Hindawi
Journal of Mathematics
Volume 2024, Article ID 8852337, 14 pages
https://doi.org/10.1155/2024/8852337

Research Article

Solitons of the Twin-Core Couplers with Fractional Beta Derivative Evolution in Optical Metamaterials via Two Distinct Methods

Meryem Odabasi Koprulu [1] and Zehra Pinar Izgi [2]

[1] *Tire Kutsan Vocational School, Ege University, Izmir 35100, Türkiye*
[2] *Department of Mathematics, Tekirdag Namik Kemal University, Tekirdag 59030, Türkiye*

Correspondence should be addressed to Meryem Odabasi Koprulu; meryem.odabasi@ege.edu.tr

Received 2 June 2023; Revised 21 November 2023; Accepted 17 February 2024; Published 27 March 2024

Academic Editor: Yusuf Pandir

The rapid advancements in metamaterial research have brought forth a new era of possibilities for controlling and manipulating light at the nanoscale. In particular, the design and engineering of optical metamaterials have created advances in the field of photonics, enabling the development of advanced devices with unprecedented functionalities. Among the myriad of intriguing metamaterial structures, the nonlinear directional couplers with beta derivative evolution have emerged as a significant avenue of exploration, offering remarkable potential for light propagation and manipulation. This study obtains the solitary wave solutions for twin-core couplers having spatial-temporal fractional beta derivative evolution by using two different methods, the Bernoulli method and the complete polynomial discriminant system method. By graphing some of the obtained solutions, the effect of the beta derivative has been shown. The findings would be beneficial to understand physical behaviours in nonlinear optics, particularly twin-core couplers with optical metamaterials.

1. Introduction

Nonlinear models encountered as a consequence of mathematical modelling of real-life problems and the analysis of these models is significant concerns of applied mathematics. Many models in the field of physics and engineering reach significance when their solutions are analysed and allow to interpret the event they represent. Recently, an issue that has been increasing in importance is metamaterials and their models [1]. Metamaterials can be defined as a new class of artificial materials that do not exist in naturally occurring materials and exhibit extraordinary properties. Metamaterials are used to control and manipulate light and sound and have applications in many areas such as solar energy management, public security, medical device manufacturing, remote aviation, and sensor identification. On the other hand, studies on metamaterials that can provide protection against radiation and seismic movements can be used in the field of sustainable energy, and make data transfer faster are continuing.

In the realm of modern optics, the ability to control and manipulate light has emerged as a compelling and consequential pursuit. The advent of metamaterials, engineered materials with extraordinary properties not found in nature, has revolutionized this field, enabling unprecedented control over the behaviour of light at the nanoscale. By intricately designing the structure and composition of metamaterials, researchers have unlocked remarkable possibilities for light manipulation, leading to a myriad of applications ranging from advanced photonic devices to information processing. Optical metamaterials (OMMs) [2, 3] with negative refractive index can be used in invisibility cloak, super-resolution imaging and efficient energy generation applications.

Within this expansive landscape of metamaterial research, twin-core couplers (TCCs) with beta derivative evolution [4] have emerged as a captivating avenue for exploration. These couplers possess the unique ability to channel light along distinct pathways within metamaterials, enhancing the precision and versatility of light manipulation. By harnessing the principles of beta derivative evolution, these couplers offer the potential for compact and efficient devices that can guide, switch, and modulate light with unprecedented control.

The concept of fractional derivative is a mathematical tool that extends the concept of differentiation to noninteger orders. It is a generalization of the classical derivative, which corresponds to the case when the order of differentiation is an integer. Fractional calculus plays an important role in modelling because in classical dynamic systems, it is not suitable to express the effect of physical problems due to their long-term occurrence [5]. In these stages, processes are expressed using fractional derivatives (e.g., Riesz derivative, Caputo derivative, conformable derivative, Hadamard derivative, Riemann–Liouville derivative, etc.). However, except for the beta derivative, most of these derivatives do not satisfy some fundamental theorems of calculus. The beta derivative [6] has found applications in various scientific disciplines, including physics, engineering, signal processing, and optimization. It allows for the characterization of nonlocal and memory-dependent effects, which cannot be captured by traditional integer-order derivatives.

In the context of OMMs and TCCs, the beta derivative evolution refers to the use of fractional order derivatives to describe and manipulate the propagation of light within the metamaterial structure. The inclusion of the beta derivative term in the mathematical model accounts for the nonlocal and memory effects that arise in these complex systems, enabling a more accurate representation of the light-guiding characteristics and facilitating the design and optimization of twin-core couplers with enhanced functionalities.

The study and application of the beta derivative in OMMs offer a deeper understanding of the underlying physics and provide a mathematical framework to analyse the dynamics of light propagation. By incorporating the beta derivative evolution into the modelling and analysis of twin-core couplers, researchers can explore novel avenues for manipulating light and realize advanced photonic devices with tailored properties and improved performance.

Soliton solutions are key elements in the fields of engineering and applied sciences because of the applications in optical tools such as metamaterials and couplers [7–18]. Some recent studies focus on a property demonstrated by solitons in symmetric optical couplers and symmetry-breaking transitions in quiescent and moving solitons in fractional couplers [19]. Likewise, nonlinear directional couplers (NLDCs), due to the applications in nonlinear optics and signal processing, are gaining much more interest. To cover soliton solutions of the NLDCs, several computational methods have been proposed and applied by researchers [20–22]. Along with them, TCCs studied by authors [23, 24] in terms of integer order derivatives.

As a result of the details above, the study is focused on the soliton solutions for TCCs with beta derivative evolution (BDE) in OMMs. By using two different methods, the Bernoulli method [25] and complete polynomial discriminant system method (CPDS) [26, 27], soliton solutions of TCCs having Kerr law nonlinearity via BDE and OMMs have been reported. By deriving analytical solutions, which are bell-shaped, kink-shaped and wings, for these equations, we believe that we will provide valuable insights into the fundamental properties such as energy, momentum, etc., and limitations of twin-core couplers.

2. Essential Concepts of β-Derivative

The beta derivative is defined by Atangana et al. [6] as

$$ {}_{0}^{A}D_{\zeta}^{\beta}H(\zeta) = \lim_{\varepsilon \longrightarrow 0} \frac{H\left(\zeta + \varepsilon\left(\zeta + 1/(\Gamma(\beta))\right)^{1-\beta}\right) - H(\zeta)}{\varepsilon}. \quad (1) $$

The definition satisfies all fundamental properties of the conventional derivative. The various properties are as follows:

$$ \begin{aligned} &\text{(i) } {}_{0}^{A}D_{\zeta}^{\beta}(aH(\zeta) + bG(\zeta)) = a{}_{0}^{A}D_{\zeta}^{\beta}H(\zeta) + \mathrm{ba}{}_{0}^{A}D_{\zeta}^{\beta}G(\zeta) \\ &\text{(ii) } {}_{0}^{A}D_{\zeta}^{\beta}(c) = 0 \\ &\text{(iii) } {}_{0}^{A}D_{\zeta}^{\beta}(H(\zeta).G(\zeta)) = G(\zeta){}_{0}^{A}D_{\zeta}^{\beta}H(\zeta) + H(\zeta){}_{0}^{A}D_{\zeta}^{\beta}G(\zeta) \\ &\text{(iv) } {}_{0}^{A}D_{\zeta}^{\beta}\left(\frac{H(\zeta)}{G(\zeta)}\right) = \frac{G(\zeta){}_{0}^{A}D_{\zeta}^{\beta}H(\zeta) - H(\zeta){}_{0}^{A}D_{\zeta}^{\beta}G(\zeta)}{G^{2}(\zeta)}, \end{aligned} \quad (2) $$

where a, b, c are real numbers, H and G are differentiable functions with $0 < \beta \leq 1$. Letting $\varepsilon = (\zeta + 1/\Gamma(\beta))^{\beta-1}h$, $h \longrightarrow 0$ when $\varepsilon \longrightarrow 0$, then one can obtain

$$ {}_0^A D_\zeta^\beta H(\zeta) = \left(\zeta + \frac{1}{\Gamma(\beta)}\right)^{1-\beta} \frac{\mathrm{dH}(\zeta)}{d\zeta}. \tag{3} $$

The beta derivative captures the fractional order behaviour of a function, providing a powerful tool for modelling and analysing systems with complex dynamics. The study and application of the beta derivative in optical metamaterials offer a deeper understanding of the underlying physics and provide a mathematical framework to analyse the dynamics of light propagation. Because of the mentioned advantages, a model consists of beta derivative is preferred in this study.

