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Research Article

Distance-Based Fractional Dimension of Certain Wheel Networks

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Metric dimension is one of the distance-based parameters which are used to find the position of the robot in a network space by utilizing lesser number of notes and minimum consumption of time. It is also used to characterize the chemical compounds. The metric dimension has a wide range of applications in the field of computer science such as integer programming, radar tracking, pattern recognition, robot navigation, and image processing. A vertex x in a network $\mathbb{W}$ resolves the adjacent pair of vertices uv if x attains an unequal distance from end points of uv. A local resolving neighbourhood set $R_L(uv)$ is a set of vertices of $\mathbb{W}$ which resolve uv. A mapping $\alpha: V(\mathbb{W}) \longrightarrow [0, 1]$ is called local resolving function of $\mathbb{W}$ if $\alpha(R_L(uv)) \geq 1$ for any adjacent pair of vertices of uv of $\mathbb{W}$ and the minimal value of $\alpha(R_L(uv))$ for all local resolving functions α of $\mathbb{W}$ is called local fractional metric dimension of $\mathbb{W}$. In this paper, we have studied the local fractional metric dimension of wheel-related networks such as web-wheel network, subdivision of wheel network, line network of subdivision of wheel network, and double-wheel network and also examined their boundedness.

1. Introduction and Preliminaries

The notion of metric dimension (MD) was introduced in the 1970s independently by Slater and Harary [1, 2]. NP-hardness and complexity of the MD problem were briefly studied in [3, 4]. MD is substantially applied in different fields such as robot navigation [5], in pharmaceutical chemistry [6], image processing [1], and in computer science [7]. In 2000, Chartrand et al. characterized all the connected networks that have a specific value of MD [6]. Liu et al. computed MD of tenser product of path, cycles, and the constant MD of Toeplitz networks [8, 9]. Barragán-Ramírezet al. defined the concept of local MD, and they also computed the local MD of the strong product of some connected networks [10].

The term fractional metric dimension (FMD) is defined by Currie and Oellermann to find the solution of certain IPP [11] and Feher et al. computed the optimal solution of IPP by using FMD [12]. In 2011, Arumugam and Mathew introduced the term FMD in networking theory [13], and the notion of local fractional metric dimension (LFMD) is defined by Aisyah et al. [14], for more about FMD see [15, 16]. Javaid et al. played an important role in the field of LFMD as they have established bounds of LFMD and characterized some connected networks those obtain the exact value of LFMD. Furthermore, they developed a computational technique to evaluate the lower bound of LFMD [17, 18].

A network $\mathbb{W}$ is an ordered pair (V, E), where the set V composing of the nodes called the vertex set $V(\mathbb{W})$ and E is the set of the links among these nodes is called the edge set $E(\mathbb{W})$. A path is a sequence of vertices in which each one adjacent to the next. The number of edges in the minimal path between two vertices u and v is called distance between them donated by $d(u, v)$.

The local resolving neighbourhood (LRN) set $R_L(uv)$ is defined as $R_L(uv) = \{x \in V(\mathbb{W}): d(x, u) \neq d(x, v)\}$. An upper local resolving function (ULRF) $\alpha: V(\mathbb{W}) \longrightarrow [0, 1]$

and $\alpha(R_L(uv)) \geq 1$, where $\alpha(R_L(uv)) = \sum_{x \in R_L(uv)} \alpha(x)$. A function is known as lower local resolving function (LLRF) if β: $V(\mathbb{W}) \longrightarrow [0, 1]$, where $\beta(R_L(uv)) \leq 1$ for each $R_L(uv)$ of $\mathbb{W}$, where $\alpha(R_L(uv)) = \sum_{x \in R_L(uv)} \beta(x)$. Then, LFMD is defined as

$$\begin{aligned} D_{\mathrm{LF}}^{-}(\mathbb{W}) &= \min\{|\alpha| \text{ is the upper local minimal resolving function of } \mathbb{W}\}, \\ D_{\mathrm{LF}}^{+}(\mathbb{W}) &= \max\{|\beta| \text{ is the lower local mximal resolving function of } \mathbb{W}\}. \end{aligned} \tag{1}$$

