,p-1)=1$. Then, for any nonprincipal character χ mod p, one has the identity*

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{\text{ma}^{h+1}+a}{p}\right)\right|^2=p^h\cdot\left(p^2-p-1\right). \tag{12}$$

Theorem 2. *Let p be an odd prime, h is an integer with $(h+1)\mid(p-1)$, and χ is any nonprincipal character* mod p. *If χ is a $(h+1)$-th character* mod p *(that is, there exists a character χ_1* mod p *such that $\chi=\chi_1^{h+1}$), then one has*

$$\sum_{m=1}^{p-1}\left|\sum_{a_1=1}^{p-1}\cdots\sum_{a_h=1}^{p-1}\chi\left(a_1+\cdots+a_h+m\overline{a_1\cdots a_h}\right)\right|^2\cdot\left|\sum_{a=1}^{p-1}e\left(\frac{\text{ma}^{h+1}+a}{p}\right)\right|^2=(h+1)\cdot p^{h+2}+O\left(h^2\cdot p^{h+1}\right). \tag{13}$$

Hence, it is meaningful in further exploring the problem of value distribution of $S(h,p)$ on certain special sets. It may be possible to characterize some profound properties of $S(h,p)$.

In this paper, we give some accurate calculating formulas for $H(k,h;p)$ with $k=4,6$ and $r=1$ or $k=4$ and $r=3$. That is, we use the identities (5) and (6) of the classical Gauss sums and analytic methods to prove the following three interesting conclusions.

Theorem 3. *Let p be an odd prime, then we have the identities*

$$\sum_{\substack{a=1}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ a+b+c+d\equiv 0 \bmod p}}^{p-1} S(abcd,p)=\begin{cases} p\cdot h_p^2, & \text{if } p\equiv 3 \bmod 4;\\ \dfrac{2p}{\pi^2}\cdot\left(\alpha^2(p)-\beta^2(p)\right)\cdot|L(1,\chi_4)|^2, & \text{if } p\equiv 5 \bmod 8;\\ 0, & \text{otherwise,}\end{cases} \tag{14}$$

where h_p denotes the class number of the imaginary quadratic field $\mathbb{Q}(\sqrt{-p})$, χ_4 denotes any four-order primitive character modulo p, $\alpha(p)$ and $\beta(p)$ are defined as in (5), and $L(s,\chi)$ denotes the Dirichlet L-function corresponding to character χ modulo p.

Theorem 4. *Let p be an odd prime, then we have the identities*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1} S(abcdef,p)=\begin{cases} -p^2\cdot h_p^2, & \text{if } p\equiv 11 \bmod 12;\\ \dfrac{p}{\pi^2}\cdot\left(4pd^2-d^4-2p^2\right)\cdot|L(1,\psi)|^2-p^2\cdot h_p^2, & \text{if } p\equiv 7 \bmod 12;\\ 0, & \text{otherwise,}\end{cases} \tag{15}$$

where ψ denotes any six-order primitive character modulo p, $4p=d^2+27\cdot b^2$, and d is uniquely determined by $d\equiv 1 \bmod 3$.

Theorem 5. *Let p be an odd prime with $3\nmid(p-1)$, then we have the identities*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ a^3+b^3\equiv c^3+d^3 \bmod p}}^{p-1} S(abcd,p)=\begin{cases} p\cdot h_p^2, & \text{if } p\equiv 11 \bmod 12;\\ \dfrac{2p}{\pi^2}\cdot\left(\alpha^2(p)-\beta^2(p)\right)\cdot|L(1,\chi_4)|^2, & \text{if } p\equiv 5 \bmod 12.\end{cases} \tag{16}$$

Some notes: since $S(-h,p)=-S(h,p)$, so for any odd number $k>1$, we have identities

$$\begin{aligned}&\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{\substack{a_k=1\\ a_1+a_2+\cdots+a_k\equiv 0 \bmod p}}^{p-1} S(a_1a_2\cdots a_k,p)\\ &=\sum_{a_1=1}^{p-1}\sum_{a_2=1}^{p-1}\cdots\sum_{\substack{a_k=1\\ a_1^3+a_2^3+\cdots+a_k^3\equiv 0 \bmod p}}^{p-1} S(a_1a_2\cdots a_k,p)=0.\end{aligned} \tag{17}$$

Therefore, we only consider the case with an even number of variables.