3. Twin-Core Couplers with Fractional Beta Derivative Evolution

The TCCs equations with spatial-temporal BD evolution is given by [4, 28].

$$ \begin{aligned} {}_0^A D_t^\beta S + \alpha_1 {}_0^A D_{xx}^\beta S + F\left(|S|^2\right)S = {} & l_1 {}_0^A D_{xx}^\beta\left(|S|^2\right)S + m_1\left(|S|^2\right){}_0^A D_{xx}^\beta S \\ & + g_1 S^2 {}_0^A D_{xx}^\beta S^* + k_1 T, \end{aligned} \tag{4} $$

$$ \begin{aligned} {}_0^A D_t^\beta T + \alpha_2 {}_0^A D_{xx}^\beta T + F\left(|T|^2\right)T = {} & l_2 {}_0^A D_{xx}^\beta\left(|T|^2\right)T + m_2\left(|T|^2\right){}_0^A D_{xx}^\beta T \\ & + g_2 T^2 {}_0^A D_{xx}^\beta T^* + k_2 S, \end{aligned} \tag{5} $$

where $0 < \beta \leq 1$.

Within that case $S(x,t)$ and $T(x,t)$ are complex valued functions describing the optical problems in two cores, respectively. ${}_0^A D_t^\beta$ and ${}_0^A D_x^\beta$ represent the beta derivatives with respect to time and space, correspondingly. g_1, g_2, α_1, and α_2 are coefficients of dispersion while m_1, m_2, l_1, and l_2 are related to trapping in the phase hole. k_1 and k_2 are coupling coefficient of TCCs.

To transfer the equations (4) and (5) into nonlinear ODE the following beneficial transformations can be used:

$$ \begin{aligned} S(x,t) &= Q_1(\delta) e^{i\phi(x,t)}, \\ T(x,t) &= Q_2(\delta) e^{i\phi(x,t)}, \end{aligned} \tag{6} $$

where

$$ \begin{aligned} \delta &= \frac{l}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{v}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta, \\ \phi(x,t) &= -\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta + \varphi_0. \end{aligned} \tag{7} $$

Here, for $p = 1,2$ the $Q_p(\delta)$ shows the amplitude component of the wave, v is velocity, l is a constant, κ is the frequency, $\phi(x,t)$ is the phase component, ω represents the wave number, and φ_0 is the phase constant. Also, phases of TCCs are identical. This step-matching condition is useful for extracting soliton solutions for the integrability of TCCs.

$$ \left(v - 2\kappa l \alpha_p\right)Q_p' + 2\kappa l\left(3l_p + m_p - g_p\right)Q_p^2 Q_p' = 0, \quad p = 1, 2. \tag{8} $$

The consequence of letting two coefficients of linearly independent functions to zero is

$$ v = 2\kappa l \alpha_p, \tag{9} $$

$$ 3l_p + m_p - g_p = 0. \tag{10} $$

If one equates two values of the speed of soliton, it yields

$$ \alpha_1 = \alpha_2 = \alpha. \tag{11} $$

Then equation (9) can be written as

$$ v = 2\kappa l \alpha. \tag{12} $$

Additionally, real parts of the examined equations are extracted as

$$ \begin{aligned} & \alpha_p l^2 Q_p'' - \left(\kappa^2 \alpha_p + \omega\right)Q_p + H\left(Q_p^2\right)Q_p + \left(g_p + l_p + m_p\right)\kappa^2 Q_p^3 - 6l^2 l_p Q_p\left(Q_p'\right)^2 \\ & - l^2\left(g_p + 3l_p + m_p\right)Q_p^2 Q_p'' - k_p Q_{p^*} = 0, \end{aligned} \tag{13} $$

where $\alpha_1 = \alpha_2 = \alpha$, $p^* = 3 - p$ and $p = 1, 2$. Owing to balancing procedure with $Q_p = Q_{p^*}$ equation (13) is turned into

$$\alpha l^2 Q_p'' - \left(k_p + \kappa^2\alpha + \omega\right)Q_p + H\left(Q_p^2\right)Q_p + 2\left(g_p - l_p\right)\kappa^2 Q_p^3 - 6l^2 l_p Q_p\left(Q_p'\right)^2 - 2l^2 g_p Q_p^2 Q_p'' = 0. \tag{14}$$

For twin-core couplers with Kerr law nonlinearity equations (4) and (5) can be reduced as below with $H(h) = \text{sh}$:

$$\begin{aligned} {}_0^A D_t^\beta S + \alpha_1 {}_0^A D_{xx}^\beta S + s_1|S|^2 S &= l_1 {}_0^A D_{xx}^\beta\left(|S|^2\right)S + m_1\left(|S|^2\right){}_0^A D_{xx}^\beta S \\ &\quad + g_1 S^2 {}_0^A D_{xx}^\beta S^* + k_1 T, \end{aligned} \tag{15}$$

$$\begin{aligned} {}_0^A D_t^\beta T + \alpha_2 {}_0^A D_{xx}^\beta T + s_2|T|^2 T &= l_2 {}_0^A D_{xx}^\beta\left(|T|^2\right)T + m_2\left(|T|^2\right){}_0^A D_{xx}^\beta T \\ &\quad + g_2 T^2 {}_0^A D_{xx}^\beta T^* + k_2 S. \end{aligned} \tag{16}$$

Equation (14) can be transformed by the equations (15) and (16).

$$\begin{aligned} &\alpha l^2 Q_p'' - \left(k_p + \alpha k^2 + \omega\right)Q_p + \left[s_p + 2\kappa^2\left(g_p - l_p\right)\right]Q_p^3 \\ &\quad - 6l^2 l_p Q_p\left(Q_p'\right)^2 - 2l^2 g_p Q_p^2 Q_p'' = 0. \end{aligned} \tag{17}$$

To attain traveling wave solutions, $l_p = g_p = 0$ is used. Therefore, equation (17) is now rewritten as

$$\alpha l^2 Q_p'' - \left(\alpha\kappa^2 + k_p + \omega\right)Q_p + s_p Q_p^3 = 0, \tag{18}$$

which have Kerr-Law nonlinearity. In order to interpret physical phenomena in nonlinear optics such as the dimensionless types of optical fields in the corresponding cores of optical fibres, S and T are evaluated in view of distinct mathematical functions.

