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Research Article

Exact Null Controllability of String Equations with Neumann Boundaries

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This article focuses on the exact null controllability of a one-dimensional wave equation in noncylindrical domains. Both the fixed endpoint and the moving endpoint are Neumann-type boundary conditions. The control is put on the moving endpoint. When the speed of the moving endpoint is less than the characteristic speed, we can obtain the exact null controllability of this equation by using the Hilbert uniqueness method. In addition, we get a sharper estimate on controllability time that depends on the speed of the moving endpoint.

1. Introduction

Given $T > 0$, let us consider the noncylindrical domain $\widehat{Q}_T^k$, defined by

$$\widehat{Q}_T^k = \left\{(x,t) \in \mathbb{R}^2; 0 < x < \alpha_k(t), \text{for all } t \in (0,T)\right\}, \tag{1}$$

where

$$\alpha_k(t) = 1 + kt. \tag{2}$$

Let

$$V(0,\alpha_k(t)) = \left\{\varphi \in H^1(0,\alpha_k(t)); \varphi(0) = 0\right\}, \quad \text{for } t \in [0,T], \tag{3}$$

which is a subspace of $H^1(0,\alpha_k(t))$. $[V(0,\alpha_k(t))]'$ donates its conjugate space.

Consider the motion of a string with one endpoint fixed and the other moving. It can be described by the following wave equation in the noncylindrical domain $\widehat{Q}_T^k$:

$$\begin{cases} u_{tt} - u_{xx} = 0, & \text{in } \widehat{Q}_T^k, \\ u_x(0,t) = 0, u_x(\alpha_k(t),t) = v(t), & \text{on } (0,T), \\ u(x,0) = u^0(x), u_t(x,0) = u^1(x), & \text{in } (0,1), \end{cases} \tag{4}$$

where v is the control variable, u is the state variable, and $(u^0,u^1) \in L^2(0,1) \times [V(0,1)]'$ is any given initial value. The constant k is called the speed of the moving endpoint. Using the similar method in [1, 2], system (4) has a unique solution in the sense of a transposition.

$$u \in C\left([0,T]; L^2(0,\alpha_k(t))\right) \bigcap C^1\left([0,T]; [V(0,\alpha_k(t))]'\right). \tag{5}$$

Control problems can be found everywhere in science, technology, and engineering practice. Fixed-time control has been used in areas such as multiagent systems (MASs), path following in autonomous vehicles, nonlinear parameterisation, nonholonomic systems, and robotic systems (for details, see [3]). In the physical sense, the application of exact controllability of wave equations in noncylindrical domains is also very extensive. A classical example is the interface of an ice-water mixture when temperature rises. Therefore, it is very necessary to study exact controllability of such wave equations.

The main purpose of this article is to consider the exact null controllability of (4). For the controllability problem of wave equations in cylindrical domains, it has already been studied by different authors. However, not much work has been done on the wave equations defined in noncylindrical domains. We refer to [1, 4–11] for some known results in this respect. In [1, 4], the exact controllability of a wave equation in a certain noncylindrical domain was studied. In [5], a globally distributed control was obtained by stabilization of the wave equation in a noncylindrical domain. In [6, 8–11], the exact Dirichlet boundary controllability of the following systems was discussed:

$$\begin{cases} u_{tt} - u_{yy} = 0, & \text{in } \widehat{Q}_T^k, \\ u(0,t) = 0, u(\alpha_k(t), t) = v(t), & \text{on } (0,T), \\ u(y,0) = u^0(y), u_t(y,0) = u^1(y), & \text{in } (0,1), \end{cases} \tag{6}$$

and

$$\begin{cases} u_{tt} - u_{yy} = 0, & \text{in } \widehat{Q}_T^k, \\ u(0,t) = v(t), u(\alpha_k(t), t) = 0, & \text{on } (0,T), \\ u(y,0) = u^0(y), u_t(y,0) = u^1(y), & \text{in } (0,1). \end{cases} \tag{7}$$

Reference [8] improved the exact controllability time of [6]. Reference [7] dealt with the exact controllability of a one-dimensional wave equation with mixed boundary conditions, in which a noncylindrical domain is transformed into a cylindrical domain. The system is as follows:

$$\begin{cases} u_{\mathrm{tt}} - u_{\mathrm{yy}} = 0, & \text{in } \widehat{Q}_T^k, \\ u(0,t) = 0, u_y(\alpha_k(t), t) = v(t), & \text{on } (0,T), \\ u(y,0) = u^0(y), u_t(y,0) = u^1(y), & \text{in } (0,1). \end{cases} \tag{8}$$

In this article, we consider the exact null controllability of the wave equation with Neumann-type boundary conditions by taking a direct calculation in a noncylindrical domain when $k \in (0, \sqrt{3}/2)$. But it is still the open problem and we need to overcome in the future when $k \in (\sqrt{3}/2, 1)$.