If $3|(p-1)$ in Theorem 5, then the situation is more complicated, and we cannot yet get accurate calculation results.

In addition, whether these sums have reciprocal laws is also an interesting problem.

These will be the subjects of our further research.

2. Several Lemmas

In this section, we will deduce several simple lemmas that are necessary in the proofs of our main results. Hereinafter, we shall use the knowledge of the analytic number theory, and the properties of the classical Gauss sums and Dedekind sums, all these can be found in references [1, 2, 4, 6]. Therefore, we do not repeat them here. First, we have the following:

Lemma 6. *Let p be an odd prime. Then, for any odd character χ modulo p, we have*

$$\sum_{\substack{a=1 \\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(abcd) = \begin{cases} p(p-1), & \text{if } p\equiv 3 \bmod 4 \text{ and } \chi=\chi_2 \bmod p; \\ \dfrac{p-1}{p}\cdot\tau^4(\chi_4), & \text{if } p\equiv 5 \bmod 8 \text{ and } \chi=\chi_4 \bmod p; \\ 0, & \text{otherwise,} \end{cases} \tag{18}$$

where χ_4 denotes the four-order primitive character modulo p.

Proof. From the definition of the classical Gauss sums, the properties of the trigonometric sums, and note that $\chi(-1)=-1$, and the identity

$$\sum_{a=1}^{p-1}\chi(a)=0, \tag{19}$$

we have

$$\begin{aligned}\sum_{\substack{a=1 \\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(abcd) &= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{m=1}^{p}\chi(abcd)e\left(\frac{m(a+b+c+d)}{p}\right) \\ &= \frac{1}{p}\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\chi(a)e\left(\frac{\mathrm{ma}}{p}\right)\right)^4 = \frac{\tau^4(\chi)}{p}\sum_{m=1}^{p-1}\overline{\chi}^4(m).\end{aligned} \tag{20}$$

If $p\equiv 3 \bmod 4$ and $\chi(-1)=-1$, then we have

$$\sum_{m=1}^{p-1}\overline{\chi}^4(m) = \begin{cases} p-1, & \text{if } \chi=\chi_2 \text{ is the Legendre's symbol} \bmod p; \\ 0, & \text{otherwise.} \end{cases} \tag{21}$$

If $p\equiv 1 \bmod 4$, let χ_4 denote any four-order primitive character modulo p. That is, $\chi_4^4=\chi_0$, the principal character modulo p and $\chi_4^k\neq\chi_0$ for $k=1,2,3$. Note that $\chi_2(-1)=1$, $\chi_4(-1)=-1$, if $p\equiv 5 \bmod 8$ and $\chi_4(-1)=1$, if $p\equiv 1 \bmod 8$. So, in these cases, from the properties of the characters modulo p, we have

$$\sum_{m=1}^{p-1}\overline{\chi}^4(m) = \begin{cases} p-1, & \text{if } p\equiv 5 \bmod 8 \text{ and } \chi=\chi_4; \\ 0, & \text{otherwise.} \end{cases} \tag{22}$$

Note that $\tau^2(\chi_2)=\chi_2(-1)\cdot p$, from (11), (20), and (21), we have the identities

$$\sum_{\substack{a=1 \\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(abcd) = \begin{cases} p(p-1), & \text{if } p\equiv 3 \bmod 4 \text{ and } \chi=\chi_2 \bmod p; \\ \dfrac{p-1}{p}\cdot\tau^4(\chi_4), & \text{if } p\equiv 5 \bmod 8 \text{ and } \chi=\chi_4 \bmod p; \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

This proves Lemma 6. □

Lemma 7. *Let p be an odd prime. Then, for any odd character χ modulo p, we have the identities*

$$\sum_{\substack{a=1}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1}\chi(abcdef) = \begin{cases} -p^2(p-1), & \text{if } p\equiv 11 \bmod 12 \text{ and } \chi=\chi_2 \bmod p;\\ \dfrac{(p-1)\cdot\tau^6(\chi)}{p}, & \text{if } p\equiv 7 \bmod 12 \text{ and } \chi=\chi_2 \text{ or } \psi \bmod p;\\ 0, & \text{otherwise.}\end{cases} \quad (24)$$