4. Construction of Solutions via Complete Polynomial Discriminant System Method

To summarize the CPDS method, consider a nonlinear fractional PDE which can be reduced to a nonlinear ODE by the wave transformation [26, 27]. To apply the method, one can consider the auxiliary equation

$$\left(Q'\right)^2 = \sum_{j=1}^{n} a_j Q^j, \tag{19}$$

where a_j and n are constants. If one rewrite the equation (19) in integral form and classify the roots of the polynomial in the denominator with the help of CPDS, one can obtain the exact analytical solutions of the mentioned fractional PDE.

By using the balancing principle in equation (19), $n = 4$. When one substitute this value in the auxiliary equation, and then taking the necessary derivatives to plug into the equation (18), the following system yields:

$$\begin{aligned} l^2\alpha a_1 &= 0, \\ -k_p - \alpha\kappa^2 - \omega + l^2\alpha a_2 &= 0, \\ l^2\alpha a_3 &= 0, \\ s_p + 2l^2\alpha a_4 &= 0. \end{aligned} \tag{20}$$

The solution of the system is

$$a_0 = a_0, a_1 = 0, a_2 = \frac{k_p + \alpha\kappa^2 + \omega}{l^2\alpha}, a_3 = 0, a_4 = \frac{s_p}{2l^2\alpha}. \tag{21}$$

If we rewrite the auxiliary equation with equation (21) and choose $a_0 = 0$ and then integrate this equation by using the CDS for polynomial, the following solutions are raised:

$$\begin{aligned} S_1(x,t) &= \sqrt{\frac{2\left(k_1 + \alpha\kappa^2 + \omega\right)}{s_1}}\,\text{sech}\left(\sqrt{\frac{k_1 + \alpha\kappa^2 + \omega}{l^2\alpha}}\,\delta\right) \\ &\quad \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^\beta + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^\beta + \varphi_0\right]\right), \end{aligned} \tag{22}$$

$$T_1(x,t) = \sqrt{\frac{2\left(k_2+\alpha\kappa^2+\omega\right)}{s_2}}\operatorname{sech}\left(\sqrt{\frac{k_2+\alpha\kappa^2+\omega}{l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta+\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta+\varphi_0\right]\right), \tag{23}$$

$$S_2(x,t) = \sqrt{\frac{2\left(k_1+\alpha\kappa^2+\omega\right)}{s_1}}\sec\left(\sqrt{-\frac{k_1+\alpha\kappa^2+\omega}{l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta+\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta+\varphi_0\right]\right), \tag{24}$$

$$T_2(x,t) = \sqrt{\frac{2\left(k_2+\alpha\kappa^2+\omega\right)}{s_2}}\sec\left(\sqrt{-\frac{k_2+\alpha\kappa^2+\omega}{l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta+\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta+\varphi_0\right]\right), \tag{25}$$

$$S_3(x,t) = \sqrt{-\frac{2\left(k_1+\alpha\kappa^2+\omega\right)}{s_1}}\operatorname{csch}\left(\sqrt{\frac{k_1+\alpha\kappa^2+\omega}{l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta+\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta+\varphi_0\right]\right), \tag{26}$$

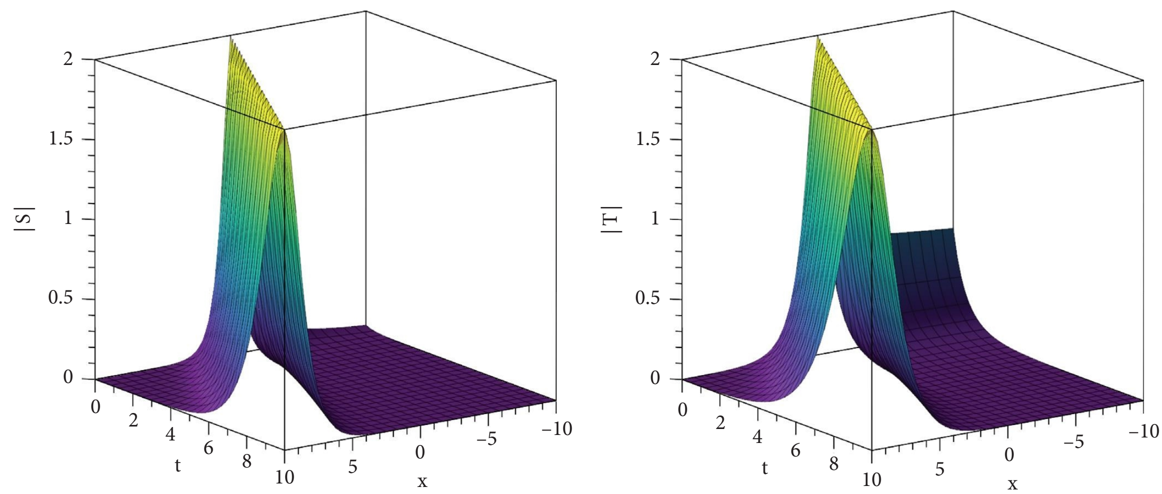
$$T_3(x,t) = \sqrt{-\frac{2\left(k_2+\alpha\kappa^2+\omega\right)}{s_2}}\operatorname{csch}\left(\sqrt{\frac{k_2+\alpha\kappa^2+\omega}{l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta+\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta+\varphi_0\right]\right), \tag{27}$$

$$S_4(x,t) = \sqrt{\frac{2\left(k_1+\alpha\kappa^2+\omega\right)}{s_1}}\csc\left(\sqrt{-\frac{k_1+\alpha\kappa^2+\omega}{l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta+\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta+\varphi_0\right]\right), \tag{28}$$

$$T_4(x,t) = \sqrt{\frac{2\left(k_2+\alpha\kappa^2+\omega\right)}{s_2}}\csc\left(\sqrt{-\frac{k_2+\alpha\kappa^2+\omega}{l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x+\frac{1}{\Gamma(\beta)}\right)^\beta+\frac{\omega}{\beta}\left(t+\frac{1}{\Gamma(\beta)}\right)^\beta+\varphi_0\right]\right), \tag{29}$$

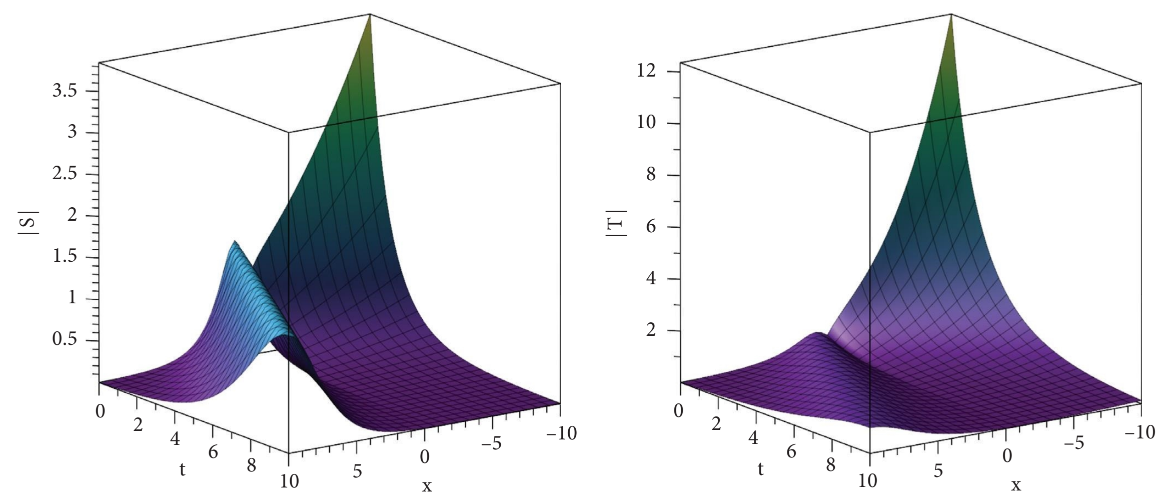
where $\delta = l/\beta(x+1/\Gamma(\beta))^\beta + v/\beta(t+1/\Gamma(\beta))^\beta$.

