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Research Article

A Solution Approach to Nonlinear Integral Equations in Generalized b-Metric Spaces

Mohammed M. M. Jaradat,[1] **Abeeda Ahmad**,[2] **Saif Ur Rehman**,[2] **Nabaa Muhammad Diaa**,[3] **Shamoona Jabeen**,[4] **Muhammad Imran Haider**,[2] **Iqra Shamas**,[2] **and Rawan A. Shlaka**[5]

[1]*Mathematics Program, Department of Mathematics, Statistics and Physics, College of Arts and Sciences, Qatar University 2017, Doha, Qatar*
[2]*Institute of Numerical Sciences, Department of Mathematics, Gomal University, Dera Ismail Khan 29220, Khyber Pakhtunkhwa, Pakistan*
[3]*Department of Construction Engineering and Project Management, Al-Noor University College, Nineveh, Iraq*
[4]*Department of Mathematics, University of Science and Technology, Bannu 28100, KP, Pakistan*
[5]*National University of Science and Technology, Dhi Qar, Nasiriyah, Iraq*

Correspondence should be addressed to Mohammed M. M. Jaradat; mmjst4@qu.edu.qa and Saif Ur Rehman; saif.urrehman27@yahoo.com

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In this paper, we study some generalized contraction conditions for three self-mappings on generalized b-metric spaces to prove the existence of some unique common fixed-point results. To unify our results, we establish a supportive example for three self-mappings to show the uniqueness of a common fixed point for a generalized contraction in the said space. In addition, we present a supportive application of nonlinear integral equations for the validation of our work. The concept presented in this paper will play an important role in the theory of fixed points in the context of generalized metric spaces with applications.

1. Introduction

Fixed-point (FP) theory is one of the interesting areas of research in mathematics and other science fields. In this theory, Banach [1] introduced a valuable and important result for the existence and uniqueness of fixed point which is known as the "Banach Contraction Principle (BCP)" and stated as "a single-valued contractive type mapping on a complete metric space (M-space) has a unique FP." BCP was later generalized in various directions, and many authors contributed to the theory of FP. Bhaskar and Lakshmikantham [2] established some FP results for a mixed monotone mapping in an ordered partial M-space using a weak contractivity type of mappings with an application. Jovanovic et al. [3] worked on common fixed point (CFP) results in M-spaces. Bojor [4] proved FP theorems for Reich-type contractions on M-spaces. Kutbi et al. [5] started to investigate CFP results for mappings with rational expressions. Batra et al. [6] presented a new extension of Kannan contractions and related FP results. Hussain [7] proved results for the solution of fractional differential equations using symmetric contraction. Debnath [8] studied Banach, Kannan, Chatterjia, and Reich-type contractive inequalities for multivalued mappings and proved CFP theorems. Rasham et al. [9] established some results for the family of multivalued mappings with the applications of functional and integral equations. Recently, Abbas et al. [10] studied the thermodynamic properties of the second-grade micropolar nanofluid flow past an exponential curved Riga stretching surface with Cattaneo–Christov double diffusion.

Furthermore, in [11], Abbas et al. discussed the thermal analysis of MHD Casson-Sutterby fluid flow over exponential stretching curved sheet.

Bakhtin [12] gave the concept of b-metric space (b-M-space). After that, Czerwik [13] presented some FP results by using b-M-spaces. In 1998 Czerwik [14] studied some nonlinear set-valued contraction results in b-M-spaces. Boriceanu et al. [15] formulated the fractal operator theory by establishing it in b-M-spaces and verified some generalized CFP results. Aydi et al. [16] worked on an FP theorem for set-valued quasi-contractions in b-M-spaces. In [17], Roshan et al. proved CFP results for four self-mappings on b-M-spaces. Shatanawi et al. [18] extended contraction conditions using comparison functions on b-M-spaces. Alqahtani et al. [19] established CFP results on an extended b-M-space. Sintunavarat and Kumam [20] presented CFP theorems in complex-valued M-spaces with their applications. Recently, Bantan et al. [21] proposed integral equations in complex-valued b-M-spaces.

In 2006, Mustafa and Sims [22] introduced the idea of generalized metric space (GM-space). Mustafa et al. [23] proved an FP theorem for self-mappings on complete GM-spaces. Abbas and Rhoades [24] discussed CFP results for noncommuting mappings without continuity in a GM-space. In [25], Hussain et al. discussed the unification of b-metric, partial metric, and GM-spaces. Gugnani et al. [26] formulated CFP results in GM-spaces and their applications. In 2012, Lakzain and Samet [27] and Mustafa et al. [28], respectively, established some FP and coincidence point results for (ψ,φ)-weakly contractive mappings in GM-spaces and ordered GM-spaces.

Aghajani et al. [29] introduced the idea of generalized b-metric space (G_b M-spaces). They proved some CFP results for four mappings satisfying a generalized weakly contractive condition in partially ordered complete b-M-spaces. Their results extended and improved several comparable results in the published literature. Roshan et al. [30], proved some CFP results for three mappings in discontinuous G_b M-spaces. Cobzas and Czerwik [31] worked on the completion of G_b M-spaces and proved some FP results. Aydi et al. [32] started to investigate a few coupled and tripled coincidence point results and also extended, complemented, and generalized several existing results in such spaces. In 2021, Gupta et al. [33] investigated various FP results on complete G_b M-spaces and proved CFP results. Mustafa et al. [28] established some coupled coincidence point results for (ψ,φ)-weakly contractive mappings in the setup of partially ordered G_b M-spaces. Makran et al. [34] provided generalized CFP results for multivalued mapping in G_b M-spaces with an application. Mebawondu and Mewomo [35] gave the concept of Suzuki-type FP results in G_b M-spaces. Recently, Mehmood et al. [36] established the notion of integral equations in complex-valued G_b M-spaces and proved some CFP results.

The main purpose of this paper is to demonstrate some results for the existence and uniqueness of CFP using three self-maps satisfying the generalized contractive conditions in G_b M-spaces with an illustrative example. Our results improve and modify many results presented in the literature. Further, we support our results by an application of the nonlinear integral equations to validate our work. This work is followed by Section 2, which consists of preliminary concepts. In Section 3, we establish some generalized CFP theorems on G_b M-spaces with an illustrative example. In Section 4, we present an application of nonlinear integral equations to support our main work. Lastly, in Section 5, we discuss the conclusion of our work.

2. Preliminaries

Definition 1 (see [29]). Let B be a nonempty set. A function $G_b\colon B\times B\times B\longrightarrow[0,\infty)$ is said to be a generalized b-metric space (G_b M-space) if the following axioms hold:

(i) $G_b(b_1,b_2,b_3)=0$ iff $b_1=b_2=b_3$

(ii) $G_b(b_1,b_1,b_2)>0$ with $b_1\neq b_2$

(iii) $G_b(b_1,b_1,b_2)\le G_b(b_1,b_2,b_3)$ with $b_3\neq b_2$

(iv) $G_b(b_1,b_2,b_3)=G_b(\{p(b_1,b_2,b_3)\})$, here, p is a permutation of b_1,b_2,b_3. (symmetry)

(v) $G_b(b_1,b_2,b_3)\le s[G_b(b_1,e,e)+G_b(e,b_2,b_3)]$

For all $b_1,b_2,b_3,e\in B$. Then, the pair (B,G_b) is said to be a G_b M-space.

Example 1. Let $B=[0,1]$ and the mapping $G_b\colon B\times B\times B\longrightarrow\mathbb{R}$ be defined as follows:

$$G_b(b_1,b_2,b_3)=\frac{1}{20}\left(|b_1-b_2|+|b_2-b_3|+|b_3-b_1|\right),\quad(1)$$

for $b_1,b_2,b_3\in B$. Then, (B,G_b) is a G_b M-space with $s=2$.

Definition 2 (see [29]). A G_b M-space is said to be symmetric if $G_b(b_1,b_2,b_2)=G_b(b_2,b_1,b_1)\ \forall\, b_1,b_2\in B$.

Proposition 3 (see [29]). *Let (B,G_b) be a G_b M-space. Then, for each $b_1,b_2,b_3,e\in B$, it follows that*

(i) $G_b(b_1,b_2,b_3)=0$ then $b_1=b_2=b_3$

(ii) $G_b(b_1,b_2,b_3)\le s(G_b(b_1,b_1,b_2)+G_b(b_1,b_1,b_3))$

(iii) $G_b(b_1,b_2,b_2)\le 2sG_b(b_1,b_1,b_2)$

(iv) $G_b(b_1,b_2,b_3)\le sG_b(b_1,e,b_3)+sG_b(e,b_2,b_3)$

Definition 4 (see [29]). Let (B,G_b) be a G_b M-space. A sequence $\{b_j\}$ in B is said to be

(i) G_b-Cauchy sequence if for any $\varepsilon>0$, $\exists\, n_0\in\mathbb{N}$ such that $G_b(b_j,b_m,b_l)<\varepsilon,\forall\, j,m,l\ge n_0$

(ii) Convergent to an element $b\in B$ if for all given $0<\varepsilon\in\mathbb{R}$, $\exists\, n_0\in\mathbb{N}$ such that $G_b(b,b_m,b_m)<\varepsilon$, whenever $m\ge n_0$

(iii) A pair (B,G_b) is said to be complete if every G_b-Cauchy sequence is G_b-convergent in B

Proposition 5 (see [29]). *Let (B,G_b) be a G_b M-space. The following statements are equivalent:*

(i) b_j *is* G_b*-convergent to* b

(ii) $G_b(b_j, b_j, b) \longrightarrow 0$ *as* $j \longrightarrow \infty$

(iii) $G_b(b_j, b, b) \longrightarrow 0$ *as* $j \longrightarrow \infty$

Proposition 6 (see [29]). *Let* (B, G_b) *be a* G_b *M-space. The following statements are equivalent:*

(i) b_j *is* G_b*-Cauchy sequence*

(ii) $G_b(b_m, b_j, b_j) \longrightarrow 0$ *as* $m, j \longrightarrow \infty$

3. Main Results

In this section, we use the approaches of Aghajani et al. [29], Gupta et al. [33], and Mustafa et al. [28] to prove some modified rational contraction theorems with illustrative examples.