Proof. From the methods of proving (11), we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1}\chi(abcdef) &= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\sum_{m=1}^{p}\chi(abcdef)\\ &\quad\cdot e\left(\frac{m(a+b+c+d+e+f)}{p}\right)\\ &= \frac{1}{p}\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma}{p}\right)\right)^6 = \frac{\tau^6(\chi)}{p}\sum_{m=1}^{p-1}\overline{\chi}^6(m).\end{aligned} \quad (25)$$

If $p\equiv 11 \bmod 12$, then for any $\chi(-1)=-1$, we have

$$\sum_{m=1}^{p-1}\overline{\chi}^6(m) = \begin{cases} p-1, & \text{if } \chi=\chi_2;\\ 0, & \text{otherwise.}\end{cases} \quad (26)$$

If $p\equiv 7 \bmod 12$, let $\psi=\chi_2\lambda$ denote any six-order primitive character modulo p, where λ denotes any three-order primitive character modulo p. Then note that $\psi(-1)=\chi_2(-1)=-1$, from the properties of the characters modulo p we have

$$\sum_{m=1}^{p-1}\overline{\chi}^6(m) = \begin{cases} p-1, & \text{if } \chi=\psi \text{ or } \chi_2;\\ 0, & \text{otherwise.}\end{cases} \quad (27)$$

If $p\equiv 1 \bmod 12$ or $p\equiv 5 \bmod 12$, then for any character $\chi(-1)=-1$ modulo p, we have

$$\sum_{m=1}^{p-1}\overline{\chi}^6(m) = 0. \quad (28)$$

Note that $\tau^2(\chi_2)=\chi_2(-1)\cdot p$, from (22), (25)–(27), we have

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1}\chi(abcdef) = \begin{cases} -p^2(p-1), & \text{if } p\equiv 11 \bmod 12 \text{ and } \chi=\chi_2 \bmod p;\\ \dfrac{(p-1)\cdot\tau^6(\chi)}{p}, & \text{if } p\equiv 7 \bmod 12 \text{ and } \chi=\chi_2 \text{ or } \psi \bmod p;\\ 0, & \text{otherwise.}\end{cases} \quad (29)$$

This proves Lemma 7. □

Lemma 8. *Let p be an odd prime with $3\nmid(p-1)$. Then, for any odd character χ modulo p, we have*

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ a^3+b^3\equiv c^3+d^3 \bmod p}}^{p-1}\chi(abcd)=\begin{cases} p(p-1), & \text{if } p\equiv 11 \bmod 12 \text{ and } \chi=\chi_2;\\ \dfrac{p-1}{p}\cdot\tau^4\left(\chi_4^h\right), & \text{if } p\equiv 5 \bmod 12 \text{ and } \chi=\chi_4 \text{ modulo} p;\\ 0, & \text{otherwise,}\end{cases} \tag{30}$$

where χ_4 denotes the four-order primitive character modulo p and $3\cdot h\equiv 1 \bmod (p-1)$.

Proof. From the definition of the classical Gauss sums, the properties of the trigonometric sums, and note that $\chi(-1)=-1$, and the identity

$$\sum_{a=1}^{p-1}\chi(a)=0 \text{ and } \sum_{a=1}^{p-1}\chi(a)e\left(\frac{-\mathrm{ma}^3}{p}\right)=-\sum_{a=1}^{p-1}\chi(a)e\left(\frac{-\mathrm{ma}^3}{p}\right), \tag{31}$$

we have

$$\begin{aligned}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{\substack{d=1\\ a^3+b^3\equiv c^3+d^3 \bmod p}}^{p-1}\chi(abcd) &= \frac{1}{p}\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{m=1}^{p-1}\chi(\mathrm{abcd})e\left(\frac{m\left(a^3+b^3-c^3-d^3\right)}{p}\right)\\ &= \frac{1}{p}\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\chi(a)e\left(\frac{\mathrm{ma}^3}{p}\right)\right)^4.\end{aligned} \tag{32}$$