Figures 1 and 2 represent the exact traveling wave solutions (22) and (23) as soliton solution with the parameters (a) $\alpha=\kappa=\omega=1=\varphi_0=1=s_1=s_2=1, l=-1, k_1=1, k_2=0.0001, \beta=0.7$ and (b) $\alpha=\kappa=\omega=1=\varphi_0=1, s_1=s_2=l=-1, k_1=-1, k_2=-1.5, \beta=0.7$, respectively. The solutions



(a) (b)

Figure 1: (a) Solution (22) (b) solution (23) for $\alpha = \kappa = \omega = 1 = \varphi_0 = 1 = s_1 = s_2 = 1, l = -1, k_1 = 1, k_2 = 0.0001, \beta = 0.7$.



(a) (b)

Figure 2: (a) Solution (22) (b) solution (23) for $\alpha = \kappa = \omega = 1 = \varphi_0 = 1, s_1 = s_2 = l = -1, k_1 = -1, k_2 = -1.5, \beta = 0.7$.

occur as a bright solitary wave solution and β parameter has an effect on the solutions.

Figure 3 represents the exact traveling wave solutions (24) and (25) with the parameters $\kappa = \omega = 1 = \varphi_0 = s_1 = s_2 = \alpha = 1, k_1 = k_2 = 2, \beta = 0.6$, $l = -1$, and $l = -2$, respectively.

Figure 4 shows the effect of the beta derivative to the graph of the solutions (22) and (23).

Then, the auxiliary equation is chosen as

$$Q' = \frac{N(Q)}{M(Q)} = \frac{a_n Q^n + \ldots + a_1 Q + a_0}{b_n Q^m + \ldots + b_1 Q + b_0}. \tag{30}$$

If we apply the balancing procedure, it yields with $n - m = 2$.

For $n = 3$ and $m = 1$,

$$Q' = \frac{a_3 Q^3 + a_2 Q^2 + a_1 Q + a_0}{b_1 Q + b_0}. \tag{31}$$

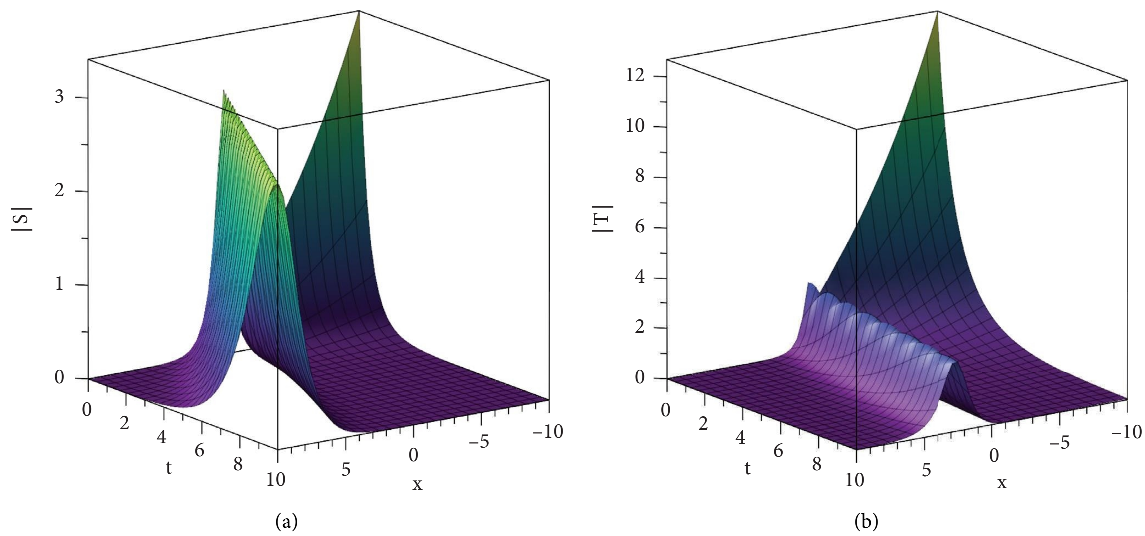


FIGURE 3: (a) Solution (24) (b) solution (25) for $\kappa = \omega = 1 = \varphi_0 = s_1 = s_2 = \alpha = 1, k_1 = k_2 = 2, \beta = 0.6$, $l = -1$, and $l = -2$, respectively.

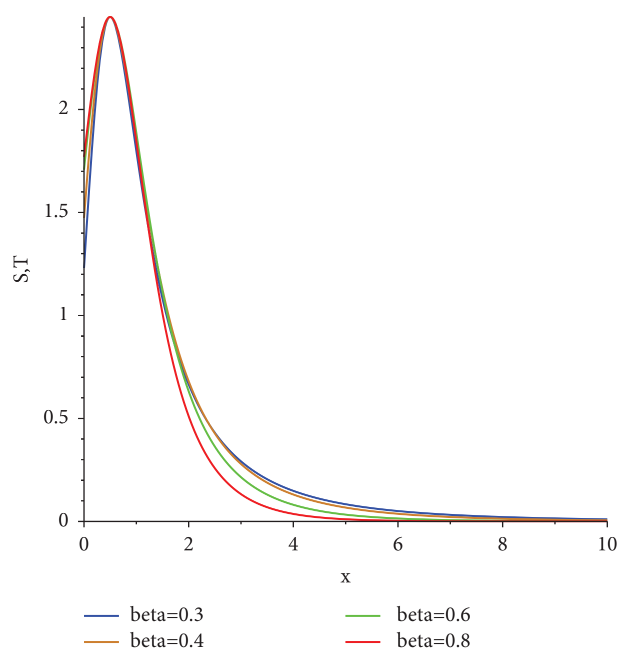


FIGURE 4: 2D graph of solutions (22) and (23) (at $t = 0.5$) for $\alpha = \kappa = \omega = 1 = \varphi_0 = 1 = s_1 = s_2 = k_1 = k_2 = 1, l = -1$, and different values of β.