This paper is organized as follows. In Section 2, we give some definitions and main theorems. In Section 3, we obtain two key inequalities by using the multiplier method used in Section 4. In Section 4, using the Hilbert uniqueness method, we give the proof of exact null controllability of (4).

2. Preliminary Work and Main Results

The goal of this paper is to study exact null controllability of (4) in the following sense.

Definition 1. Equation (4) is called to be null controllable at the time T, if for any given initial value

$$\left(u^0, u^1\right) \in L^2(0,1) \times [V(0,1)]', \tag{9}$$

one can always find a control $v \in [H^1(0,T)]'$ such that the corresponding solution u of (4) in the sense of a transposition satisfies

$$\begin{aligned} u(T) &= 0, \\ u_t(T) &= 0. \end{aligned} \tag{10}$$

Definition 2. Equation (4) is called to be exactly controllable at the time T, if for any given initial value

$$\left(u^0, u^1\right) \in L^2(0,1) \times [V(0,1)]', \tag{11}$$

and any target function

$$\left(u_d^0, u_d^1\right) \in L^2(0, \alpha_k(T)) \times [V(0, \alpha_k(T))]', \tag{12}$$

one can always find a control $v_1 \in [H^1(0,T)]'$ such that the corresponding solution u of (4) in the sense of a transposition satisfies

$$u(T) = u_d^0, u_t(T) = u_d^1. \tag{13}$$

Remark 3. Null controllability of (4) is equivalent to exact controllability of (4).

Throughout this article, we set

$$T_k^* = \frac{2}{1-k}, \tag{14}$$

for the controllability time. The specific proof will be given later in this paper.

Remark 4. It is easy to verify

$$\begin{aligned} T_0 &= \lim_{k \longrightarrow 0} T_k^* \\ &= \lim_{k \longrightarrow 0} \frac{2}{1-k} = 2. \end{aligned} \tag{15}$$

The time $T_0 = 2$ is in accordance with the controllability time obtained in [12].

The main results of this paper are the following theorems.

Theorem 5. *For any given $T > T_k^*$, equation (4) is exactly null controllable at time T in the sense of Definition 1.*

The key to Proof of Theorem 5 is two important inequalities for the following homogeneous wave equation in the noncylindrical domain $\widehat{Q}_T^k$:

$$\begin{cases} z_{tt} - z_{xx} = 0, & \text{in } \widehat{Q}_T^k, \\ z_x(0,t) = 0, z_x(\alpha_k(t), t) + 2kz_t(\alpha_k(t), t) = 0, & \text{on } (0,T), \\ z(x,0) = z^0(x), z_t(x,0) = z^1(x), & \text{in } (0,1), \end{cases} \tag{16}$$

where $(z^0, z^1) \in V(0,1) \times L^2(0,1)$ is any given initial value. By [1], we know that (16) has a unique weak solution.

$$z \in C^1\left([0,T]; V\left(0, \alpha_k(t)\right)\right) \cap C\left([0,T]; L^2\left(0, \alpha_k(t)\right)\right). \tag{17}$$

We have the following two important inequalities. The proof of the two important inequalities is given in Section 3.

Theorem 6. *Let $T > T_k^*$. For any $(z^0, z^1) \in V(0,1) \times L^2(0,1)$, there exists a constant $C > 0$ such that the corresponding solution z of (16) satisfies*

$$\begin{aligned} &C\left(\left|z^0\right|^2_{V(0,1)} + \left|z^1\right|^2_{L^2(0,1)}\right) \\ &\quad \leq \int_0^T \left|z_t\left(\alpha_k(t), t\right)\right|^2 dt \\ &\quad \leq C\left(\left|z^0\right|^2_{V(0,1)}\right| + \left|z^1\right|^2_{L^2(0,1)}\Big|\right). \end{aligned} \tag{18}$$

Remark 7. In fact, for a more general function $\alpha_k(t)$, where $0 < \alpha_k'(t) < \sqrt{3}/2$, we can obtain the same results as in this paper.

Remark 8. We denote by C a positive constant depending only on T and k, which may be different from one place to another.

3. Observability: Proof of Theorem 6

In this section, in order to prove Theorem 6, we need the following lemmas.