If $3\nmid (p-1)$, let $3\cdot h\equiv 1 \bmod (p-1)$, then we have

$$\begin{aligned}\sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\chi(a)e\left(\frac{ma^3}{p}\right)\right)^4 &= \sum_{m=1}^{p-1}\left(\sum_{a=1}^{p-1}\chi\left(a^h\right)e\left(\frac{\mathrm{ma}}{p}\right)\right)^4\\ &= \tau^4\left(\chi^h\right)\sum_{m=1}^{p-1}\overline{\chi}^{4h}(m)=\tau^4\left(\chi^h\right)\sum_{m=1}^{p-1}\overline{\chi}^4(m)\\ &= \begin{cases} p^2(p-1), & \text{if } p\equiv 11 \bmod 12 \text{ and } \chi=\chi_2 \bmod p;\\ \tau^4\left(\chi_4^k\right)(p-1), & \text{if } p\equiv 5 \bmod 12 \text{ and } \chi=\chi_4 \bmod p;\\ 0, & \text{otherwise.}\end{cases}\end{aligned} \tag{33}$$

Now, Lemma 8 follows from (28) and (32). □

Lemma 9. *Let $q>2$ be an integer, then for any integer h with $(h,q)=1$, we have the identity*

$$S(a,q)=\frac{1}{\pi^2 q}\sum_{d\mid q}\frac{d^2}{\varphi(d)}\sum_{\substack{\chi \bmod d,\\ \chi(-1)=-1.}}\chi(a)|L(1,\chi)|^2 \tag{34}$$

where $L(s,\chi)$ denotes the Dirichlet L-function corresponding to $\chi \bmod d$.

Proof. See Lemma 2 in [10]. □

3. Proofs of the Theorems

In this section, we will provide the proofs of our theorems. We start with the proof of Theorem 3. For any odd prime p and integer $1\le a\le p-1$, from Lemma 9, we have

$$S(a,p)=\frac{p}{\pi^2\cdot(p-1)}\sum_{\substack{\chi \bmod p,\\ \chi(-1)=-1.}}\chi(a)|L(1,\chi)|^2. \tag{35}$$

If $p \equiv 3 \bmod 4$, note that $\chi_2(-1) = -1$ and $|L(1,\chi_2)| = \pi/\sqrt{p} \cdot h_p$, and from (33) and Lemma 6, we have

$$\sum_{\substack{a=1\\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} S(abcd,p) = \frac{p}{\pi^2\cdot(p-1)}\sum_{\substack{\chi \bmod p,\\ \chi(-1)=-1.}}\left(\sum_{\substack{a=1\\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(abcd)\right)\cdot |L(1,\chi)|^2. \tag{36}$$

If $p \equiv 5 \bmod 8$, note that $\chi_2(-1) = 1$ and there are two four-order primitive characters χ_4 and $\overline{\chi}_4$ modulo p such that $\chi_4(-1) = \overline{\chi}_4(-1) = -1$ and $|L(1,\chi_4)|^2 = |L(1,\overline{\chi}_4)|^2$. So in this case, from (4), (5), and (33) and Lemma 6, we have

$$\begin{aligned}\sum_{\substack{a=1\\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} S(abcd,p) &= \frac{p}{\pi^2\cdot(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{\substack{a=1\\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\chi(abcd)\right)\cdot |L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot\tau^4(\chi_4)\cdot |L(1,\chi_4)|^2 + \frac{1}{\pi^2}\cdot\tau^4(\overline{\chi}_4)\cdot |L(1,\overline{\chi}_4)|^2\\
&= \frac{1}{\pi^2}\cdot\left(\left(\tau^2(\chi_4)+\tau^2(\overline{\chi}_4)\right)^2 - 2\cdot\tau^2(\chi_4)\cdot\tau^2(\overline{\chi}_4)\right)\cdot |L(1,\chi_4)|^2\\
&= \frac{1}{\pi^2}\cdot\left(4p\alpha^2(p) - 2\cdot p^2\right)\cdot |L(1,\chi_4)|^2 = \frac{2p}{\pi^2}\cdot\left(\alpha^2(p)-\beta^2(p)\right)\cdot |L(1,\chi_4)|^2.\end{aligned} \tag{37}$$

If $p \equiv 1 \bmod 8$, then for any odd character $\chi(-1) = -1$, we have $\sum_{m=1}^{p-1}\overline{\chi}^4(m) = 0$. So in this case, from Lemma 6 and the methods of proving (35), we have the identity

$$\sum_{\substack{a=1\\ a+b+c+d\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1} S(abcd,p) = 0. \tag{38}$$

Now, Theorem 3 follows from (35)–(37).