The corresponding system is

$$\begin{aligned}
l^2\alpha a_0 a_1 b_0 - l^2\alpha a_0^2 b_1 &= 0,\\
l^2\alpha a_1^2 b_0 + 2l^2\alpha a_0 a_2 b_0 - kb_0^3 - \alpha\kappa^2 b_0^3 - \omega b_0^3 - l^2\alpha a_0 a_1 b_1 &= 0,\\
3l^2\alpha a_1 a_2 b_0 + 3l^2\alpha a_0 a_3 b_0 - 3kb_0^2 b_1 - 3\alpha\kappa^2 b_0^2 b_1 - 3\omega b_0^2 b_1 &= 0,\\
2l^2\alpha a_2^2 b_0 + 4l^2\alpha a_1 a_3 b_0 + sb_0^3 + l^2\alpha a_1 a_2 b_1 + l^2\alpha a_0 a_3 b_1 - 3kb_0 b_1^2 - 3\alpha\kappa^2 b_0 b_1^2 - 3\omega b_0 b_1^2 &= 0,\\
5l^2\alpha a_2 a_3 b_0 + l^2\alpha a_2^2 b_1 + 2l^2\alpha a_1 a_3 b_1 + 3sb_0^2 b_1 - kb_1^3 - \alpha\kappa^2 b_1^3 - \omega b_1^3 &= 0,\\
3l^2\alpha a_3^2 b_0 + 3l^2\alpha a_2 a_3 b_1 + 3sb_0 b_1^2 &= 0,\\
2l^2\alpha a_3^2 b_1 + sb_1^3 &= 0.
\end{aligned} \tag{32}$$

From the solution of the system above, one can obtain the set of coefficients and corresponding solutions as follows:

Set 1:

$$a_0 = 0,\ a_1 = \frac{i\left(k + \alpha\kappa^2 + \omega\right)b_1}{\sqrt{2s\alpha}l}, a_2 = 0, a_3 = -\frac{i\sqrt{s}b_1}{\sqrt{2\alpha}l}, b_0 = 0, b_1 = b_1. \tag{33}$$

Set 2:

$$a_0 = \frac{i\left(k + \alpha\kappa^2 + \omega\right)b_0}{\sqrt{2s\alpha}l}, a_1 = 0, a_2 = -\frac{i\sqrt{s}b_0}{\sqrt{2\alpha}l}, a_3 = 0, b_0 = b_0, b_1 = 0. \tag{34}$$

For these two sets, analytical solutions of equations (4) and (5) are extracted as

$$\begin{aligned}
S_5(x,t) = {} & \sqrt{\frac{k_1 + \alpha\kappa^2 + \omega}{s_1}}\tanh\left(\sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right)\\
& \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right),
\end{aligned} \tag{35}$$

$$\begin{aligned}
T_5(x,t) = {} & \sqrt{\frac{k_2 + \alpha\kappa^2 + \omega}{s_2}}\tanh\left(\sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right)\\
& \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right),
\end{aligned} \tag{36}$$

$$\begin{aligned}
S_6(x,t) = {} & \sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{s_1}}\tan\left(\sqrt{\frac{k_1 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right)\\
& \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right),
\end{aligned} \tag{37}$$

$$T_6(x,t) = \sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{s_2}}\tan\left(\sqrt{\frac{k_2 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right), \tag{38}$$

$$S_7(x,t) = \sqrt{\frac{k_1 + \alpha\kappa^2 + \omega}{s_1}}\coth\left(\sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right), \tag{39}$$

$$T_7(x,t) = \sqrt{\frac{k_2 + \alpha\kappa^2 + \omega}{s_2}}\coth\left(\sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right), \tag{40}$$

$$S_8(x,t) = \sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{s_1}}\cot\left(\sqrt{\frac{k_1 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right), \tag{41}$$

$$T_8(x,t) = \sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{s_2}}\cot\left(\sqrt{\frac{k_2 + \alpha\kappa^2 + \omega}{2l^2\alpha}}\delta\right) \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right). \tag{42}$$

Figure 5 shows the behaviour of exact traveling wave solutions (35) and (36) with respect to the parameters $\kappa = \omega = 1 = \varphi_0 = s_1 = s_2 = k_1 = k_2 = 1, l = \alpha = -1, \beta = 0.5$ and $\beta = 0.95$, respectively.

From the solution of the system equation (32), set 3 is obtained as

Set 3:

$$a_0 = \frac{i\left(k + \alpha\kappa^2 + \omega\right)b_0}{l\sqrt{2\alpha s}}, a_1 = \frac{i\left(k + \alpha\kappa^2 + \omega\right)b_1}{l\sqrt{2\alpha s}}, a_2 = -\frac{i\sqrt{s}b_0}{l\sqrt{2\alpha}}, a_3 = -\frac{i\sqrt{s}b_1}{l\sqrt{2\alpha}}. \tag{43}$$

Corresponding solutions are as follows:

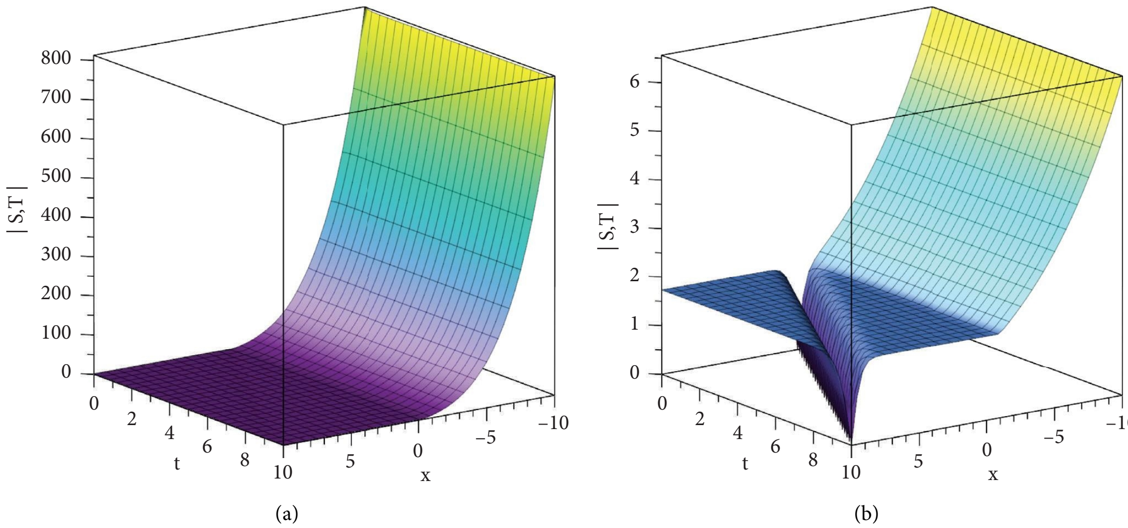


FIGURE 5: (a) Solution (35) (b) solution (36) for $\kappa = \omega = 1 = \varphi_0 = s_1 = s_2 = k_1 = k_2 = 1, l = \alpha = -1, \beta = 0.5$ and $\beta = 0.95$, respectively.

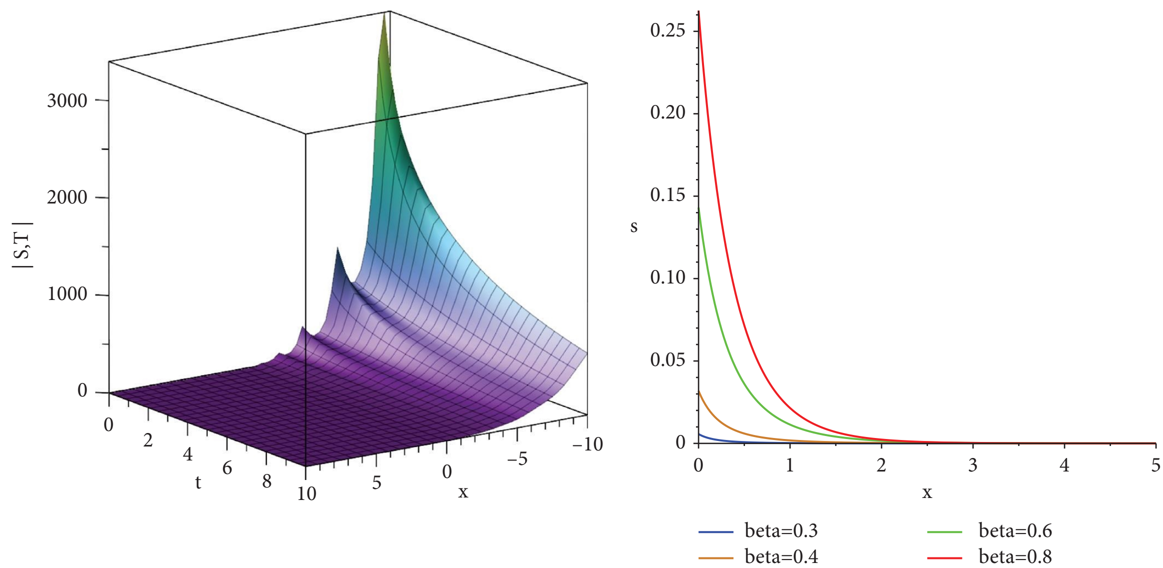
$$S_9(x,t) = \sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{2s_1}}\left[\tanh\left(\sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right) - \coth\left(\sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right)\right] \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right), \tag{44}$$