The line network $L(\mathbb{W})$ of a network $\mathbb{W}$ is defined to have as its vertices the edges of $\mathbb{W}$, with two nodes are adjacent if the corresponding edges share a node in [19]. A subdivision of a network $S(\mathbb{W})$ is obtained by adding an additional vertex into each edge of $\mathbb{W}$. Since Javaid et al. [17, 18] have established the bounds of LFMD of general networks and they have also computed the exact value of LFMD of specific networks. In this context, we have developed bounds of LFMD of some special class of generalized wheel networks. Furthermore, the bounds and exact values of LFMD are depends upon the cardinalities of the LRN of each network.

In this article, our objective is to compute the LFMD of wheel-related networks such as web-wheel, subdivision of wheel, line network of subdivision of wheel, and double-wheel networks. These networks attain different values of LFMD at different levels; therefore, it is very interesting to investigate their LFMD. In the end, a comprehensive conclusion is given as well. The article is organised as follows: Section 2 contains the preliminary concepts involving of the concepts involved in the article; in Section 3, all the main results are given in detail; and Section 4 deals with the conclusion.

2. Main Results

In this current section, we are interested in determining the LFMD of wheel-related networks, such as web-wheel network, subdivision of wheel network, and line network of subdivision of the wheel network.

2.1. LRN Set and LFMD of Subdivision of Wheel Network. The subdivision of wheel network (SW_k) is obtained by adding a vertex w_i and v_i to each edge of wheel network W_k, where $1 \leq i \leq k$. For more details, see Figure 1.

Theorem 1. *Let SW_k be a subdivision of wheel network. Then,*

$$D_{\mathrm{LF}}(\mathrm{SW}_k) = 1. \tag{2}$$

Proof. Since SW_k is a bipartite network and the cardinality of each LRN set of SW_k is equal to its vertex. Hence, $|R_L(y)| = |V(\mathrm{SW}_K)|$, $\forall y \in E(\mathrm{SW}_k)$. Now, we consider a constant LRF α: $V(\mathrm{SW}_k) \longrightarrow [0, 1]$ as $\alpha(v) = 1/3k + 1$, $\forall v \in V(\mathrm{SW}_k)$, hence $D_{\mathrm{LF}}(\mathrm{SW}_k) = \sum_{i=1}^{3k+1} 1/3k + 1 = 1$. Consequently,

$$D_{\mathrm{LF}}(\mathrm{SW}_k) = 1. \tag{3}$$

□

2.2. Line Network of Subdivision of Wheel Network $L\mathrm{SW}_k$. The network LSW_k is obtained by adding new vertex x_i in SW_k, its vertex set is $V(\mathrm{LSW}_k) = \{u_i, v_i, w_i, x_1 \colon 1 \leq i \leq k\}$ and its edge set is $E(\mathrm{LSW}_k) = \{w_i x_i, w_i v_i, u_i v_i, u_i u_{i+1}, u_i u_{i+2}, u_i u_{i+3}, \ldots, u_i u_{i+n} \colon 1 \leq i \leq k\}$. For more information, see Figure 2.

Lemma 2. *Let $L\mathrm{SW}_k$ be a line network of subdivision of wheel network. Then,*

(a) $|R_L(u_i u_{i+1})| = 8$ and $\cup_{i=1}^{3k} R_L(u_i u_{i+1}) = V(L\mathrm{SW}_k)$

(b) $|R_L(y)| \leq |R_L(u_i u_{i+1})|$ and $|\cup_{i=1}^{2k} R_L(u_i u_{i+1})| = 2k - 1$