Now, we prove Theorem 4. If $p \equiv 11 \bmod 12$, then for any $\chi(-1) = -1$, we have

$$\sum_{m=1}^{p-1}\overline{\chi}^6(m) = \begin{cases} p-1, & \text{if } \chi = \chi_2;\\ 0, & \text{otherwise.}\end{cases} \tag{39}$$

Note that $\tau^2(\chi_2) = -p$, and from (33) and Lemma 7, we have

$$\begin{aligned}&\sum_{\substack{a=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1} S(\text{abcdef},p)\\
&= \frac{p}{\pi^2\cdot(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{\substack{a=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{f=1}^{p-1}\chi(\text{abcdef})\right)\cdot |L(1,\chi)|^2\\
&= \frac{1}{\pi^2\cdot(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{m=1}^{p-1}\overline{\chi}^6(m)\cdot\tau^6(\chi)\right)\cdot |L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot\tau^6(\chi_2)\cdot |L(1,\chi_2)|^2 = -p^2\cdot h_p^2.\end{aligned} \tag{40}$$

If $p \equiv 7 \bmod 12$, let $\psi = \chi_2 \lambda$ denote any six-order primitive character modulo p, where λ denotes any three-order primitive character modulo p. Then, note that there are two six-order primitive characters $\psi, \overline{\psi}$ modulo p, $\psi(-1) = \chi_2(-1) = -1$, and $|L(1,\psi)|^2 = |L(1,\overline{\psi})|^2$, and from (6) and (33), Lemma 7, and the methods of proving (39), we have

$$\begin{aligned}
&\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1} S(abcdef, p)\\
&= \frac{p}{\pi^2\cdot(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1}\chi(abcdef)\right)\cdot|L(1,\chi)|^2\\
&= \frac{1}{\pi^2\cdot(p-1)}\sum_{\substack{\chi \bmod p\\ \chi(-1)=-1}}\left(\sum_{m=1}^{p-1}\overline{\chi}^6(m)\cdot\tau^6(\chi)\right)\cdot|L(1,\chi)|^2\\
&= \frac{1}{\pi^2}\cdot\left(\tau^6(\psi)+\tau^6(\overline{\psi})\right)\cdot|L(1,\psi)|^2+\frac{1}{\pi^2}\cdot\tau^6(\chi_2)\cdot|L(1,\chi_2)|^2\\
&= \frac{1}{\pi^2}\cdot\left(\left(\tau^3(\psi)+\tau^3(\overline{\psi})\right)^2-2\cdot\tau^3(\psi)\cdot\tau^3(\overline{\psi})\right)\cdot|L(1,\psi)|^2-p^2\cdot h_p^2\\
&= \frac{1}{\pi^2}\cdot\left(2p^3-p\left(d^2-2p\right)^2\right)\cdot|L(1,\psi)|^2-p^2\cdot h_p^2\\
&= \frac{p}{\pi^2}\cdot\left(4pd^2-d^4-2p^2\right)\cdot|L(1,\psi)|^2-p^2\cdot h_p^2.
\end{aligned} \tag{41}$$

If $p \equiv 1 \bmod 12$ or $p \equiv 5 \bmod 12$, then for any odd character χ modulo p, we have

$$\sum_{m=1}^{p-1}\overline{\chi}^6(m) = 0. \tag{42}$$

This time, we have the identity

$$\sum_{a=1}^{p-1}\sum_{b=1}^{p-1}\sum_{c=1}^{p-1}\sum_{d=1}^{p-1}\sum_{e=1}^{p-1}\sum_{\substack{f=1\\ a+b+c+d+e+f\equiv 0 \bmod p}}^{p-1} S(\mathrm{abcdef}, p) = 0. \tag{43}$$

Now, Theorem 4 follows from (39)–(41).