$$T_9(x,t) = \sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{2s_2}}\left[\tanh\left(\sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right) - \coth\left(\sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right)\right] \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right), \tag{45}$$

$$S_{10}(x,t) = \pm\sqrt{\frac{k_1 + \alpha\kappa^2 + \omega}{2s_1}}\left[\tan\left(\sqrt{-\frac{k_1 + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right) + \cot\left(\sqrt{-\frac{k + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right)\right] \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right), \tag{46}$$

$$T_{10}(x,t) = \pm\sqrt{\frac{k_2 + \alpha\kappa^2 + \omega}{2s_2}}\left[\tan\left(\sqrt{-\frac{k_2 + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right) + \cot\left(\sqrt{-\frac{k + \alpha\kappa^2 + \omega}{4l^2\alpha}}\delta\right)\right] \times \exp\left(i\left[-\frac{\kappa}{\beta}\left(x + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \frac{\omega}{\beta}\left(t + \frac{1}{\Gamma(\beta)}\right)^{\beta} + \varphi_0\right]\right). \tag{47}$$

Figure 6 shows the exact analytical solutions (44) and (45) with the parameters $\omega = 5, \varphi_0 = \ k_1 = k_2 = 1, l = 10, s_1 = s_2 = \alpha = \kappa = -1, \beta = 0.5$. The 3D plot and 2D plot for $t = 0.5$ of the solution are shown in Figures 6(a) and 6(b), respectively.



(a) (b)

Figure 6: (a) 3D, (b) 2D plot (at $t = 0.5$) of the solutions (44) and (45) for $\omega = 5, \varphi_0 = k_1 = k_2 = 1, l = 10, s_1 = s_2 = \alpha = \kappa = -1, \beta = 0.5$.

5. Construction of Solutions via the Bernoulli Method

Bernoulli method [25, 29] is one of the ansatz-based methods used to solve nonlinear evolution equations. To apply the method Bernoulli type differential equation is considered as the auxiliary equation:

$$z' = \mathrm{Pz} + \mathrm{Rz}^2, \tag{48}$$

where $z = z(\delta)$, P, and R are parameters. Additionally, the solution assumption is determined via balancing principle $Q = \sum_{n=0}^{N=1} a_{ip} z^i = a_{0p} + a_{1p} z$. Substituting Bernoulli type differential equation and the assumption into equation (18) and equalling the coefficients of each power of $z(\delta)$ to zero, nonlinear algebraic system is hold. The solution set of the system is obtained

$$P = \pm \frac{1}{l}\sqrt{-\frac{2\alpha_p\kappa^2 + 2\omega + 2k_p}{\alpha_p}}, a_{0p} = \pm\sqrt{-\frac{-\alpha_p\kappa^2 - \omega - k_p}{s_p}}, a_{1p} = \mp \frac{lR\alpha_p\sqrt{s_p\left(2\alpha_p\kappa^2 + 2\omega + 2k_p\right)}}{s_p\sqrt{\alpha_p\left(-\alpha_p\kappa^2 - \omega - k_p\right)}}. \tag{49}$$

When the solution of the Bernoulli type differential equation, the obtained parameters are substituted into the assumption of the solution, the solutions are obtained as

$$S(x,t) = \frac{\left(-\mathrm{lRP}\alpha_1\sqrt{-\left(2\alpha_1\kappa^2 + 2\omega + 2k_1\right)/\alpha_1} + \left(\alpha_1\kappa^2 + \omega + k_1\right)\left(e^{-P\delta}C_1P - R\right)\right)e^{i\left(-\kappa/\beta\left(x + (1/\Gamma(\beta))\right)^{\beta} + \omega/\beta\left(t + 1/(\Gamma(\beta))\right)^{\beta} + \varphi_0\right)}}{\left(e^{-P\delta}C_1P - R\right)s_1\sqrt{\left(\alpha_1\kappa^2 + \omega + k_1\right)/s_1}}, \tag{50}$$

$$T(x,t) = \frac{\left(-\mathrm{lRP}\alpha_2\sqrt{-\left(2\alpha_2\kappa^2 + 2\omega + 2k_2\right)/\alpha_2} + \left(\alpha_2\kappa^2 + \omega + k_2\right)\left(e^{-P\delta}C_1P - R\right)\right)e^{i\left(-\kappa/\beta\left(x + (1/\Gamma(\beta))\right)^{\beta} + \omega/\beta\left(t + 1/(\Gamma(\beta))\right)^{\beta} + \varphi_0\right)}}{\left(e^{-P\delta}C_1P - R\right)s_2\sqrt{\left(\alpha_2\kappa^2 + \omega + k_2\right)/s_2}}. \tag{51}$$

Figure 7 represents the exact wave solution as soliton solution with the parameters $\alpha_2 = -1$, $s_2 = 1$, $\omega = 1$, $c = -1$, $\kappa = 1$, $k_2 = 1$, $\beta = 0.9$, $R = 1$, $l = -1$, $\theta_0 = 1$, $C_1 = 1$. The 3D, 2D plots (at $t = 0.5$ for various values of β) of solution (51) are shown in Figures 7(a) and 7(b), respectively.