Proof. Let LSW_k be a web wheel network, where $k + 1 \pmod k = 1$

(a) $R_L(u_i u_{i+1}) = \{u_i, u_{i+1}, v_i, v_{i+1}, w_i, w_{i+1}, w_{i+2}, w_{i+3}\}$, $R_L(u_i u_{i+1}) = \{u_i, u_{i+1}, v_i, v_{i+1}, w_i, w_{i+1}, w_{i+2}, w_{i+3}\}$, and $\cup_{i=1}^{3k} R_L(u_i u_{i+1}) = V(\mathrm{LSW}_k)$, therefore $|\cup_{i=1}^{3k} R_L(u_i u_{i+1})| = 3k$

(b) The LRN sets other than $R_L(u_i u_{i+1})$ are $R_L(u_i v_i) = V(\mathrm{LSW}_k) - \{x_{i+1}, w_{n+i-1}\}$, $R_L(v_i w_i) = V(\mathrm{LSW}_k) - \{v_{i+1}, w_{i+1}, w_{i+2}, x_i\}$, $R_L(v_i x_i) = V(\mathrm{LSW}_k) - \{w_i, x_{k+i-3}, w_{k+i-1}, x_{k+i-1}\}$, $R_L(w_i x_i) = \{v_{i+1}, v_{i+4}, w_i, w_{i+1}, w_{i+2}, w_{i+3}, w_{i+4}, x_i, x_{i+1}, x_{i+2}, x_{i+3}, x_{i+4}\}$ and $R_L(x_i w_{i+1}) = \{u_i, u_{i+1}, v_i, v_{i+1}, v_{i+2}, v_{i+4}, w_i, w_{i+1}, w_{i+2}, w_{i+3}, w_{i+4}, x_i, x_{i+1}, x_{i+2}, x_{i+4}\}$

Table 1 clears the order of each $R_L(y)$. □

Theorem 3. *Let $L\mathrm{SW}_k$ be line network of subdivision of wheel network. Then,*

$$\frac{3k}{3k-2} \leq D_{\mathrm{LF}}(\mathrm{LSW}_k) \leq \frac{3k}{8}. \tag{4}$$

Proof. In order to prove the theorem, we have divided into particular case (Case A) and general case (Case B).

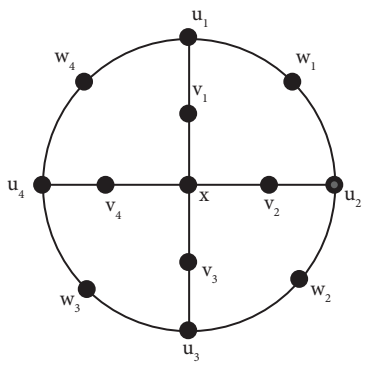

Figure 1: Subdivision of wheel network SW_4.

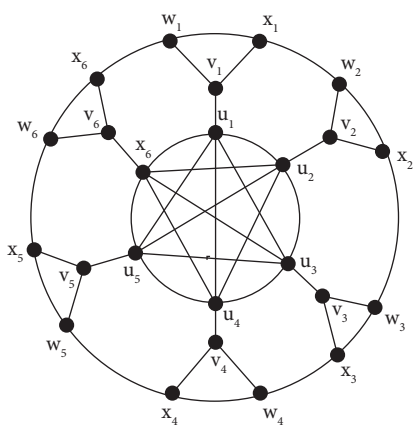

Figure 2: Line network subdivision of wheel network LSW_6.

Table 1: LRN sets and their comparison.

LRN set	Comparison
$R_L(u_iv_i)$	$3k-2>8$
$R_L(u_ix)$	$2k-3>8$
$R_L(w_ix_i)$	$12>8$
$R_L(x_iw_{i+1})$	$15>8$
$R_L(w_iv_i)$	$3k-4>8$

Case A.