Now, we prove Theorem 5. If $p \equiv 5 \bmod 12$, note that $\chi_2(-1) = 1$ and there are two four-order primitive characters χ_4 and $\overline{\chi}_4$ modulo p such that $\chi_4(-1) = \overline{\chi}_4(-1) = -1$, $\tau^2(\chi_4)\cdot\tau^2(\overline{\chi}_4) = \tau^2(\chi_4^h)\cdot\tau^2(\overline{\chi}_4^h) = p^2$, and $|L(1,\chi_4^h)|^2 = |L(1,\overline{\chi}_4^h)|^2 = |L(1,\chi_4)|^2 = |L(1,\overline{\chi}_4)|^2$ for any integer h with $(h, p-1) = 1$. So in this case, from (33), Lemma 8, and the methods of proving Theorem 3, we may immediately deduce Theorem 5.

This completes the proofs of our all results.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

All the authors have equally contributed to this work. All the authors have read and approved the final manuscript.

Acknowledgments

The author would like to thank the referees for their very helpful and detailed comments. This work was supported by the N. S. F. (12161031) of P. R. China, the N. S. F. (2021MS01003) of Inner Mongolia (CN), and the Higher Education Scientific Research Project (NJZY21173) of Inner Mongolia (CN).

References

[1] T. M. Apostol, *Introduction to Analytic Number Theory*, Springer-Verlag, New York, 1976.

[2] K. Ireland and M. Rosen, *A Classical Introduction to Modern Number Theory*, Springer-Verlag, New York, NY, USA, 1982.

[3] Z. Chen and W. Zhang, "On the fourth-order linear recurrence formula related to classical Gauss sums," *Open Mathematics*, vol. 15, no. 1, pp. 1251–1255, 2017.

[4] W. P. Zhang and H. L. Li, *Elementary Number Theory*, Shaanxi Normal University Press, Xi'an, China, 2013.

[5] L. Chen, "On classical Gauss sums and some of their properties," *Symmetry*, vol. 10, no. 11, p. 625, 2018.

[6] T. M. Apostol, *Modular Functions and Dirichlet Series in Number Theory*, Springer-Verlag, New York, NY, USA, 1976.

[7] H. Rademacher, "On the transformation of log$\eta\tau$," *Journal of the Indian Mathematical Society*, vol. 19, pp. 25–30, 1955.

[8] H. Rademacher and E. Grosswald, "Dedekind sums, carus mathematical monographs," *Math. Assoc. Amer*, 1972.

[9] L. Carlitz, "The reciprocity theorem for Dedekind sums," *Pacific Journal of Mathematics*, vol. 3, pp. 523–527, 1953.

[10] W. P. Zhang, "On the mean values of Dedekind Sums," *Journal de Théorie des Nombres de Bordeaux*, vol. 8, no. 2, pp. 429–442, 1996.

[11] J. B. Conrey, E. Fransen, R. Klein, and C. Scott, "Mean values of Dedekind sums," *Journal of Number Theory*, vol. 56, no. 2, pp. 214–226, 1996.

[12] L. J. Mordell, "The reciprocity formula for Dedekind sums," *American Journal of Mathematics*, vol. 73, no. 3, pp. 593–598, 1951.

[13] W. P. Zhang, "A note on the mean square value of the Dedekind sums," *Acta Mathematica Hungarica*, vol. 86, no. 4, pp. 275–289, 2000.

[14] W. P. Zhang, "On the number of the solutions of one kind congruence equation mod," *Journal of Northwest University*, vol. 46, pp. 313–316, 2016.

[15] W. P. Zhang, "The hybrid mean value involving Dedekind sums and one kind sums analogous to Kloosterman sums," *Journal of Shanxi Normal University (Philosophy and Social Sciences edition)*, vol. 45, pp. 1–4, 2017.

[16] G. V. Milovanovic and Y. Simsek, "Dedekind and hardy type sums and trigonometric sums induced by quadrature formulas," in *Trigonometric Sums and Their Applications*, A. Raigorodskii and M. T. Rassias, Eds., pp. 183–228, Springer, Cham, 2020.

[17] Y. Simsek, "q-Dedekind type sums related to q-zeta function and basic L-series," *Journal of Mathematical Analysis and Applications*, vol. 318, no. 1, pp. 333–351, 2006.

[18] Y. Simsek, "On analytic properties and character analogs of Hardy sums," *Taiwanese Journal of Mathematics*, vol. 13, no. 1, pp. 253–268, 2009.

[19] J. Li and X. Zhang, "On the character sums analogous to high dimensional Kloosterman sums," *AIMS Mathematics*, vol. 7, no. 1, pp. 294–305, 2021.