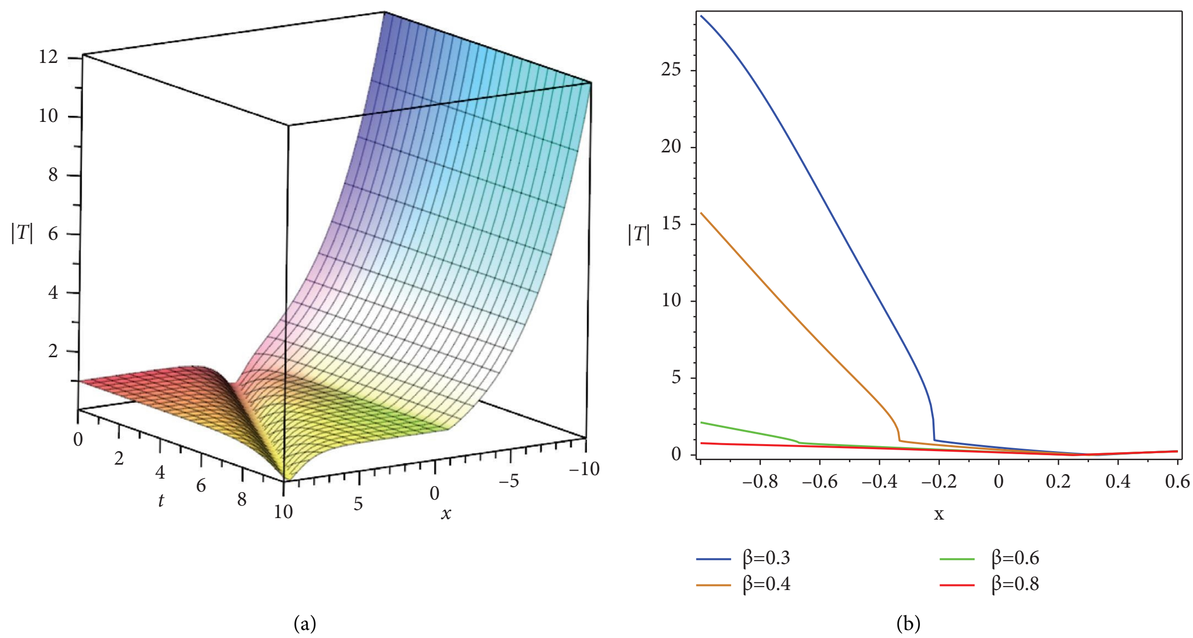


FIGURE 7: (a) 3D, (b) 2D plot (at $t = 0.5$) of the modulus of the exact wave solution (51) when $\alpha_2 = -1$, $s_2 = 1$, $\omega = 1$, $c = -1$, $\kappa = 1$, $k_2 = 1$, $\beta = 0.9$, $R = 1$, $l = -1$, $\theta_0 = 1$, $C_1 = 1$.

The obtained results are for the equations (4) and (5) under some assumptions/restrictions. Now, the solutions are obtained without any assumptions/restrictions for the general case for equations (4) and (5). With the help of the beneficial transformations, equations (4) and (5) are reduced to two nonlinear ODEs as corresponding to imaginary and real parts, respectively. Applying the Bernoulli method procedure, following solution set for parameters is obtained

$$P = 0, v = 2\alpha_p \kappa \mathrm{l}, a_{0p} = 0, l_p = -\frac{2g_p}{3}, m_p = 3g_p, \tag{52}$$

$$\omega = -\alpha_p \kappa^2 - k_p, \quad a_{1p} = \pm \frac{lR\sqrt{-6\alpha_p\left(10\kappa^2 g_p + 3s_p\right)}}{10\kappa^2 g_p + 3s_p}. \tag{53}$$

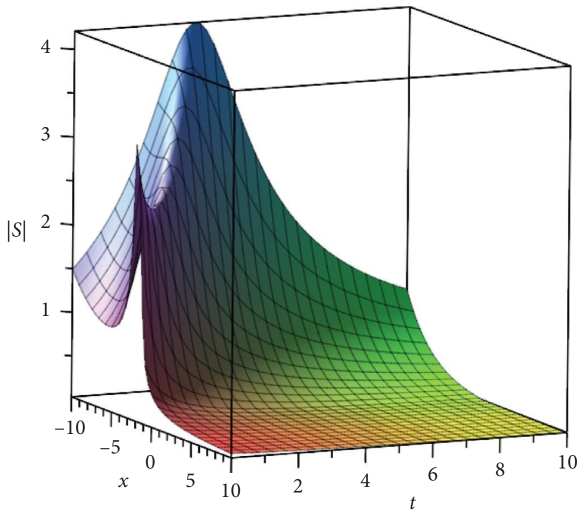
When the solution of the Bernoulli type differential equation, the obtained parameters is substituted into the assumption of the solution, the solutions are obtained

$$S(x,t) = \frac{\mathrm{lR}\sqrt{-6\alpha_1\left(10\kappa^2 g_1 + 3s_1\right)}e^{-i\left(\kappa/\beta(x+1/(\Gamma(\beta)))^{\beta}+\left(\left(\alpha_p\kappa^2+k_p\right)/\beta\right)(t+1/(\Gamma(\beta)))^{\beta}-\varphi_0\right)}}{\left(10\kappa^2 g_1 + 3s_1\right)\left(R\delta - C_1\right)}, \tag{54}$$

$$T(x,t) = \frac{\mathrm{lR}\sqrt{-6\alpha_2\left(10\kappa^2 g_2 + 3s_2\right)}e^{-i\left(\kappa/\beta(x+1/(\Gamma(\beta)))^{\beta}+\left(\left(\alpha_p\kappa^2+k_p\right)/\beta\right)(t+1/(\Gamma(\beta)))^{\beta}-\varphi_0\right)}}{\left(10\kappa^2 g_2 + 3s_2\right)\left(R\delta - C_1\right)}. \tag{55}$$

Figure 8 represents the exact wave solution as soliton solution with the parameters $\alpha_1 = -1$, $s_1 = 1$, $g_1 = 0.5$, $\omega = 1$, $\kappa = 1$, $k_1 = -2$, $\beta = 0.9$, $R = 1$, $l = -1$, $\theta_0 = 1$, $C_1 = 1$. The 3D and 2D plots (at $t = 0.5$ for various values of β) of are shown in Figures 8(a) and 8(b), respectively.

Different patterns, dynamic behaviours and types of solitary wave solutions can be seen from the graphs sketched for different parameters to understand the physical behaviours of the obtained solutions. One can also see that the beta fractional parameter has



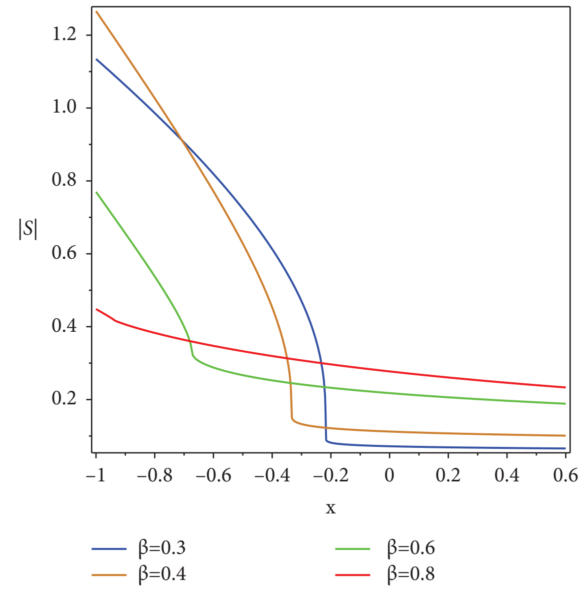


FIGURE 8: (a) 3D, (b) 2D plot (at $t = 0.5$) of the modulus of the exact wave solution (54) when $\alpha_1 = -1$, $s_1 = 1$, $g_1 = 0.5$, $\omega = 1$, $\kappa = 1$, $k_1 = -2$, $\beta = 0.9$, $R = 1$, $l = -1$, $\theta_0 = 1$, $C_1 = 1$.

a great impact on the wave structure and the amplitude is affected by the different values of the dispersion coefficient.

6. Conclusions

The twin-core couplers with fractional BDE including Kerr nonlinearity have been considered. To obtain analytical solutions, two different methods, namely, the Bernoulli method and complete polynomial discriminant system method have been applied. Several types of soliton solutions have been successfully obtained by these methods. Several exact soliton solutions of the model have been attained by the CPDS method under some restrictions. Besides, by using the Bernoulli method the solutions are obtained without any assumptions/restrictions for the general case. For stability, three options are available: Lyapunov stability, orbital stability, and Vakhitov–Kolokolov criterion [19]. In general, the obtained solution is considered as a soliton/travelling wave solution of a nonlinear partial differential equation, and then the stability of the particular motion $Q_p(x - \mathrm{ct})$ has to be compared with a general class of motion. When compare our solutions with [4, 28], one can see that some of the results are new. It is believed that some of the findings in this study are not seen in the literature till now. From integrated photonic circuits to optical sensing and beyond, these couplers hold immense potential for advancing various fields of science and technology. By leveraging the analytical solutions presented in this paper, researchers can unravel the complex dynamics of light propagation within these structures and pave the way for transformative advancements in the field of light manipulation. By using the analytical frameworks, researchers can gain deeper insights into the underlying physics, establish design guidelines, and open up avenues for further innovation and optimization of these couplers in OMMs.