The possible LRN sets of LSW_3 are as follows:

$$\begin{aligned}
R_L(w_1v_1) &= V(LSW_3)-\{v_2,w_2,x_1\},\\
R_L(w_2v_2) &= V(LSW_3)-\{v_3,w_3,x_2\},\\
R_L(w_3v_3) &= V(LSW_3)-\{v_1,w_1,x_3\},\\
R_L(u_1u_2) &= V(LSW_3)-\{u_3,v_3,w_3,x_3\},\\
R_L(u_2u_3) &= V(LSW_3)-\{u_1,v_1,w_1,x_1\},\\
R_L(u_3u_1) &= V(LSW_3)-\{u_2,v_2,w_2,x_2\},\\
R_L(v_1x_1) &= V(LSW_3)-\{v_3,w_1,w_3,x_3\},\\
R_L(v_2x_2) &= V(LSW_3)-\{v_1,w_2,w_1,x_1\},\\
R_L(v_3x_3) &= V(LSW_3)-\{v_2,w_3,w_2,x_2\},\\
R_L(w_1x_1) &= V(LSW_3)-\{u_1,u_2,u_3,v_1\},\\
R_L(w_2x_2) &= V(LSW_3)-\{u_2,u_3,u_1,v_2\},\\
R_L(w_3x_3) &= V(LSW_3)-\{u_3,u_1,u_2,v_3\},\\
R_L(x_1w_2) &= V(LSW_3)-\{u_3,v_3\},\\
R_L(x_2w_3) &= V(LSW_3)-\{u_1,v_1\},\\
R_L(x_3w_1) &= V(LSW_3)-\{u_2,v_2\},\\
R_L(u_1v_1) &= V(LSW_3)-\{w_3,x_2\},\\
R_L(u_2v_2) &= V(LSW_3)-\{w_1,x_3\},\\
R_L(u_3v_3) &= V(LSW_3)-\{w_2,x_1\}.
\end{aligned} \tag{5}$$

It is clear from above LRN sets that $|R_L(x_iw_{i+1})|=10$; now consider α: $V(LSW_3)\longrightarrow[0,1]$ as maximal LRF defined by $\alpha(x)=1/10\,\forall x\in V(LSW_3)$, hence $D_{LF}(LSW_3)\geq\sum_{i=1}^{12}1/10=6/5$. Likewise, $|R_L(u_iu_{i+1})|=8$ and $|R_L(u_iu_{i+1})|<|R_L(y)|$, $R_L(y)$ are other LRN sets of LSW_3; now consider β: $V(LSW_3)\longrightarrow[0,1]$ as minimal LRF defined by $\beta(v)=1/8$ to each $v\in V(LSW_3)$ hence $D_{LF}(LSW_3)\leq\sum_{i=1}^{12}1/8=3/2$.

$$6/5\leq D_{LF}(LSW_3)\leq 3/2. \tag{6}$$

Case 2.

For $k\geq 3$ with the reference of Lemma 2 $|R_L(u_iv_i)|=4k-2$ and $|R_L(u_iv_i)|\geq|R_L(y)|$. Moreover, the cardinality of each LRN set is not same. Therefore, we consider a maximal LLRF α: $V(LSW_k)\longrightarrow[0,1]$ defined by $\alpha(x)=1/4k-2$, $\forall x\in V(LSW_k)$ hence $D_{LF}(LSW_k)\geq\sum_{i=1}^{4k}1/4k-2=2k/2k-1$. Likewise, $|R_L(u_iu_{i+1})|=8$ and $|R_L(u_iu_{i+1}||\leq R_L(y)|,\forall y\in E(LSW_k)$. Now, we consider a minimal LRF β: $V(LSW_k)\longrightarrow[0,1]$ defined by $\beta(x)=1/8\,\forall x\in V(LSW_k)$, hence $D_{LF}(LSW_k)\leq\sum_{i=1}^{4k}1/8=k/2$.

$$\frac{2k}{2k-1}\leq D_{LF}(LSW_k)\leq\frac{k}{2}. \tag{7}$$

□

2.3. Double-Wheel Network. A double-wheel network DW_k is obtained from wheel network W_k by joining all the vertices of outer cycle with central vertex and each other. The vertex set $V(DW_k)=\{u_i,v_i,x\colon 1\leq i\leq k\}$ and $E(DW_k)=\{u_iu_{i+1},u_ix,v_ix,v_iv_{i+1}\colon 1\leq i\leq k\}$. For more details about double-wheel network, see Figure 3.