Theorem 7. *Let* (B, G_b) *be a* G_b *M-space with coefficient* $s > 1$ *and* T_1, T_2, T_3: $B \longrightarrow B$ *be three self-mappings which satisfy*

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \gamma_1 G_b(b_1, b_2, T_2b_2) + \gamma_2 G_b(T_1b_1, T_1b_1, b_2) + \gamma_3 \min\left\{\begin{array}{c} G_b(b_1, b_2, T_2b_2), G_b(T_1b_1, T_1b_1, b_2), G_b(T_1b_1, b_2, b_2), G_b(T_2b_2, T_2b_2, b_3), \\ G_b(T_2b_2, b_3, b_3), \left(\dfrac{G_b(b_2, T_2b_2, T_2b_2).G_b(b_3, T_3b_3, T_3b_3)}{1 + G_b(T_1b_1, b_3, b_3)}\right) \end{array}\right\}, \quad (2)$$

for all $b_1, b_2, b_3 \in B$, $\gamma_1, \gamma_2, \gamma_3 \in [0, 1)$ *with* $s\gamma_1 < 1$ *and* $2s^2\gamma_2 < 1$. *Then, the three self-mappings* T_1, T_2, *and* T_3 *have a CFP in B. Moreover, if* $(\gamma_1 + 2s\gamma_2) < 1$, *then* T_1, T_2, *and* T_3 *have a unique CFP in B.*

Proof. Fix $b_0 \in B$. We now define an iterative sequence in B as follows:

$$\begin{aligned} b_{3j+1} &= T_1 b_{3j}, \\ b_{3j+2} &= T_2 b_{3j+1}, \text{and} \\ b_{3j+3} &= T_3 b_{3j+2}, \quad \forall\, j \geq 0. \end{aligned} \quad (3)$$

By using (2), we have

$$\begin{aligned} G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+3}\right) &= G_b\left(T_1b_{3j}, T_2b_{3j+1}, T_3b_{3j+2}\right) \leq \gamma_1 G_b\left(b_{3j}, b_{3j+1}, T_2b_{3j+1}\right) + \gamma_2 G_b\left(T_1b_{3j}, T_1b_{3j}, b_{3j+1}\right) \\ &\quad + \gamma_3 \min\left\{\begin{array}{c} G_b\left(b_{3j}, b_{3j+1}, T_2b_{3j+1}\right), G_b\left(T_1b_{3j}, T_1b_{3j}, b_{3j+1}\right), G_b\left(T_1b_{3j}, b_{3j+1}, b_{3j+1}\right), \\ G_b\left(T_2b_{3j+1}, T_2b_{3j+1}, b_{3j+2}\right), G_b\left(T_2b_{3j+1}, b_{3j+2}, b_{3j+2}\right), \\ \left(\dfrac{G_b\left(b_{3j+1}, T_2b_{3j+1}, T_2b_{3j+1}\right).G_b\left(b_{3j+2}, T_3b_{3j+2}, T_3b_{3j+2}\right)}{1 + G_b\left(T_1b_{3j}, b_{3j+2}, b_{3j+2}\right)}\right) \end{array}\right\} \\ &= \gamma_1 G_b\left(b_{3j}, b_{3j+1}, b_{3j+2}\right) + \gamma_2 G_b\left(b_{3j+1}, b_{3j+1}, b_{3j+1}\right) \\ &\quad + \gamma_3 \min\left\{\begin{array}{c} G_b\left(b_{3j}, b_{3j+1}, b_{3j+2}\right), G_b\left(b_{3j+1}, b_{3j+1}, b_{3j+1}\right), G_b\left(b_{3j+1}, b_{3j+1}, b_{3j+1}\right), \\ G_b\left(b_{3j+2}, b_{3j+2}, b_{3j+2}\right), G_b\left(b_{3j+2}, b_{3j+2}, b_{3j+2}\right), \\ \left(\dfrac{G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+2}\right).G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+3}\right)}{1 + G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+2}\right)}\right) \end{array}\right\}. \end{aligned} \quad (4)$$

After simplification, we obtain

$$G_b(b_{3j+1}, b_{3j+2}, b_{3j+3}) \leq \gamma_1 G_b(b_{3j}, b_{3j+1}, b_{3j+2}). \tag{5}$$

Similarly, again by the view of (2),

$$\begin{aligned} G_b(b_{3j+2}, b_{3j+3}, b_{3j+4}) &= G_b(T_2 b_{3j+1}, T_3 b_{3j+2}, T_1 b_{3(j+1)}) \leq \gamma_1 G_b(b_{3j+1}, b_{3j+2}, T_3 b_{3j+2}) + \gamma_2 G_b(T_2 b_{3j+1}, T_2 b_{3j+1}, b_{3j+2}) \\ &+ \gamma_3 \min \left\{ \begin{array}{c} G_b(b_{3j+1}, b_{3j+2}, T_3 b_{3j+2}), G_b(T_2 b_{3j+1}, T_2 b_{3j+1}, b_{3j+2}), G_b(T_2 b_{3j+1}, b_{3j+2}, b_{3j+2}), \\ G_b(T_3 b_{3j+2}, T_3 b_{3j+2}, b_{3(j+1)}), G_b(T_3 b_{3j+2}, b_{3(j+1)}, b_{3(j+1)}), \\ \left(\dfrac{G_b(b_{3j+2}, T_3 b_{3j+2}, T_3 b_{3j+2}).G_b(b_{3(j+1)}, T_1 b_{3(j+1)}, T_1 b_{3(j+1)})}{1 + G_b(T_2 b_{3j+1}, b_{3(j+1)}, b_{3(j+1)})} \right) \end{array} \right\} \\ &= \gamma_1 G_b(b_{3j+1}, b_{3j+2}, b_{3j+3}) + \gamma_2 G_b(b_{3j+2}, b_{3j+2}, b_{3j+2}) \\ &+ \gamma_3 \min \left\{ \begin{array}{c} G_b(b_{3j+1}, b_{3j+2}, b_{3j+3}), G_b(b_{3j+2}, b_{3j+2}, b_{3j+2}), G_b(b_{3j+2}, b_{3j+2}, b_{3j+2}), \\ G_b(b_{3j+3}, b_{3j+3}, b_{3(j+1)}), G_b(b_{3j+3}, b_{3(j+1)}, b_{3(j+1)}), \\ \left(\dfrac{G_b(b_{3j+2}, b_{3j+3}, b_{3j+3}).G_b(b_{3(j+1)}, b_{3j+4}, b_{3j+4})}{1 + G_b(b_{3j+2}, b_{3(j+1)}, b_{3(j+1)})} \right) \end{array} \right\}. \end{aligned} \tag{6}$$

After simplification, we obtain

$$G_b(b_{3j+2}, b_{3j+3}, b_{3j+4}) \leq \gamma_1 G_b(b_{3j+1}, b_{3j+2}, b_{3j+3}). \tag{7}$$

By a similar argument as in above, we can show that

$$G_b(b_{3j+3}, b_{3j+4}, b_{3j+5}) \leq \gamma_1 G_b(b_{3j+2}, b_{3j+3}, b_{3j+4}). \tag{8}$$

Now, from (5), (7), and (8), we conclude that

$$G_b(b_{3j+1}, b_{3j+2}, b_{3j+3}) \leq \gamma_1 G_b(b_{3j}, b_{3j+1}, b_{3j+2}) \leq \cdots \leq {\gamma_1}^{3j+1} G_b(b_0, b_1, b_2) \longrightarrow 0, \text{as } j \longrightarrow \infty. \tag{9}$$

Hence, we have proved that the sequence $\{b_j\}$ is contractive under the G_b M-space for three self-mappings. Therefore,

$$\lim_{j \longrightarrow \infty} G_b(b_j, b_{j+1}, b_{j+2}) = 0. \tag{10}$$

Next, we will show that $\{b_j\}$ is a G_b-Cauchy sequence in B. For all $j, m \in \mathbb{N}$, and $m > j$, using the rectangle inequality and (9), we have

$$\begin{aligned} G_b(b_j, b_m, b_m) &\leq s[G_b(b_j, b_{j+1}, b_{j+1}) + G_b(b_{j+1}, b_m, b_m)] \\ &\leq sG_b(b_j, b_{j+1}, b_{j+1}) + s^2 G_b(b_{j+1}, b_{j+2}, b_{j+2}) + s^3 G_b(b_{j+2}, b_{j+3}, b_{j+3}) + \cdots + s^m G_b(b_{m-1}, b_m, b_m) \\ &\leq sG_b(b_j, b_{j+1}, b_{j+2}) + s^2 G_b(b_{j+1}, b_{j+2}, b_{j+3}) + s^3 G_b(b_{j+2}, b_{j+3}, b_{j+4}) + \cdots + s^m G_b(b_{m-1}, b_m, b_{m+1}) \\ &\leq s{\gamma_1}^j G_b(b_0, b_1, b_1) + s^2 {\gamma_1}^{j+1} G_b(b_0, b_1, b_1) + s^3 {\gamma_1}^{j+2} G_b(b_0, b_1, b_1) + \cdots + s^{m-j-1} {\gamma_1}^{m-1} G_b(b_0, b_1, b_1) \\ &\leq s{\gamma_1}^j [G_b(b_0, b_1, b_1) + s{\gamma_1}^1 G_b(b_0, b_1, b_1) + s^2 {\gamma_1}^2 G_b(b_0, b_1, b_1) + \cdots + s^{m-1} {\gamma_1}^{m-j-1} G_b(b_0, b_1, b_1)]. \end{aligned} \tag{11}$$

Since $\gamma_1 < 1$, we have

$$G_b\left(b_j, b_m, b_m\right) \leq \frac{s\gamma_1{}^j}{1 - s\gamma_1} G_b\left(b_0, b_1, b_1\right) \longrightarrow 0, \text{as } j \longrightarrow \infty. \tag{12}$$