Data Availability

The data used to support the findings of this study are available from the corresponding author upon request.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] A. Valipour, M. H. Kargozarfard, M. Rakhshi, A. Yaghootian, and H. M. Sedighi, "Metamaterials and their applications: an overview," *Proceedings of the Institution of Mechanical Engineers, Part L: Journal of Materials: Design and Applications*, vol. 236, no. 11, pp. 2171–2210, 2022.

[2] W. Cai and V. Shalaev, *Optical Metamaterials: Fundamentals and Applications*, Springer, New York, NY, USA, 2010.

[3] M. Iwanaga, "Photonic metamaterials: a new class of materials for manipulating light waves," *Science and Technology of Advanced Materials*, vol. 13, no. 5, p. 053002, 2012.

[4] M. F. Uddin, M. G. Hafez, Z. Hammouch, H. Rezazadeh, and D. Baleanu, "Traveling wave with beta derivative spatial–temporal evolution for describing the nonlinear directional couplers with metamaterials via two distinct methods," *Alexandria Engineering Journal*, vol. 60, no. 1, pp. 1055–1065, 2021.

[5] J. Singh, D. Kumar, M. Al Qurashi, and D. Bleanu, "A new fractional model for giving up smoking dynamics," *Advances in Difference Equations*, vol. 88, 2017.

[6] A. Atangana, D. Baleanu, and A. Alsaedi, "Analysis of time–fractional Hunter-Saxton equation: a model of neumatic liquid crystal," *Open Physics*, vol. 14, no. 1, pp. 145–149, 2016.

[7] X. N. Liu, G. K. Hu, C. T. Sun, and G. L. Huang, "Wave propagation characterization and design of two-dimensional elastic chiral metacomposite," *Journal of Sound and Vibration*, vol. 330, no. 11, pp. 2536–2553, 2011.

[8] J. H. Oh, Y. E. Kwon, H. J. Lee, and Y. Y. Kim, "Elastic metamaterials for independent realization of negativity in density and stiffness," *Scientific Reports*, vol. 6, no. 1, Article ID 23630, 2016.

[9] A. Biswas, M. Mirzazadeh, M. Eslami, D. Milovic, and M. Belic, "Solitons in optical metamaterials by functional variable method and first integral approach," *Frequenz*, vol. 68, no. 11–12, pp. 525–530, 2014.

[10] M. Ekici, M. Mirzazadeh, Q. Zhou, S. P. Moshokoa, A. Biswas, and M. Belic, "Solitons in optical metamaterials with fractional temporal evolution," *Optik*, vol. 127, no. 22, pp. 10879–10897, 2016.

[11] L. Zeng and J. Zeng, "Fractional quantum couplers," *Chaos, Solitons and Fractals*, vol. 140, Article ID 110271, 2020.

[12] Y. Pandir and A. Ekin, "New solitary wave solutions of the Korteweg-de Vries (KdV) equation by new version of the trial equation method," *Electronic Journal of Applied Mathematics*, vol. 1, no. 1, pp. 101–113, 2023.

[13] L. Zeng, Y. Zhu, B. A. Malomed et al., "Quadratic fractional solitons," *Chaos, Solitons and Fractals*, vol. 154, Article ID 111586, 2022.

[14] B. A. Malomed, "Optical solitons and vortices in fractional media: a mini-review of recent results," *Photonics*, vol. 8, no. 9, p. 353, 2021.

[15] Y. Gurefe, "The generalized Kudryashov method for the nonlinear fractional partial differential equations with the beta-derivative," *Revista Mexicana de Física*, vol. 66, no. 6 Nov-Dec, pp. 771–781, 2020.

[16] N. Raza, I. G. Murtaza, S. Sial, and M. Younis, "On solitons: the biomolecular nonlinear transmission line models with constant and time variable coefficients," *Waves in Random and Complex Media*, vol. 28, no. 3, pp. 553–569, 2018.

[17] U. Afzal, N. Raza, and I. G. Murtaza, "On soliton solutions of time fractional form of Sawada–Kotera equation," *Nonlinear Dynamics*, vol. 95, no. 1, pp. 391–405, 2019.

[18] N. Raza, U. Afzal, A. R. Butt, and H. Rezazadeh, "Optical solitons in nematic liquid crystals with Kerr and parabolic law nonlinearities," *Optical and Quantum Electronics*, vol. 51, no. 4, p. 107, 2019.

[19] D. V. Strunin and B. A. Malomed, "Symmetry-breaking transitions in quiescent and moving solitons in fractional couplers," *Physical Review*, vol. 107, no. 6, Article ID 064203, 2023.

[20] C. Q. Dai and Y. Y. Wang, "Controllable combined Peregrine soliton and Kuznetsov–Ma soliton in $${\varvec{\mathcal {PT}}}$$ PT-symmetric nonlinear couplers with gain and loss," *Nonlinear Dynamics*, vol. 80, no. 1-2, pp. 715–721, 2015.

[21] A. Z. Chowdhury, "Performance study of silica-on-silicon based multi mode interference (MMI) optical coupler," *Photonic Sensors*, vol. 4, no. 1, pp. 34–42, 2014.

[22] B. Q. Li, Y. L. Ma, and T. M. Yang, "The oscillating collisions between the three solitons for a dual-mode fiber coupler system," *Superlattices and Microstructures*, vol. 110, pp. 126–132, 2017.

[23] A. H. Arnous, M. Ekici, S. P. Moshokoa, M. Zaka Ullah, A. Biswas, and M. Belic, "Solitons in nonlinear directional couplers with optical metamaterials by trial function scheme," *Acta Physica Polonica A*, vol. 132, no. 4, pp. 1399–1410, 2017.

[24] J. Vega-Guzman, M. F. Mahmood, Q. Zhou et al., "Solitons in nonlinear directional couplers with optical metamaterials," *Nonlinear Dynamics*, vol. 87, no. 1, pp. 427–458, 2017.

[25] N. K. Vitanov, "Application of simplest equations of Bernoulli and Riccati kind for obtaining exact traveling-wave solutions for a class of PDEs with polynomial nonlinearity," *Communications in Nonlinear Science and Numerical Simulation*, vol. 15, no. 8, pp. 2050–2060, 2010.

[26] L. Yang, X. R. Hou, and Z. B. Zeng, "A complete discrimination system for polynomials," *Science in China (Series E)*, vol. 39, no. 6, pp. 628–646, 1996.

[27] C. S. Liu, "Applications of complete discrimination system for polynomial for classifications of traveling wave solutions to nonlinear differential equations," *Computer Physics Communications*, vol. 181, no. 2, pp. 317–324, 2010.

[28] A. Asad, M. B. Riaz, and Y. Geng, "Sensitive demonstration of the twin-core couplers including Kerr law non-linearity via beta derivative evolution," *Fractal and Fractional*, vol. 6, no. 12, p. 697, 2022.

[29] Z. Pinar Izgi, "Rogue waves and solitons of the generalized modified nonlinear Schrödinger equations," *Mathematics and Computers in Simulation*, vol. 208, pp. 535–549, 2023.