Lemma 4. *Let* DW_k *be a double-wheel network. Then,*

(a) $|R_L(x)|=|R_L(u_iu_{i+1})|=|R_L(v_iv_{i+1})|=8$ *and* $\cup_{i=1}^{2k}R_L(x)=2k$

(b) $|R_L(y)|\leq|R_L(u_iu_{i+1})|$ *and* $|\cup_{i=1}^{2k+1}R_L(u_iu_{i+1})|=2k$

Proof. Let DW_k be a double, where $k+1\,(\mathrm{mod}\,k)=1$

(a) $R_L(u_iu_{i+1})=R_L(v_iv_{i+1})=\{u_i,u_{i+1},\ u_{i+2},u_{k+i-1},\ v_i,v_{i+1},v_{i+2},v_{k+i-1}\}$, and $\cup_{i=1}^{2k}R_L(x)=V(DW_k)-\{x\}$, therefore $|\cup_{i=1}^{2k}R_L(x)|=2k$

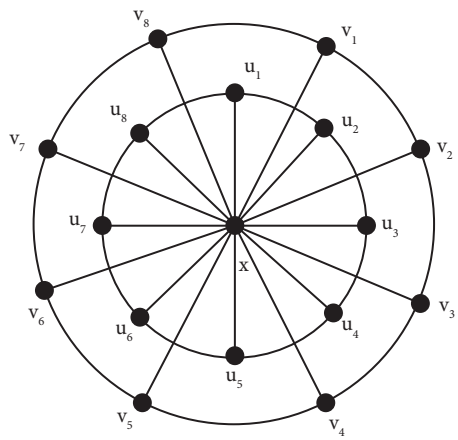

Figure 3: Double-wheel network DW_8.

(b) The $R_L(v_i x) = V(DW_k)$, $R_L(u_i x) = V(DW_k) - \{u_{i+1}, u_{k+i-1}\}$

Table 2 clears the order of each $R_L(y)$. □

Theorem 5. *Let* DW_3 *be a double-wheel network. Then,*

$$1 < D_{LF}(DW_3) \leq \frac{7}{3}. \tag{8}$$

Proof. The possible LRN sets of DW_3 are

$$\begin{aligned} R_L(u_1 x) &= \{u_1, v_1, x\}, \\ R_L(u_2 x) &= \{u_2, v_2, x\}, \\ R_L(u_3 x) &= \{u_3, v_3, x\}, \\ R_L(u_1 u_2) &= \{u_1, u_2, v_1, v_2\}, \\ R_L(u_2 u_3) &= \{u_2, u_3, v_2, v_3\}, \\ R_L(u_3 u_1) &= \{u_3, u_1, v_3, v_1\}, \\ R_L(u_1 v_1) &= V(DW_3), \\ R_L(u_2 v_2) &= V(DW_3), \\ R_L(u_3 v_3) &= V(DW_3). \end{aligned} \tag{9}$$

It is clear from above LRN sets that $|R_L(u_i x)| = 7$ now consider a maximal LRF α: $V(DW_3) \longrightarrow [0, 1]$ defined by $\alpha(v) = 1/7\, \forall v \in V(DW_3)$ hence $D_{LF}(DW_3) > \sum_{i=1}^{7} 1/7 = 1$. Likewise, $|R_L(u_i v_i)| = 3$. and $|R_L(u_i u_{i+1}| \leq |R_L(y)|$, where $R_L(y)$ are other LRN sets of DW_3 now consider a minimal LRF β: $V(DW_3) \longrightarrow [0, 1]$ defined by $\beta(v) = 1/3\, \forall v \in V(DW_3)$ hence $D_{LF}(DW_3) \leq \sum_{i=1}^{7} 1/3 = 7/3$.

$$1 < D_{LF}(DW_3) \leq \frac{7}{3}. \tag{10}$$

□

Table 2: LRN sets and their comparison.