By using Proposition 3 (ii), we have $G_b(b_j, b_l, b_m) \leq s[G_b(b_j, b_m, b_m) + G_b(b_l, b_m, b_m)]$ for $j, l, m \in \mathbb{N}$ with $j < l < m$. If we take the limit as $j, l, m \longrightarrow \infty$, we get $G_b(b_j, b_l, b_m) \longrightarrow 0$. Hence, $\{b_j\}$ is a G_b-Cauchy sequence. Since (B, G_b) is complete, there is $\delta \in B$, such that $b_j \longrightarrow \delta$ as $j \longrightarrow \infty$ or $\lim_{j\longrightarrow\infty} b_j = \delta$. We now show that $T_1\delta = \delta$ by contrary case, let $T_1\delta \neq \delta$. Then, by using the rectangular property of (B, G_b) and by the view of (2), we have that

$$\begin{aligned}
G_b(T_1\delta, \delta, \delta) &\leq sG_b\left(T_1\delta, b_{3j+2}, b_{3j+2}\right) + sG_b\left(b_{3j+2}, \delta, \delta\right) \leq sG_b\left(T_1\delta, b_{3j+2}, b_{3j+3}\right) + sG_b\left(b_{3j+2}, \delta, \delta\right)\\
&= sG_b\left(b_{3j+2}, \delta, \delta\right) + sG_b\left(T_1\delta, T_2 b_{3j+1}, T_3 b_{3j+2}\right) \leq sG_b\left(b_{3j+2}, \delta, \delta\right) + s\gamma_1 G_b\left(\delta, b_{3j+1}, T_2 b_{3j+1}\right)\\
&\quad + s\gamma_2 G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right) + s\gamma_3 \min\left\{\begin{array}{c}
G_b\left(\delta, b_{3j+1}, T_2 b_{3j+1}\right), G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right), G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right),\\[2ex]
G_b\left(T_2 b_{3j+1}, T_2 b_{3j+1}, b_{3j+2}\right), G_b\left(T_2 b_{3j+1}, b_{3j+2}, b_{3j+2}\right),\\[2ex]
\left(\dfrac{G_b\left(b_{3j+1}, T_2 b_{3j+1}, T_2 b_{3j+1}\right).G_b\left(b_{3j+2}, T_3 b_{3j+2}, T_3 b_{3j+2}\right)}{1 + G_b\left(T_1\delta, b_{3j+2}, b_{3j+2}\right)}\right)
\end{array}\right\}\\
&= sG_b\left(b_{3j+2}, \delta, \delta\right) + s\gamma_1 G_b\left(\delta, b_{3j+1}, b_{3j+2}\right) + s\gamma_2 G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right)\\
&\quad + s\gamma_3 \min\left\{\begin{array}{c}
G_b\left(\delta, b_{3j+1}, b_{3j+2}\right), G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right), G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right),\\[2ex]
G_b\left(b_{3j+2}, b_{3j+2}, b_{3j+2}\right), G_b\left(b_{3j+2}, b_{3j+2}, b_{3j+2}\right),\\[2ex]
\left(\dfrac{G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+2}\right).G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+3}\right)}{1 + G_b\left(T_1\delta, b_{3j+2}, b_{3j+2}\right)}\right)
\end{array}\right\}.
\end{aligned} \tag{13}$$

After simplification, we obtain

$$\begin{aligned}
G_b\left(T_1\delta, b_{3j+2}, b_{3j+3}\right) \leq{}& sG_b\left(b_{3j+2}, \delta, \delta\right) + s\gamma_1 G_b\left(\delta, b_{3j+1}, b_{3j+2}\right)\\
&+ s\gamma_2 G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right).
\end{aligned} \tag{14}$$

Now, by taking limit $j \longrightarrow \infty$ and by using Proposition 3 (iii), we obtain

$$G_b(T_1\delta, \delta, \delta) \leq 2s^2\gamma_2 G_b(T_1\delta, \delta, \delta), \tag{15}$$

which implies that $(1 - 2s^2\gamma_2)G_b(T_1\delta, \delta, \delta) \leq 0$ is a contradiction, since $(1 - 2s^2\gamma_2) > 0$. Thus, $G_b(T_1\delta, \delta, \delta) = 0$, which yields that $T_1\delta = \delta$.

Next, we show that $T_2\delta = \delta$ by contrary case. Let $T_2\delta \neq \delta$, then again by using the rectangular property of (B, G_b) and by the view of (2), we have that

$$\begin{aligned}G_b(\delta, T_2\delta, \delta) &\leq sG_b(\delta, \delta, b_{3j+3}) + sG_b(b_{3j+3}, b_{3j+3}, T_2\delta) \leq sG_b(\delta, \delta, b_{3j+3}) + sG_b(b_{3j+1}, T_2\delta, b_{3j+3})\\ &= sG_b(\delta, \delta, b_{3j+3}) + sG_b(T_1 b_{3j}, T_2\delta, T_3 b_{3j+2}) \leq sG_b(\delta, \delta, b_{3j+3}) + s\gamma_1 G_b(b_{3j}, \delta, T_2\delta)\\ &\quad + s\gamma_2 G_b(T_1 b_{3j}, T_1 b_{3j}, \delta) + s\gamma_3 \min\left\{\begin{array}{c} G_b(b_{3j}, \delta, T_2\delta), G_b(T_1 b_{3j}, T_1 b_{3j}, \delta), G_b(T_1 b_{3j}, \delta, \delta),\\ G_b(T_2\delta, T_2\delta, b_{3j+2}), G_b(T_2\delta, b_{3j+2}, b_{3j+2}),\\ \left(\dfrac{G_b(\delta, T_2\delta, T_2\delta).G_b(b_{3j+2}, T_3 b_{3j+2}, T_3 b_{3j+2})}{1 + G_b(T_1 b_{3j}, b_{3j+2}, b_{3j+2})}\right)\end{array}\right\}\\ &= sG_b(\delta, \delta, b_{3j+3}) + s\gamma_1 G_b(b_{3j}, \delta, T_2\delta) + s\gamma_2 G_b(b_{3j+1}, b_{3j+1}, \delta)\\ &\quad + s\gamma_3 \min\left\{\begin{array}{c} G_b(b_{3j}, \delta, T_2\delta), G_b(b_{3j+1}, b_{3j+1}, \delta), G_b(b_{3j+1}, \delta, \delta),\\ G_b(T_2\delta, T_2\delta, b_{3j+2}), G_b(T_2\delta, b_{3j+2}, b_{3j+2}),\\ \left(\dfrac{G_b(\delta, T_2\delta, T_2\delta).G_b(b_{3j+2}, b_{3j+3}, b_{3j+3})}{1 + G_b(b_{3j+1}, b_{3j+2}, b_{3j+2})}\right)\end{array}\right\}.\end{aligned} \tag{16}$$

Now, by taking limit $j \longrightarrow \infty$, we obtain

$$G_b(\delta, T_2\delta, \delta) \leq s\gamma_1 G_b(\delta, \delta, T_2\delta). \tag{17}$$

Hence, $(1 - s\gamma_1)G_b(\delta, T_2\delta, \delta) \leq 0$ is a contradiction, since $(1 - s\gamma_1) > 0$. Thus, $G_b(\delta, T_2\delta, \delta) = 0$, which yields that $T_2\delta = \delta$.

Now, we have to show that $T_3\delta = \delta$ by contrary case. Let $T_3\delta \neq \delta$, then by using the rectangular property of (B, G_b) and by the view of (2), we have that

$$\begin{aligned}G_b(\delta, \delta, T_3\delta) &\leq sG_b(\delta, \delta, b_{3j+2}) + sG_b(b_{3j+2}, b_{3j+2}, T_3\delta) \leq sG_b(\delta, \delta, b_{3j+2}) + sG_b(b_{3j+1}, b_{3j+2}, T_3\delta)\\ &= sG_b(\delta, \delta, b_{3j+2}) + sG_b(T_1 b_{3j}, T_2 b_{3j+1}, T_3\delta) \leq sG_b(\delta, \delta, b_{3j+2}) + s\gamma_1 G_b(b_{3j}, b_{3j+1}, T_2 b_{3j+1})\\ &\quad + s\gamma_2 G_b(T_1 b_{3j}, T_1 b_{3j}, b_{3j+1}) + s\gamma_3 \min\left\{\begin{array}{c} G_b(b_{3j}, b_{3j+1}, T_2 b_{3j+1}), G_b(T_1 b_{3j}, T_1 b_{3j}, b_{3j+1}), G_b(T_1 b_{3j}, b_{3j+1}, b_{3j+1}),\\ G_b(T_2 b_{3j+1}, T_2 b_{3j+1}, \delta), G_b(T_2 b_{3j+1}, \delta, \delta),\\ \left(\dfrac{G_b(b_{3j+1}, T_2 b_{3j+1}, T_2 b_{3j+1}).G_b(\delta, T_3\delta, T_3\delta)}{1 + G_b(T_1 b_{3j}, \delta, \delta)}\right)\end{array}\right\}\\ &= sG_b(\delta, \delta, b_{3j+2}) + s\gamma_1 G_b(b_{3j}, b_{3j+1}, b_{3j+2}) + s\gamma_2 G_b(b_{3j+1}, b_{3j+1}, b_{3j+1})\\ &\quad + s\gamma_3 \min\left\{\begin{array}{c} G_b(b_{3j}, b_{3j+1}, b_{3j+2}), G_b(b_{3j+1}, b_{3j+1}, b_{3j+1}), G_b(b_{3j+1}, b_{3j+1}, b_{3j+1}),\\ G_b(b_{3j+2}, b_{3j+2}, \delta), G_b(b_{3j+2}, \delta, \delta),\\ \left(\dfrac{G_b(b_{3j+1}, b_{3j+2}, b_{3j+2}).G_b(\delta, T_3\delta, T_3\delta)}{1 + G_b(b_{3j+1}, \delta, \delta)}\right)\end{array}\right\}.\end{aligned} \tag{18}$$

After simplification, we obtain

$$G_b(\delta, \delta, T_3\delta) \le sG_b(\delta, \delta, b_{3j+2}) + s\gamma_1 G_b(b_{3j}, b_{3j+1}, b_{3j+2}). \tag{19}$$

Now, by taking limit $j \longrightarrow \infty$, we obtain $G_b(\delta, \delta, T_3\delta) \le 0$ which is a contradiction. Thus, $T_3\delta = \delta$. Hence, we have proved that "δ" is a CFP of T_1, T_2, and T_3; that is, $T_1\delta = T_2\delta = T_3\delta = \delta$.

To this end, we prove the uniqueness of the CFP. Assume that $\delta^* \in B$ is another CFP of the mappings T_1, T_2, and T_3; that is, $T_1\delta^* = T_2\delta^* = T_3\delta^* = \delta^*$. Then, from (2), we have that

$$\begin{aligned} G_b(\delta, \delta^*, \delta^*) &= G_b(T_1\delta, T_2\delta^*, T_3\delta^*) \le \gamma_1 G_b(\delta, \delta^*, T_2\delta^*) + \gamma_2 G_b(T_1\delta, T_1\delta, \delta^*) \\ &\quad + \gamma_3 \min\left\{ \begin{array}{c} G_b(\delta, \delta^*, T_2\delta^*), G_b(T_1\delta, T_1\delta, \delta^*), G_b(T_1\delta, \delta^*, \delta^*), \\ G_b(T_2\delta^*, T_2\delta^*, \delta^*), G_b(T_2\delta^*, \delta^*, \delta^*), \\ \dfrac{G_b(\delta^*, T_2\delta^*, T_2\delta^*).G_b(\delta^*, T_3\delta^*, T_3\delta^*)}{1 + G_b(T_1\delta, \delta^*, \delta^*)}, \end{array} \right\} \\ &= \gamma_1 G_b(\delta, \delta^*, \delta^*) + \gamma_2 G_b(\delta, \delta, \delta^*). \end{aligned} \tag{20}$$

Now, by using Proposition 3 (iii), we get that

$$G_b(\delta, \delta^*, \delta^*) \le \gamma_1 G_b(\delta, \delta^*, \delta^*) + 2s\gamma_2 G_b(\delta, \delta^*, \delta^*), \tag{21}$$

which implies that $(1 - \gamma_1 - 2s\gamma_2)G_b(\delta, \delta^*, \delta^*) \le 0$ is a contradiction, since $(1 - \gamma_1 - 2s\gamma_2) > 0$. Therefore, $G_b(\delta, \delta^*, \delta^*) = 0$, and so $\delta = \delta^*$. The proof is complete.