LRN set	Comparison
$R_L(v_i x)$	$2k + 1 > 4$
$R_L(u_i x)$	$2k - 1 > 4$

Theorem 6. *Let* DW_5 *be a double-wheel network. Then,*

$$D_{LF}(DW_5) = 5. \tag{11}$$

Proof. The possible LRN sets of DW_5 are

$$\begin{aligned} R_L(v_1 x) &= \{u_1, u_2, u_3, v_1, x\}, \\ R_L(v_2 x) &= \{u_2, u_3, u_4, v_2, x\}, \\ R_L(v_3 x) &= \{u_3, u_4, u_5, v_3, x\}, \\ R_L(v_4 x) &= \{u_4, u_5, u_1, v_4, x\}, \\ R_L(v_5 x) &= \{u_5, u_1, u_2, v_5, x\}, \\ R_L(u_1 x) &= \{u_1, v_1, v_2, v_3, x\}, \\ R_L(u_2 x) &= \{u_2, v_2, v_3, v_4, x\}, \\ R_L(u_3 x) &= \{u_3, v_3, v_4, v_5, x\}, \\ R_L(u_4 x) &= \{u_4, v_4, v_5, v_1, x\}, \\ R_L(u_5 x) &= \{u_2, v_5, v_1, v_2, x\}, \\ R_L(u_1 u_2) &= \{u_1, u_2\}, \\ R_L(u_2 u_3) &= \{u_2, u_3\}, \\ R_L(u_3 u_4) &= \{u_3, u_4\}, \\ R_L(u_4 u_5) &= \{u_2, u_3\}, \\ R_L(u_5 u_1) &= \{u_1, u_5\}, \\ R_L(v_1 v_2) &= \{v_1, v_2\}, \\ R_L(v_2 v_3) &= \{v_2, v_3\}, \\ R_L(v_3 v_4) &= \{v_3, v_4\}, \\ R_L(v_4 v_5) &= \{v_4, v_5\}, \\ R_L(v_5 v_1) &= \{v_1, v_5\}. \end{aligned} \tag{12}$$

Since $|R_L(x)| = |R_L(u_i u_{i+1})| = |R_L(v_i v_{i+1})| = 2$ and $|R_L(x)| \leq |R_L(y)|, \forall y \in E(DW_5)$, we consider a constant LRF α: $V(DW_5) \longrightarrow [0, 1]$ defined by $\alpha(v) = 1/2\ \forall v \in \{u_i u_{i+1}\} \cup \{v_i v_{i+1}\}$. Therefore, $D_{LF}(DW_5) = \sum_{i=1}^{10} 1/2 = 5$.

$$D_{LF}(DW_5) = 5. \tag{13}$$

□

Theorem 7. *Let* DW_k *be a double-wheel network. Then,*

$$1 < D_{LF}(DW_k) \leq \frac{k}{2}. \tag{14}$$

Table 3: Bounds of wheel-related networks (WW_k, SW_k, LSW_k, and DW_k).

Network	LFMD	Lower bound	Upper bound	Comment
WW_k	$1 < D_{LF}(WW_k) \leq 2k + 1/8.$	1	2	Bounded
LSW_k	$3k/3k - 2 \leq D_{LF}(SW_k) \leq 3k/8.$	1	∞	Unbounded
DW_k	$1 < D_{LF}(DW_k) \leq k/2.$	1	∞	Unbounded

Proof. In order to prove the theorem, we have divided into a particular case (Case A) and general case (Case B).

Case A

$$\begin{aligned}
R_L(u_1u_2) &= \{u_1, u_2, u_3, u_6\},\\
R_L(u_2u_3) &= \{u_2, u_3, u_4, u_1\},\\
R_L(u_3u_4) &= \{u_3, u_4, u_5, u_2\},\\
R_L(u_4u_5) &= \{u_4, u_5, u_6, u_3\},\\
R_L(u_5u_6) &= \{u_5, u_6, u_1, u_4\},\\
R_L(u_6u_1) &= \{u_5, u_1, u_2, u_5\},\\
R_L(v_1v_2) &= \{v_1, v_2, v_3, v_6\},\\
R_L(v_2v_3) &= \{v_2, v_3, v_4, v_1\},\\
R_L(v_3v_4) &= \{v_3, v_4, v_5, v_2\},\\
R_L(v_4v_5) &= \{v_4, v_5, v_6, v_3\},\\
R_L(v_5v_6) &= \{v_5, v_6, v_1, v_4\},\\
R_L(v_6v_1) &= \{v_6, v_1, v_2, v_5\},\\
R_L(u_1x) &= V(DW_6),\\
R_L(u_2x) &= V(DW_6),\\
R_L(u_3x) &= V(DW_6),\\
R_L(u_4x) &= V(DW_6),\\
R_L(u_5x) &= V(DW_6),\\
R_L(u_6x) &= V(DW_6),\\
R_L(v_1x) &= V(DW_6),\\
R_L(v_2x) &= V(DW_6),\\
R_L(v_3x) &= V(DW_6),\\
R_L(v_4x) &= V(DW_6),\\
R_L(v_5x) &= V(DW_6),\\
R_L(v_6x) &= V(DW_6).
\end{aligned} \tag{15}$$

From above LRN sets that $|R_L(v_ix)| = |R_L(u_ix)| = 13$ now consider a maximal ULRF α: $V(DW_6) \longrightarrow [0, 1]$ by $\alpha(v) = 1/13\ \forall v \in V(LSW_6)$ hence $D_{LF}(DW_6) \geq \sum_{i=1}^{13} 1/13 = 1$. Likewise, $|R_L(u_iu_{i+1})| = |R_L(v_iv_{i+1})| = 4$ and $|R_L(u_iu_{i+1}| \leq |R_L(y)|$, where $R_L(y)$ are other LRN sets of DW_6 now consider a minimal LLRF β: $V(DW_6) \longrightarrow [0, 1]$ defined by $\beta(v) = 1/4\ \forall v \in V(DW_5)$ hence $D_{LF}(DW_6) \leq \sum_{i=1}^{12} 1/4 = 3$.

$$1 < D_{LF}(DW_6) \leq 3. \tag{16}$$

Case B.

For $k \geq 6$ with the reference of Lemma 4, $|R_L(u_iv_i)| = 2k + 1$ and $|R_L(u_iv_i)| \geq |R_L(y)|, \forall y \in E(DW_k)$. Now, we consider a maximal LLRF α: $V(DW_k) \longrightarrow [0, 1]$ defined by $\alpha(v) = 1/2k + 1$ to each $v \in V(DW_k)$ hence $D_{LF}(DW_k) > \sum_{i=1}^{2k+1} 1/2k + 1 = 1$. Likewise, $|R_L(u_i\ u_{i+1})| = 4$ and $|R_L(u_iu_{i+1})| \leq |R_L(y)|$. Again, we consider a minimal LRF β: $V(DW_k) \longrightarrow [0, 1]$ defined by $\beta(v) = 1/4$ $\forall v \in V(DW_k)$ hence $D_{LF}(DW_k) \leq \sum_{i=1}^{2k} 1/4 = k/2$.

$$1 < D_{LF}(DW_k) \leq \frac{k}{2}. \tag{17}$$

□

3. Conclusion

In this article, we have obtained the sharp bounds of the local fractional metric dimension of wheel-related networks such as the web-wheel network, subdivision of wheel network, line network of subdivision of wheel network, and double-wheel network. It has been proved that link networks of subdivision of wheel network (LSW_k) and double-wheel network (DW_k) remain unbounded when the order of these networks approaches to ∞. Moreover, the LFMD of subdivision of wheel network is exactly 1, and in future, it would be very interesting to investigate the LFMD of all the wheel-related networking attaining an exact value.

The boundedness and unboundedness other than the subdivision of wheel networks is also obtained in Table 3.

Data Availability

The data used to support the findings of this study are included within this article. However, the reader may contact the corresponding author for more details on the data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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