By taking $\gamma_2 = 0$, in Theorem 7, we get Corollary 8. □

Corollary 8. *Let (B, G_b) be a G_b M-space with coefficient $s > 1$ and T_1, T_2, T_3: $B \longrightarrow B$ be three self-mappings which satisfy*

$$G_b(T_1b_1, T_2b_2, T_3b_3) \le \gamma_1 G_b(b_1, b_2, T_2b_2) + \gamma_3 \min\left\{ \begin{array}{c} G_b(b_1, b_2, T_2b_2), G_b(T_1b_1, T_1b_2, b_2), G_b(T_1b_1, b_2, b_2), G_b(T_2b_2, T_2b_2, b_3), \\ G_b(T_2b_2, b_3, b_3), \left(\dfrac{G_b(b_2, T_2b_2, T_2b_2).G_b(b_3, T_3b_3, T_3b_3)}{1 + G_b(T_1b_1, b_3, b_3)}\right) \end{array} \right\}, \tag{22}$$

for all $b_1, b_2, b_3 \in B$, and $\gamma_1, \gamma_3 \in [0, 1)$ with $s\gamma_1 < 1$. Then, the three self-mappings T_1, T_2, and T_3 have a unique CFP in B.

By specializing $\gamma_1 = 0$, in Theorem 7, we get Corollary 9.

Corollary 9. *Let (B, G_b) be a G_b M-space with coefficient $s > 1$ and T_1, T_2, T_3: $B \longrightarrow B$ be three self-mappings which satisfy*

$$G_b(T_1b_1, T_2b_2, T_3b_3) \le \gamma_2 G_b(T_1b_1, T_1b_1, b_2) + \gamma_3 \min\left\{ \begin{array}{c} G_b(b_1, b_2, T_2b_2), G_b(T_1b_1, T_1b_1, b_2), G_b(T_1b_1, b_2, b_2), G_b(T_2b_2, T_2b_2, b_3), \\ G_b(T_2b_2, b_3, b_3), \left(\dfrac{G_b(b_2, T_2b_2, T_2b_2).G_b(b_3, T_3b_3, T_3b_3)}{1 + G_b(T_1b_1, b_3, b_3)}\right) \end{array} \right\}, \tag{23}$$

for all $b_1, b_2, b_3 \in B$, and $\gamma_2, \gamma_3 \in [0, 1)$ with $2s^2\gamma_2 < 1$. Then, the three self-mappings T_1, T_2, and T_3 have a unique CFP in B.

Theorem 10. *Let (B, G_b) be a complete G_b M-space with coefficient $s > 1$ and T_1, T_2, T_3: $B \longrightarrow B$ be three self-mappings which satisfy*

$$G_b(T_1b_1,T_2b_2,T_3b_3)\le \gamma_1 G_b(b_1,b_2,T_2b_2)+\gamma_2 G_b(T_1b_1,b_2,b_2)+\gamma_3\max\left\{\begin{array}{c} G_b(b_1,b_2,T_2b_2),G_b(T_1b_1,T_1b_1,b_2),G_b(T_1b_1,b_2,b_2),G_b(T_2b_2,T_2b_2,b_3),\\ G_b(T_2b_2,b_3,b_3),\left(\dfrac{G_b(b_2,T_2b_2,T_2b_2).G_b(b_3,T_3b_3,T_3b_3)}{1+G_b(T_1b_1,b_3,b_3)}\right)\end{array}\right\}, \tag{24}$$

$\forall\, b_1,b_2,b_3\in B$, $\gamma_1,\gamma_2,\gamma_3\ge 0$ with $(\gamma_1+\gamma_3)<1$, $(s\gamma_1+s\gamma_3)<(s\gamma_1+2s^2\gamma_3)<1$, and $(s\gamma_2+s\gamma_3)<(s\gamma_2+2s^2\gamma_3)<1$. Then, T_1,T_2, and T_3 have a CFP in B. Moreover, if $(\gamma_1+\gamma_2+2s\gamma_3)<1$, then T_1,T_2, and T_3 have a unique CFP in B.

Proof. Fix $b_0\in B$. We now define the iterative sequences in B as follows:

$$\begin{aligned} b_{3j+1}&=T_1b_{3j},\\ b_{3j+2}&=T_2b_{3j+1},\quad \forall j\ge 0,\\ b_{3j+3}&=T_3b_{3j+2}.\end{aligned} \tag{25}$$

Now, by using (24), we have

$$\begin{aligned} G_b(b_{3j+1},b_{3j+2},b_{3j+3})&=G_b(T_1b_{3j},T_2b_{3j+1},T_3b_{3j+2})\le \gamma_1G_b(b_{3j},b_{3j+1},T_2b_{3j+1})+\gamma_2G_b(T_1b_{3j},b_{3j+1},b_{3j+1})\\ &\quad+\gamma_3\max\left\{\begin{array}{c} G_b(b_{3j},b_{3j+1},T_2b_{3j+1}),G_b(T_1b_{3j},T_1b_{3j},b_{3j+1}),G_b(T_1b_{3j},b_{3j+1},b_{3j+1}),\\ G_b(T_2b_{3j+1},T_2b_{3j+1},b_{3j+2}),G_b(T_2b_{3j+1},b_{3j+2},b_{3j+2}),\\ \left(\dfrac{G_b(b_{3j+1},T_2b_{3j+1},T_2b_{3j+1}).G_b(b_{3j+2},T_3b_{3j+2},T_3b_{3j+2})}{1+G_b(T_1b_{3j},b_{3j+2},b_{3j+2})}\right)\end{array}\right\}\\ &=\gamma_1G_b(b_{3j},b_{3j+1},b_{3j+2})+\gamma_2G_b(b_{3j+1},b_{3j+1},b_{3j+1})\\ &\quad+\gamma_3\max\left\{\begin{array}{c} G_b(b_{3j},b_{3j+1},b_{3j+2}),G_b(b_{3j+1},b_{3j+1},b_{3j+1}),G_b(b_{3j+1},b_{3j+1},b_{3j+1}),\\ G_b(b_{3j+2},b_{3j+2},b_{3j+2}),G_b(b_{3j+2},b_{3j+2},b_{3j+2}),\\ \left(\dfrac{G_b(b_{3j+1},b_{3j+2},b_{3j+2}).G_b(b_{3j+2},b_{3j+3},b_{3j+3})}{1+G_b(b_{3j+1},b_{3j+2},b_{3j+2})}\right)\end{array}\right\}.\end{aligned} \tag{26}$$

However,

$$\frac{G_b(b_{3j+1},b_{3j+2},b_{3j+2})}{1+G_b(b_{3j+1},b_{3j+2},b_{3j+2})}\le 1. \tag{27}$$

And so,

$$\frac{G_b(b_{3j+1},b_{3j+2},b_{3j+2}).G_b(b_{3j+2},b_{3j+3},b_{3j+3})}{1+G_b(b_{3j+1},b_{3j+2},b_{3j+2})}\le G_b(b_{3j+2},b_{3j+3},b_{3j+3}). \tag{28}$$

Thus, by combining (26) and (28), we obtain

$$G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)\leq\gamma_1G_b\left(b_{3j},b_{3j+1},b_{3j+2}\right)+\gamma_3\max\left\{G_b\left(b_{3j},b_{3j+1},b_{3j+2}\right),G_b\left(b_{3j+2},b_{3j+3},b_{3j+3}\right)\right\}. \tag{29}$$

By using Definition 1 (iii), we obtain

$$G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)\leq\gamma_1G_b\left(b_{3j},b_{3j+1},b_{3j+2}\right)+\gamma_3\max\left\{G_b\left(b_{3j},b_{3j+1},b_{3j+2}\right),G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)\right\}. \tag{30}$$

Now, there are two possibilities:

Possibility I. If $\max\{G_b(b_{3j},b_{3j+1},b_{3j+2}), G_b(b_{3j+1},b_{3j+2},b_{3j+3})\}=G_b(b_{3j},b_{3j+1},b_{3j+2})$, then (30) becomes

$$G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)\leq\lambda_1G_b\left(b_{3j},b_{3j+1},b_{3j+2}\right),\text{where }\lambda_1=\left(\gamma_1+\gamma_3\right)<1. \tag{31}$$

Possibility II. If the $\max\left\{G_b(b_{3j},b_{3j+1},b_{3j+2}), G_b(b_{3j+1},b_{3j+2},b_{3j+3})\right\}=G_b(b_{3j+1},b_{3j+2},b_{3j+3})$, then (30) becomes

$$G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)\leq\lambda_2G_b\left(b_{3j},b_{3j+1},b_{3j+2}\right),\text{where }\quad\lambda_2=\frac{\gamma_1}{1-\gamma_3}<1. \tag{32}$$

From both cases, we obtain that

$$G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)\leq\lambda G_b\left(b_{3j},b_{3j+1},b_{3j+2}\right),\text{where }\lambda=\max\left\{\lambda_1,\lambda_2\right\}<1. \tag{33}$$

Similarly, again by using (24),

$$\begin{aligned}
G_b\left(b_{3j+2},b_{3j+3},b_{3j+4}\right)&=G_b\left(T_2b_{3j+1},T_3b_{3j+2},T_1b_{3(j+1)}\right)\leq\gamma_1G_b\left(b_{3j+1},b_{3j+2},T_3b_{3j+2}\right)+\gamma_2G_b\left(T_2b_{3j+1},b_{3j+2},b_{3j+2}\right)\\
&\quad+\gamma_3\max\left\{\begin{array}{c}G_b\left(b_{3j+1},b_{3j+2},T_3b_{3j+2}\right),G_b\left(T_2b_{3j+1},T_2b_{3j+1},b_{3j+2}\right),G_b\left(T_2b_{3j+1},b_{3j+2},b_{3j+2}\right),\\
G_b\left(T_3b_{3j+2},T_3b_{3j+2},b_{3(j+1)}\right),G_b\left(T_3b_{3j+2},b_{3(j+1)},b_{3(j+1)}\right),\\
\left(\dfrac{G_b\left(b_{3j+2},T_3b_{3j+2},T_3b_{3j+2}\right).G_b\left(b_{3(j+1)},T_1b_{3(j+1)},T_1b_{3(j+1)}\right)}{1+G_b\left(T_2b_{3j+1},b_{3(j+1)},b_{3(j+1)}\right)}\right)\end{array}\right\}\\
&=\gamma_1G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)+\gamma_2G_b\left(b_{3j+2},b_{3j+2},b_{3j+2}\right)\\
&\quad+\gamma_3\max\left\{\begin{array}{c}G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right),G_b\left(b_{3j+2},b_{3j+2},b_{3j+2}\right),G_b\left(b_{3j+2},b_{3j+2},b_{3j+2}\right),\\
G_b\left(b_{3j+3},b_{3j+3},b_{3(j+1)}\right),G_b\left(b_{3j+3},b_{3(j+1)},b_{3(j+1)}\right),\\
\left(\dfrac{G_b\left(b_{3j+2},b_{3j+3},b_{3j+3}\right).G_b\left(b_{3(j+1)},b_{3(j+1)+1},b_{3(j+1)+1}\right)}{1+G_b\left(b_{3j+2},b_{3(j+1)},b_{3(j+1)}\right)}\right)\end{array}\right\}\\
&=\gamma_1G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right)+\gamma_3\max\left\{G_b\left(b_{3j+1},b_{3j+2},b_{3j+3}\right),\left(\frac{G_b\left(b_{3j+2},b_{3j+3},b_{3j+3}\right).G_b\left(b_{3j+3},b_{3j+4},b_{3j+4}\right)}{1+G_b\left(b_{3j+2},b_{3j+3},b_{3j+3}\right)}\right)\right\}.
\end{aligned} \tag{34}$$

However,

$$\frac{G_b\left(b_{3j+3}, b_{3j+4}, b_{3j+4}\right)}{1+G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+3}\right)} \leq 1. \tag{35}$$

And so,

$$\frac{G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+3}\right).G_b\left(b_{3j+3}, b_{3j+4}, b_{3j+4}\right)}{1+G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+3}\right)} \leq G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+4}\right). \tag{36}$$

By combining (34) and (36), we obtain

$$G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+4}\right) \leq \gamma_1 G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+3}\right) + \gamma_3 \max\left\{G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+3}\right), G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+4}\right)\right\}. \tag{37}$$

By applying a similar argument as in the above two cases, we obtain

$$G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+4}\right) \leq \lambda G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+3}\right), \text{where } \lambda = \max\{\lambda_1, \lambda_2\} < 1. \tag{38}$$

By a similar argument as above, one can show that

$$G_b\left(b_{3j+3}, b_{3j+4}, b_{3j+5}\right) \leq \lambda G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+4}\right), \text{where } \lambda = \max\{\lambda_1, \lambda_2\} < 1. \tag{39}$$

Now, from (33), (38), and (39), we conclude that

$$G_b\left(b_{3j+3}, b_{3j+4}, b_{3j+5}\right) \leq \lambda G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+4}\right) \leq \cdots \leq \lambda^{3j+3} G_b\left(b_0, b_1, b_2\right) \longrightarrow 0, \text{as } j \longrightarrow \infty. \tag{40}$$

Hence, we have proved that the sequence $\left\{b_j\right\}$ is contractive under the G_b M-space for three self-mappings. Therefore,

$$\operatorname*{Lim}_{j\longrightarrow\infty} G_b\left(b_j, b_{j+1}, b_{j+2}\right) = 0. \tag{41}$$

Next, we will show that $\left\{b_j\right\}$ is a G_b-Cauchy sequence in B. For all $j, m \in \mathbb{N}$, and $m > j$, using the rectangle inequality, we have

$$\begin{aligned}
G_b\left(b_j, b_m, b_m\right) &\leq s\left[G_b\left(b_j, b_{j+1}, b_{j+1}\right) + G_b\left(b_{j+1}, b_m, b_m\right)\right] \\
&\leq sG_b\left(b_j, b_{j+1}, b_{j+1}\right) + s^2 G_b\left(b_{j+1}, b_{j+2}, b_{j+2}\right) + s^3 G_b\left(b_{j+2}, b_{j+3}, b_{j+3}\right) + \cdots + s^m G_b\left(b_{m-1}, b_m, b_m\right) \\
&\leq sG_b\left(b_j, b_{j+1}, b_{j+2}\right) + s^2 G_b\left(b_{j+1}, b_{j+2}, b_{j+3}\right) + s^3 G_b\left(b_{j+2}, b_{j+3}, b_{j+4}\right) + \cdots + s^m G_b\left(b_{m-1}, b_m, b_{m+1}\right) \\
&\leq s\lambda^j G_b\left(b_0, b_1, b_2\right) + s^2\lambda^{j+1} G_b\left(b_0, b_1, b_2\right) + s^3\lambda^{j+2} G_b\left(b_0, b_1, b_2\right) + \cdots + s^{m-j}\lambda^{m-1} G_b\left(b_0, b_1, b_2\right) \\
&\leq s\lambda^j G_b\left(b_0, b_1, b_2\right) + s^2\lambda^{j+1} G_b\left(b_0, b_1, b_2\right) + s^3\lambda^{j+2} G_b\left(b_0, b_1, b_2\right) + \cdots + s^{m+j-1}\lambda^{m-1} G_b\left(b_0, b_1, b_2\right) \\
&\leq s\lambda^j\left[G_b\left(b_0, b_1, b_2\right) + s\lambda^1 G_b\left(b_0, b_1, b_2\right) + s^2\lambda^2 G_b\left(b_0, b_1, b_2\right) + \cdots + s^{m-1}\lambda^{m-1} G_b\left(b_0, b_1, b_2\right)\right].
\end{aligned} \tag{42}$$

Since $\lambda < 1$, so the above inequality yields that

$$G_b\left(b_j, b_m, b_m\right) \leq \frac{s\lambda^j}{1 - s\lambda} G_b\left(b_0, b_1, b_2\right) \longrightarrow 0, \text{as } j \longrightarrow \infty. \tag{43}$$

By using Proposition 3 (ii), we have $G_b\left(b_j, b_l, b_m\right) \leq s\left[G_b\left(b_j, b_m, b_m\right) + G_b\left(b_l, b_m, b_m\right)\right]$ for $j, l, m \in \mathbb{N}$ with $j < l < m$. If we take the limit as $j, l, m \longrightarrow \infty$, we get $G_b\left(b_j, b_l, b_m\right) \longrightarrow 0$. Hence, $\{b_j\}$ is a G_b-Cauchy sequence. Since B is a complete G_b-metric space, there is $\delta \in B$, such that $b_j \longrightarrow \delta$ as $j \longrightarrow \infty$ or $\lim_{j \longrightarrow \infty} b_j = \delta$. We now show that $T_1\delta = \delta$ by contrary case, let $T_1\delta \neq \delta$. Then, by using the rectangular property of G_b-metric space and by the view of (24), we have that

$$\begin{aligned}
G_b\left(T_1\delta, \delta, \delta\right) &= sG_b\left(T_1\delta, b_{3j+2}, b_{3j+2}\right) + sG_b\left(b_{3j+2}, \delta, \delta\right) \leq sG_b\left(T_1\delta, b_{3j+2}, b_{3j+3}\right) + sG_b\left(b_{3j+2}, \delta, \delta\right) \\
&= sG_b\left(b_{3j+2}, \delta, \delta\right) + sG_b\left(T_1\delta, T_2 b_{3j+1}, T_3 b_{3j+2}\right) \leq sG_b\left(b_{3j+2}, \delta, \delta\right) + s\gamma_1 G_b\left(\delta, b_{3j+1}, T_2 b_{3j+1}\right) \\
&\quad + s\gamma_2 G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right) + s\gamma_3 \max\left\{\begin{array}{c}
G_b\left(\delta, b_{3j+1}, T_2 b_{3j+1}\right), G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right), G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right), \\
G_b\left(T_2 b_{3j+1}, T_2 b_{3j+1}, b_{3j+2}\right), G_b\left(T_2 b_{3j+1}, b_{3j+2}, b_{3j+2}\right), \\
\left(\dfrac{G_b\left(b_{3j+1}, T_2 b_{3j+1}, T_2 b_{3j+1}\right).G_b\left(b_{3j+2}, T_3 b_{3j+2}, T_3 b_{3j+2}\right)}{1 + G_b\left(T_1\delta, b_{3j+2}, b_{3j+2}\right)}\right)
\end{array}\right\} \\
&= sG_b\left(b_{3j+2}, \delta, \delta\right) + s\gamma_1 G_b\left(\eta, b_{3j+1}, b_{3j+2}\right) + s\gamma_2 G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right) \\
&\quad + s\gamma_3 \max\left\{\begin{array}{c}
G_b\left(\delta, b_{3j+1}, b_{3j+2}\right), G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right), G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right), \\
G_b\left(b_{3j+2}, b_{3j+2}, b_{3j+2}\right), G_b\left(b_{3j+2}, b_{3j+2}, b_{3j+2}\right), \\
\left(\dfrac{G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+2}\right).G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+3}\right)}{1 + G_b\left(T_1\delta, b_{3j+2}, b_{3j+2}\right)}\right)
\end{array}\right\}.
\end{aligned} \tag{44}$$

After simplification, we obtain

$$\begin{aligned}
G_b\left(T_1\delta, \delta, \delta\right) &\leq sG_b\left(b_{3j+2}, \delta, \delta\right) + s\gamma_1 G_b\left(\delta, b_{3j+1}, b_{3j+2}\right) + s\gamma_2 G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right) \\
&\quad + s\gamma_3 \max\left\{\begin{array}{c}
G_b\left(\delta, b_{3j+1}, b_{3j+2}\right), G_b\left(T_1\delta, T_1\delta, b_{3j+1}\right), G_b\left(T_1\delta, b_{3j+1}, b_{3j+1}\right), \\
\left(\dfrac{G_b\left(b_{3j+1}, b_{3j+2}, b_{3j+2}\right).G_b\left(b_{3j+2}, b_{3j+3}, b_{3j+3}\right)}{1 + G_b\left(T_1\delta, b_{3j+2}, b_{3j+2}\right)}\right)
\end{array}\right\}.
\end{aligned} \tag{45}$$

By taking limit $j \longrightarrow \infty$, we obtain

$$G_b\left(T_1\delta, \delta, \delta\right) = s\gamma_2 G_b\left(T_1\delta, \delta, \delta\right) + s\gamma_3 \max\left\{G_b\left(T_1\delta, T_1\delta, \delta\right), G_b\left(T_1\delta, \delta, \delta\right)\right\}. \tag{46}$$

To this end, we have two possibilities to consider:

Possibility I. If $G_b(T_1\delta, T_1\delta, \delta)$ be the maximum term, then

$$G_b(T_1\delta,\delta,\delta) = s\gamma_2 G_b(T_1\delta,\delta,\delta) + s\gamma_3 G_b(T_1\delta,T_1\delta,\delta) \le s\gamma_2 G_b(T_1\delta,\delta,\delta) + 2s^2\gamma_3 G_b(T_1\delta,\delta,\delta). \tag{47}$$

And so, $(1 - s\gamma_2 - 2s^2\gamma_3)G_b(T_1\delta,\delta,\delta) \le 0$ is a contradiction, since $(1 - s\gamma_2 - 2s^2\gamma_3) > 0$. Thus, $T_1\delta = \delta$.

Possibility II. If $G_b(T_1\delta, \delta, \delta)$ be the maximum term, then

$$G_b(T_1\delta,\delta,\delta) = s\gamma_2 G_b(T_1\delta,\delta,\delta) + s\gamma_3 G_b(T_1\delta,\delta,\delta). \tag{48}$$

And so, $(1 - s\gamma_2 - s\gamma_3)G_b(T_1\delta,\delta,\delta) \le 0$ is a contradiction, since $(1 - s\gamma_2 - s\gamma_3) > 0$. Thus, $T_1(\delta) = \delta$. Hence, from both possibilities, we get that $T_1\delta = \delta$.

Next, we show that $T_2\delta = \delta$ by contrary case and let $T_2\delta \ne \delta$. Then, by using the rectangular inequality of G_b-metric space and by the view of (13), we have that

$$\begin{aligned}
G_b(\delta,T_2\delta,\delta) &\le sG_b\left(T_2\delta,b_{3j+3},b_{3j+3}\right) + sG_b\left(b_{3j+3},\delta,\delta\right) \le sG_b\left(b_{3j+1},T_2\delta,b_{3j+3}\right) + sG_b\left(b_{3j+3},\delta,\delta\right)\\
&= sG_b\left(b_{3j+3},\delta,\delta\right) + sG_b\left(T_1b_{3j},T_2\delta,T_3b_{3j+2}\right) \le sG_b\left(b_{3j+3},\delta,\delta\right) + s\gamma_1 G_b\left(b_{3j},\delta,T_2\delta\right) + s\gamma_2 G_b\left(T_1b_{3j},\delta,\delta\right)\\
&\quad + s\gamma_3 \max\left\{\begin{array}{c} G_b\left(b_{3j},\delta,T_2\delta\right), G_b\left(T_1b_{3j},T_1b_{3j},\delta\right), G_b\left(T_1b_{3j},\delta,\delta\right),\\ G_b\left(T_2\delta,T_2\delta,b_{3j+2}\right), G_b\left(T_2\delta,b_{3j+2},b_{3j+2}\right),\\ \left(\dfrac{G_b(\delta,T_2\delta,T_2\delta).G_b\left(b_{3j+2},T_3b_{3j+2},T_3b_{3j+2}\right)}{1+G_b\left(T_1b_{3j},b_{3j+2},b_{3j+2}\right)}\right)\end{array}\right\}\\
&= sG_b\left(b_{3j+3},\delta,\delta\right) + s\gamma_1 G_b\left(b_{3j},\delta,T_2\delta\right) + s\gamma_2 G_b\left(b_{3j+1},\delta,\delta\right)\\
&\quad + s\gamma_3 \max\left\{\begin{array}{c} G_b\left(b_{3j},\delta,T_2\delta\right), G_b\left(b_{3j+1},b_{3j+1},\delta\right), G_b\left(b_{3j+1},\delta,\delta\right),\\ G_b\left(T_2\delta,T_2\delta,b_{3j+2}\right), G_b\left(T_2\delta,b_{3j+2},b_{3j+2}\right),\\ \left(\dfrac{G_b(\delta,T_2\delta,T_2\delta).G_b\left(b_{3j+2},b_{3j+3},b_{3j+3}\right)}{1+G_b\left(b_{3j+1},b_{3j+2},b_{3j+2}\right)}\right)\end{array}\right\}.
\end{aligned} \tag{49}$$

By taking limit $j \longrightarrow \infty$ and after simplification, we obtain

$$G_b(\delta,T_2\delta,\delta) \le s\gamma_1 G_b(\delta,\delta,T_2\delta) + s\gamma_3 \max\{G_b(\delta,\delta,T_2\delta), G_b(T_2\delta,T_2\delta,\delta)\}. \tag{50}$$

Now, as above, there are two possibilities:

Possibility I. If $\{G_b(\delta,\delta,T_2\delta), G_b(T_2\delta,T_2\delta,\delta)\} = G_b(\delta,\delta,T_2\delta)$, then (50) implies

$$G_b(\delta,T_2\delta,\delta) \le s\gamma_1 G_b(\delta,\delta,T_2\delta) + s\gamma_3 G_b(\delta,\delta,T_2\delta). \tag{51}$$

And so, $(1 - s\gamma_1 - s\gamma_3)G_b(\delta,T_2\delta,\delta) \le 0$ is a contradiction, since $(1 - s\gamma_1 - s\gamma_3) > 0$. Thus, $T_2\delta = \delta$.

Possibility II. If $\{G_b(\delta,\delta,T_2\delta), G_b(T_2\delta,T_2\delta,\delta)\} = G_b(T_2\delta,T_2\delta,\delta)$, then (28) implies

$$\begin{aligned} G_b(\delta,T_2\delta,\delta) &\le s\gamma_1 G_b(\delta,\delta,T_2\delta) + s\gamma_3 G_b(T_2\delta,T_2\delta,\delta) \\ &\le s\gamma_1 G_b(\delta,\delta,T_2\delta) + 2s^2\gamma_3 G_b(T_2\delta,\delta,\delta). \end{aligned} \tag{52}$$

And so, $(1 - s\gamma_1 - 2s^2\gamma_3)G_b(\delta,T_2\delta,\delta) \le 0$ is a contradiction, since $(1 - s\gamma_1 - 2s^2\gamma_3) > 0$. Thus, $T_2\delta = \delta$. Hence, from both possibilities, we get that $T_2\delta = \delta$.

Now, we have to show that $T_3\delta = \delta$ by contrary case and let $T_3\delta \ne \delta$. By using (24), we have that

$$\begin{aligned} G_b(\delta,\delta,T_3\delta) &\le sG_b(\delta,\delta,b_{3j+1}) + sG_b(b_{3j+1},b_{3j+1},T_3\delta) \le sG_b(\delta,\delta,b_{3j+1}) + sG_b(b_{3j+1},b_{3j+2},T_3\delta) \\ &= sG_b(\delta,\delta,b_{3j+1}) + sG_b(T_1b_{3j},T_2b_{3j+1},T_3\delta) \le sG_b(\delta,\delta,b_{3j+1}) + s\gamma_1 G_b(b_{3j},b_{3j+1},T_2b_{3j+1}) \\ &\quad + s\gamma_2 G_b(T_1b_{3j},b_{3j+1},b_{3j+1}) + s\gamma_3 \max\left\{\begin{array}{c} G_b(b_{3j},b_{3j+1},T_2b_{3j+1}), G_b(T_1b_{3j},T_1b_{3j},b_{3j+1}), G_b(T_1b_{3j},b_{3j+1},b_{3j+1}), \\ G_b(T_2b_{3j+1},T_2b_{3j+1},\delta), G_b(T_2b_{3j+1},\delta,\delta), \\ \left(\dfrac{G_b(b_{3j+1},T_2b_{3j+1},T_2b_{3j+1}).G_b(\delta,T_3\delta,T_3\delta)}{1+G_b(T_1b_{3j},\delta,\delta)}\right) \end{array}\right\} \\ &= sG_b(\delta,\delta,b_{3j+1}) + s\gamma_1 G_b(b_{3j},b_{3j+1},b_{3j+2}) + s\gamma_2 G_b(b_{3j+1},b_{3j+1},b_{3j+1}) \\ &\quad + s\gamma_3 \max\left\{\begin{array}{c} G_b(b_{3j},b_{3j+1},b_{3j+2}), G_b(b_{3j+1},b_{3j+1},b_{3j+1}), G_b(b_{3j+1},b_{3j+1},b_{3j+1}), \\ G_b(b_{3j+2},b_{3j+2},\delta), G_b(b_{3j+2},\delta,\delta), \\ \left(\dfrac{G_b(b_{3j+1},b_{3j+2},b_{3j+2}).G_b(\delta,T_3\delta,T_3\delta)}{1+G_b(b_{3j+1},\delta,\delta)}\right) \end{array}\right\} \end{aligned} \tag{53}$$

After simplification, we obtain

$$G_b(\delta,\delta,T_3\delta) \le sG_b(\delta,\delta,b_{3j+1}) + s\gamma_1 G_b(b_{3j},b_{3j+1},b_{3j+2}) + s\gamma_3 \max\left\{\begin{array}{c} G_b(b_{3j},b_{3j+1},b_{3j+2}), G_b(b_{3j+2},b_{3j+2},\delta), G_b(b_{3j+2},\delta,\delta), \\ \left(\dfrac{G_b(b_{3j+1},b_{3j+2},b_{3j+2}).G_b(\delta,T_3\delta,T_3\delta)}{1+G_b(b_{3j+1},\delta,\delta)}\right) \end{array}\right\}. \tag{54}$$

By taking limit $j \longrightarrow \infty$, we get $G_b(\delta,\delta,T_3\delta) \le 0$, which is a contradiction. Thus, $T_3\delta = \delta$. Hence, it is proved that "δ" is a CFP of T_1, T_2, and T_3, that is, $T_1\delta = T_2\delta = T_3\delta = \delta$.

We now show the uniqueness. Assume that $\delta^* \in B$ is another CFP of the mappings T_1, T_2, and T_3 such that

$$T_1\delta^* = T_2\delta^* = T_3\delta^* = \delta^*. \quad (55)$$

Then, from (24), we have that

$$\begin{aligned} G_b(\delta,\delta^*,\delta^*) &= G_b(T_1\delta,T_2\delta^*,T_3\delta^*) \le \gamma_1 G_b(\delta,\delta^*,T_2\delta^*) + \gamma_2 G_b(T_1\delta,\delta^*,\delta^*) \\ &\quad + \gamma_3 \max\left\{\begin{array}{c} G_b(\delta,\delta^*,T_2\delta^*), G_b(T_1\delta,T_1\delta,\delta^*), G_b(T_1\delta,\delta^*,\delta^*), \\ G_b(T_2\delta^*,T_2\delta^*,\delta^*), G_b(T_2\delta^*,\delta^*,\delta^*), \\ \dfrac{G_b(\delta^*,T_2\delta^*,T_2\delta^*).G_b(\delta^*,T_3\delta^*,T_3\delta^*)}{1+G_b(T_1\delta,\delta^*,\delta^*)}, \end{array}\right\} \\ &= \gamma_1 G_b(\delta,\delta^*,\delta^*) + \gamma_2 G_b(\delta,\delta^*,\delta^*) + \gamma_3 \max\left\{\begin{array}{c} G_b(\delta,\delta^*,\delta^*), G_b(\delta,\delta,\delta^*), G_b(\delta,\delta^*,\delta^*), G_b(\delta^*,\delta^*,\delta^*), \\ G_b(\delta^*,\delta^*,\delta^*), \dfrac{G_b(\delta^*,\delta^*,\delta^*).G_b(\delta^*,\delta^*,\delta^*)}{1+G_b(\delta,\delta^*,\delta^*)} \end{array}\right\}. \end{aligned} \quad (56)$$

After simplification, we obtain

$$G_b(\delta,\delta^*,\delta^*) \le (\gamma_1+\gamma_2)G_b(\delta,\delta^*,\delta^*) + \gamma_3 \max\{G_b(\delta,\delta^*,\delta^*), G_b(\delta,\delta,\delta^*)\}. \quad (57)$$

By using Proposition 3 (iii), we have

$$G_b(\delta,\delta^*,\delta^*) \le (\gamma_1+\gamma_2)G_b(\delta,\delta^*,\delta^*) + \gamma_3 \max\{G_b(\delta,\delta^*,\delta^*), 2sG_b(\delta,\delta^*,\delta^*)\}. \quad (58)$$

Note that $2sG_b(\delta,\delta^*,\delta^*)$ is a maximum term in (58); therefore,

$$G_b(\delta,\delta^*,\delta^*) \le (\gamma_1+\gamma_2+2s\gamma_3)G_b(\delta,\delta^*,\delta^*), \quad (59)$$

which implies that $(1-\gamma_1-\gamma_2-2s\gamma_3)G_b(\delta,\delta^*,\delta^*) \le 0$ is a contradiction, since $(1-\gamma_1-\gamma_2-2s\gamma_3) > 0$; therefore, $\delta = \delta^*$. Hence, the mappings T_1, T_2, and T_3 have a unique CFP in B. The proof is complete.

By taking $\gamma_2 = 0$, in Theorem 10, we get Corollary 11. □

Corollary 11. *Let (B, G_b) be a complete G_b M-space with coefficient $s > 1$ and T_1, T_2, T_3: $B \longrightarrow B$ be three self-mappings which satisfy*

$$G_b(T_1b_1,T_2b_2,T_3b_3) \le \gamma_1 G_b(b_1,b_2,T_2b_2) + \gamma_3 \max\left\{\begin{array}{c} G_b(b_1,b_2,T_2b_2), G_b(T_1b_1,T_1b_1,b_2), G_b(T_1b_1,b_2,b_2), G_b(T_2b_2,T_2b_2,b_3), \\ G_b(T_2b_2,b_3,b_3), \left(\dfrac{G_b(b_2,T_2b_2,T_2b_2).G_b(b_3,T_3b_3,T_3b_3)}{1+G_b(T_1b_1,b_3,b_3)}\right) \end{array}\right\}, \quad (60)$$

$\forall b_1, b_2, b_3 \in B$, $\gamma_1, \gamma_3 \ge 0$ with $(\gamma_1+\gamma_3) < 1$ and $(s\gamma_1 + s\gamma_3) < (s\gamma_1 + 2s^2\gamma_3) < 1$. Moreover, if $(\gamma_1 + 2s\gamma_3) < 1$, then the three self-mappings T_1, T_2, and T_3 have a unique CFP in B.

Specializing $\gamma_1 = 0$, in Theorem 10, we get Corollary 12.

Corollary 12. *Let (B, G_b) be a complete G_b M-space with coefficient $s > 1$ and $T_1, T_2, T_3\colon B \longrightarrow B$ be three self-mappings which satisfy*

$$G_b(T_1b_1, T_2b_2, T_3b_3) \le \gamma_2 G_b(T_1b_1, b_2, b_2) + \gamma_3 \max\left\{\begin{array}{c} G_b(b_1, b_2, T_2b_2), G_b(T_1b_1, T_1b_1, b_2), G_b(T_1b_1, b_2, b_2), G_b(T_2b_2, T_2b_2, b_3), \\ G_b(T_2b_2, b_3, b_3), \left(\dfrac{G_b(b_2, T_2b_2, T_2b_2).G_b(b_3, T_3b_3, T_3b_3)}{1 + G_b(T_1b_1, b_3, b_3)}\right) \end{array}\right\}, \tag{61}$$

$\forall\, b_1, b_2, b_3 \in B$, $\gamma_2, \gamma_3 \ge 0$ with $(s\gamma_2 + s\gamma_3) < (s\gamma_2 + 2s^2\gamma_3) < 1$. Moreover, if $(\gamma_2 + 2s\gamma_3) < 1$, then the three self-mappings T_1, T_2, and T_3 have a unique CFP in B.

Example 2. Let (B, G_b) be a G_b *M*-space where $B = [0, \infty)$ and $G_b\colon B \times B \times B \longrightarrow \mathbb{R}$ is defined as follows:

$$G_b(b_1, b_2, b_3) = \max\{|b_1 - b_2|, |b_2 - b_3|, |b_3 - b_1|\}, \tag{62}$$

for all $b_1, b_2, b_3 \in B$. Now, we define the three self-mappings, that is, $T_1, T_2, T_3\colon B \longrightarrow B$ by

$$\begin{aligned} T_1b_1 &= \begin{cases} \dfrac{b_1}{10} + \dfrac{9}{10}, & \text{for } b_1 \in [0,\ 1], \\ \dfrac{1}{7}(5b_1 + 22), & \text{for } b_1 \in (1, \infty), \end{cases} \\ T_2b_2 &= \begin{cases} \dfrac{4b_2}{5} + \dfrac{9}{10}, & \text{for } b_2 \in [0,\ 1], \\ \dfrac{1}{3}(2b_2 + 11), & \text{for } b_2 \in (1, \infty), \end{cases} \\ T_3b_3 &= \begin{cases} \dfrac{3b_3}{5} + \dfrac{9}{10}, & \text{for } b_3 \in [0,\ 1], \\ \dfrac{1}{8}(7b_3 + 11), & \text{for } b_3 \in (1, \infty). \end{cases} \end{aligned} \tag{63}$$

Now, first, we calculate all the terms of (24) and we have that

$$\begin{aligned} G_b(T_1b_1, T_2b_2, T_3b_3) &= \max\{|T_1b_1 - T_2b_2|, |T_2b_2 - T_3b_3|, |T_3b_3 - T_1b_1|\} \\ &= \frac{1}{10}\max\{|b_1 - 8b_2|, |8b_2 - 3b_3|, |3b_3 - b_1|\}. \end{aligned} \tag{64}$$

Also,

$$\begin{aligned} G_b(b_1, b_2, T_2b_2) &= \max\{|b_1 - b_2|, |b_2 - T_2b_2|, |T_2b_2 - b_1|\} \\ &= \frac{1}{5}\max\left\{5|b_1 - b_2|, \left|b_2 + \frac{9}{2}\right|, \left|4b_2 + \frac{9}{2} - 5b_1\right|\right\}. \end{aligned} \tag{65}$$

Similarly, we can calculate the remaining terms of (13) as follows:

$$\begin{aligned}
&G_b(T_1b_1, T_1b_1, b_2) = G_b(T_1b_1, b_2, b_2) = |T_1b_1 - b_2| = \frac{1}{10}|b_1 + 9 - 10b_2|,\\
&G_b(T_2b_2, T_2b_2, b_3) = G_b(T_2b_2, b_3, b_3) = |T_2b_2 - b_3| = \frac{1}{10}|8b_2 + 9 - 10b_3|,\\
&G_b(b_2, T_2b_2, T_2b_2) = |T_2b_2 - b_2| = \frac{1}{10}|9 - 2b_2|, G_b(b_3, T_2b_2, T_2b_2) = |T_2b_2 - b_3| = \frac{1}{10}|9 - 7b_3|,
\end{aligned} \tag{66}$$

and $G_b(T_1b_1, b_3, b_3) = |T_1b_1 - b_3| = 1/10|b_1 + 9 - 10b_3|$. Now, from (24), (64), and (65), we have that

$$\begin{aligned}
G_b(T_1b_1, T_2b_2, T_3b_3) &= \frac{1}{10}\max\{|b_1 - 8b_2|, |8b_2 - 3b_3|, |3b_3 - b_1|\} \le \frac{2}{15}\max\left\{5|b_1 - b_2|, \left|b_2 + \frac{9}{2}\right|, \left|4b_2 + \frac{9}{2} - 5b_1\right|\right\}\\
&= \frac{2}{3}\left(\frac{1}{5}\max\left\{5|b_1 - b_2|, \left|b_2 + \frac{9}{2}\right|, \left|4b_2 + \frac{9}{2} - 5b_1\right|\right\}\right)\\
&\le \frac{2}{3}G_b(b_1, b_2, T_2b_2) + \frac{1}{20}G_b(T_1b_1, b_2, b_2)\\
&\quad + \frac{1}{30}\max\left\{\begin{array}{c} G_b(b_1, b_2, T_2b_2), G_b(T_1b_1, T_1b_1, b_2), G_b(T_1b_1, b_2, b_2), G_b(T_2b_2, T_2b_2, b_3),\\ G_b(T_2b_2, b_3, b_3), \left(\dfrac{G_b(b_2, T_2b_2, T_2b_2).G_b(b_3, T_3b_3, T_3b_3)}{1 + G_b(T_1b_1, b_3, b_3)}\right)\end{array}\right\}.
\end{aligned} \tag{67}$$

Hence, it is proved that all the conditions of Theorem 10 are satisfied with $\gamma_1 = 2/3$, $\gamma_2 = 1/20$, $\gamma_3 = 1/30$ and $s = 5/4 > 1$, that is, $(\gamma_1 + \gamma_3) = 7/10 < 1$, $(s\gamma_1 + s\gamma_3) = 7/8 < (s\gamma_1 + 2s^2\gamma_3) = 15/16 < 1$, $(s\gamma_2 + s\gamma_3) = 5/48 < (s\gamma_2 + 2s^2\gamma_3) = 11/96 < 1$, and $(\gamma_1 + \gamma_2 + 2s\gamma_3) = 4/5 < 1$. The three self-mappings T_1, T_2, and T_3 have a unique CFP in B, which is $11 \in [0, \infty)$.

4. Application

In this section, we present an application to nonlinear integral equations (NLIEs) to support our results. The considered system of NLIEs is of the form as follows:

$$\begin{aligned}
b_1(\mu) &= \int_{h_1}^{h_2} \tau_1(\mu, \nu, b_1(\nu))d\nu,\\
b_2(\mu) &= \int_{h_1}^{h_2} \tau_2(\mu, \nu, b_2(\nu))d\nu,\\
b_3(\mu) &= \int_{h_1}^{h_2} \tau_3(\mu, \nu, b_3(\nu))d\nu,
\end{aligned} \tag{68}$$

where $\mu \in [h_1, h_2]$, for all $b_1, b_2, b_3 \in B$ where $B = C([h_1, h_2], \mathbb{R})$ is the set of all real-valued continuous functions on $[h_1, h_2]$ and τ_1, τ_2, τ_3: $[h_1, h_2] \times [h_1, h_2] \times \mathbb{R} \longrightarrow \mathbb{R}$. A G_b−metric G_b: $B \times B \times B \longrightarrow \mathbb{R}$ be defined as follows:

$$G_b(b_1, b_2, b_3) = \left(\|b_1 - b_2\| + \|b_2 - b_3\| + \|b_3 - b_1\|\right) \text{ for all } b_1, b_2, b_3 \in B. \tag{69}$$

Then, easily one can prove that (B, G_b) is a complete G_b−metric space. Now, we establish a theorem based on NLIEs to achieve the previous results on the existence of a common solution to support our work.

Theorem 13. *Let $T_1, T_2, T_3: B \longrightarrow B$ be three self-mappings and let there exist $\beta \in (0,1)$ satisfying*

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \beta \mathbb{M}(\mathbb{T}, b_1, b_2, b_3) \quad \text{for all } b_1, b_2, b_3 \in B, \tag{70}$$

where $\mathbb{T} = T_1, T_2, T_3$ and

$$\mathbb{M}(\mathbb{T}, b_1, b_2, b_3) = \max\{M_1(\mathbb{T}, b_1, b_2, b_3), M_2(\mathbb{T}, b_1, b_2, b_3), M_3(\mathbb{T}, b_1, b_2, b_3)\}. \tag{71}$$

Now, we define $M_1(\mathbb{T}, b_1, b_2, b_3) = k_1(\mathbb{T}, b_1, b_2, b_3)$, $M_2(\mathbb{T}, b_1, b_2, b_3) = k_2(\mathbb{T}, b_1, b_2, b_3)$, and

$$M_3(\mathbb{T}, b_1, b_2, b_3) = \min\left\{\begin{array}{l} k_1(\mathbb{T}, b_1, b_2, b_3), k_2(\mathbb{T}, b_1, b_2, b_3), k_3(\mathbb{T}, b_1, b_2, b_3), \\ k_4(\mathbb{T}, b_1, b_2, b_3), k_5(\mathbb{T}, b_1, b_2, b_3), k_6(\mathbb{T}, b_1, b_2, b_3) \end{array}\right\}, \tag{72}$$

where

$$\begin{aligned} k_1(\mathbb{T}, b_1, b_2, b_3) &= \|b_1 - b_2\| + \|b_2 - T_2b_2\| + \|T_2b_2 - b_1\|, \\ k_2(\mathbb{T}, b_1, b_2, b_3) &= k_3(\mathbb{T}, b_1, b_2, b_3) = 2\|T_1b_1 - b_2\|, \\ k_4(\mathbb{T}, b_1, b_2, b_3) &= k_5(\mathbb{T}, b_1, b_2, b_3) = 2\|T_2b_2 - b_3\|, \\ k_6(\mathbb{T}, b_1, b_2, b_3) &= \frac{4\|T_2b_2 - b_2\|.\|T_3b_3 - b_3\|}{1 + 2(\|T_1b_1 - b_3\|)}, \end{aligned} \tag{73}$$

for all $b_1, b_2, b_3 \in B$. Then, the system of NLIEs (68) has a unique common solution.

Proof. The integral operators $T_1, T_2, T_3: B \longrightarrow B$ be defined as follows:

$$\begin{aligned} (T_1b_1)(\mu) = T_1b_1 &= \int_{h_1}^{h_2} \tau_1(\mu, \nu, b_1(\nu))d\nu, \\ (T_2b_2)(\mu) = T_2b_2 &= \int_{h_1}^{h_2} \tau_2(\mu, \nu, b_2(\nu))d\nu, \\ (T_3b_3)(\mu) = T_3b_3 &= \int_{h_1}^{h_2} \tau_3(\mu, \nu, b_3(\nu))d\nu. \end{aligned} \tag{74}$$

Now, we apply Theorem 7. Then, we may have the following three cases:

(1) If $M_1(\mathbb{T}, b_1, b_2, b_3)$ be the maximum term in (71), then, from (69) and (70), we have that

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \beta(\|b_1 - b_2\| + \|b_2 - T_2b_2\| + \|T_2b_2 - b_1\|), \quad \forall b_1, b_2, b_3 \in B. \tag{75}$$

Thus, the mappings T_1, T_2, and T_3 satisfy all the conditions of Theorem 7 with $\beta = \gamma_1$ and $\gamma_2 = \gamma_3 = 0$ in (2). Then, the given NLIEs, i.e., (35) have a unique common solution in B.

(2) If $M_2(\mathbb{T}, b_1, b_2, b_3)$ be the maximum term in (71), then, from (69) and (70), we have that

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \beta(2\|T_1b_1 - b_2\|), \quad \forall b_1, b_2, b_3 \in B. \tag{76}$$

Thus, the mappings T_1, T_2, and T_3 satisfy all the conditions of Theorem 7 with $\beta = \gamma_2$ and $\gamma_1 = \gamma_3 = 0$, in (2). Then, the given NLIEs (35) have a unique common solution in B.

(3) If $M_3(\mathbb{T}, b_1, b_2, b_3)$ be the maximum term in (71), then

$$\mathbb{M}(\mathbb{T}, b_1, b_2, b_3) = \min\left\{\begin{array}{l} k_1(\mathbb{T}, b_1, b_2, b_3), k_2(\mathbb{T}, b_1, b_2, b_3), k_3(\mathbb{T}, b_1, b_2, b_3), \\ k_4(\mathbb{T}, b_1, b_2, b_3), k_5(\mathbb{T}, b_1, b_2, b_3), k_6(\mathbb{T}, b_1, b_2, b_3) \end{array}\right\}. \tag{77}$$

Then further, we may have occurrence of the following four subcases:

$\mathbf{3}_{\mathbf{(i)}}$ if $k_1(\mathbb{T}, b_1, b_2, b_3)$ be the minimum term in (77), then $\mathbb{M}(\mathbb{T}, b_1, b_2, b_3) = k_1(\mathbb{T}, b_1, b_2, b_3)$. Now, from (69) and (70), we have that

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \beta\left(\|b_1 - b_2\| + \|b_2 - T_2b_2\| + \|T_2b_2 - b_1\|\right), \quad \forall b_1, b_2, b_3 \in B. \tag{78}$$

$\mathbf{3}_{\mathbf{(ii)}}$ if $k_2(\mathbb{T}, b_1, b_2, b_3)$ be the minimum term in (77) and $k_2(\mathbb{T}, b_1, b_2, b_3) = k_3(\mathbb{T}, b_1, b_2, b_3)$, then $\mathbb{M}(\mathbb{T}, b_1, b_2, b_3) = k_2(\mathbb{T}, b_1, b_2, b_3)$. Now, from (69) and (70), we have that

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \beta\left(2\|T_1b_1 - b_2\|\right), \quad \forall b_1, b_2, b_3 \in B. \tag{79}$$

$\mathbf{3}_{\mathbf{(iii)}}$ if $k_4(\mathbb{T}, b_1, b_2, b_3)$ be the minimum term in (77) and $k_4(\mathbb{T}, b_1, b_2, b_3) = k_5(\mathbb{T}, b_1, b_2, b_3)$, then $\mathbb{M}(\mathbb{T}, b_1, b_2, b_3) = k_4(\mathbb{T}, b_1, b_2, b_3)$. Now, from (69) and (70), we have that

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \beta\left(2\|T_2b_2 - b_3\|\right), \quad \forall b_1, b_2, b_3 \in B. \tag{80}$$

$\mathbf{3}_{\mathbf{(vi)}}$ if $k_6(\mathbb{T}, b_1, b_2, b_3)$ be the minimum term in (77), then $\mathbb{M}(\mathbb{T}, b_1, b_2, b_3) = k_6(\mathbb{T}, b_1, b_2, b_3)$. Now, from (69) and (70), we have that

$$G_b(T_1b_1, T_2b_2, T_3b_3) \leq \beta\left(\frac{4\|T_2b_2 - b_2\|.\|T_3b_3 - b_3\|}{1 + 2\left(\|T_1b_1 - b_3\|\right)}\right), \quad \forall b_1, b_2, b_3 \in B. \tag{81}$$

Thus, the subcases ($\mathbf{3}_{\mathbf{(i)}}$–$\mathbf{3}_{\mathbf{(iv)}}$) satisfying all the conditions of Theorem 7 with $\beta = \gamma_3$ and $\gamma_1 = \gamma_2 = 0$ in (1) are satisfied. Then, the given system of NLIEs, i.e., (68), has a unique common solution in B. □

5. Conclusion

In this paper, we established some CFP theorems for three self-mappings on complete G_b M-spaces. We proved the uniqueness of CFP by using some generalized rational-type contraction conditions in G_b M-spaces without the continuity of self-mappings. We presented an illustrative example of a unique CFP for three self-mappings to justify our results. In addition, we presented an application of nonlinear integral equations to get the existing results for a unique common solution to support our work. By using this concept, one can define various rational-type contraction conditions for three or more single-valued and multivalued mappings in the context of generalized metric spaces such as generalized b-metric spaces, complex-valued generalized metric spaces, and complex-valued generalized b-metric spaces with applications of different types of differential equations and nonlinear integral equations.

Data Availability

No datasets were generated or analyzed during this current study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Authors' Contributions

The authors equally contributed to the finding and writing of this research work.

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