We define the following weighted energy function for (16):

$$E(t) = \frac{1}{2}\int_0^{\alpha_k(t)} \left[\left|z_t(x,t)\right|^2 + \left|z_x(x,t)\right|^2\right] dx, \quad \text{for } t \geq 0, \tag{19}$$

where z is the solution of (16). It follows that

$$E(0) = \frac{1}{2}\int_0^1 \left[\left|z^1(x)\right|^2 + \left|z_x^0(x)\right|^2\right] dx. \tag{20}$$

Lemma 9. *For any $(z^0, z^1) \in V(0,1) \times L^2(0,1)$ and $t \in [0,T]$, the corresponding solution z of (16) satisfies*

$$E(t) - E(0) = \frac{k\left(4k^2 - 3\right)}{2}\int_0^t \left|z_s\left(\alpha_k(s), s\right)\right|^2 ds. \tag{21}$$

Proof. Multiplying the first equation of (16) by $z_s(x,s)$ and integrating on $(0, \alpha_k(s)) \times (0,t)$, we derive

$$\begin{aligned} 0 &= \int_0^t \int_0^{\alpha_k(s)} \left[z_{ss}(x,s) - z_{xx}(x,s)\right] z_s(x,s) \mathrm{d}x\mathrm{d}s \\ &= \frac{1}{2}\int_0^t \int_0^{\alpha_k(s)} \left[\left|z_s(x,s)\right|^2 + \left|z_x(x,s)\right|^2\right]_s \mathrm{d}x\mathrm{d}s \\ &\quad - \int_0^t \int_0^{\alpha_k(s)} \left(z_s(x,s) z_x(x,s)\right)_x \mathrm{d}x\mathrm{d}s. \end{aligned} \tag{22}$$

Since $\alpha_{k,s}(s) = k$, it follows from this abovementioned equality that

$$\begin{aligned} 0 &= \frac{1}{2}\int_0^t \frac{\partial}{\partial s}\int_0^{\alpha_k(s)} \left[\left|z_s(x,s)\right|^2 + \left|z_x(x,s)\right|^2\right] \mathrm{d}x\mathrm{d}s \\ &\quad - \frac{k}{2}\int_0^t \left[\left|z_s\left(\alpha_k(s), s\right)\right|^2 + \left|z_x\left(\alpha_k(s), s\right)\right|^2\right] \mathrm{d}s \\ &\quad - \int_0^t z_s(x,s) z_x(x,s)\Big|_0^{\alpha_k(s)} \mathrm{d}s \\ &= \frac{1}{2}\int_0^{\alpha_k(t)} \left[\left|z_t(x,t)\right|^2 + \left|z_x(x,t)\right|^2\right] dx \\ &\quad - \frac{1}{2}\int_0^1 \left[\left|z_t(x,0)\right|^2 + \left|z_x(x,0)\right|^2\right] dx \\ &\quad - \frac{k}{2}\int_0^t \left[\left|z_s\left(\alpha_k(s), s\right)\right|^2 + \left|z_x \alpha_k(s), s\right|^2\right] ds \\ &\quad - \int_0^t z_s\left(\alpha_k(s), s\right) z_x\left(\alpha_k(s), s\right) ds \\ &\quad + \int_0^t z_s(0,s) z_x(0,s) ds. \end{aligned} \tag{23}$$

Considering $z_x(0,s) = 0$ and the definition of $E(t)$ and $E(0)$, it follows from (23) that

$$\begin{aligned} &E(t) - E(0) \\ &\quad = \frac{k}{2}\int_0^t \left[\left|z_s\left(\alpha_k(s), s\right)\right|^2 + \left|z_x\left(\alpha_k(s), s\right)\right|^2\right] ds \\ &\quad\quad + \int_0^t z_s\left(\alpha_k(s), s\right) z_x\left(\alpha_k(s), s\right) ds. \end{aligned} \tag{24}$$

Note that

$$z_x\left(\alpha_k(s), s\right) = -2k z_s\left(\alpha_k(s), s\right). \tag{25}$$

Therefore, we derive that

$$
\begin{aligned}
&E(t)-E(0)\\
&=\frac{k}{2}\int_0^t\left[\left|z_s(\alpha_k(s),s)\right|^2+\left|-2kz_s(\alpha_k(s),s)\right|^2\right]ds\\
&\quad-2k\int_0^t z_s(\alpha_k(s),s)z_s(\alpha_k(s),s)ds\\
&=\frac{k\left(4k^2-3\right)}{2}\int_0^t\left|z_s(\alpha_k(s),s)\right|^2ds.
\end{aligned}
\tag{26}
$$

□

Remark 10. For $k\in(0,\sqrt{3}/2)$, according to (21), it is easy to check that

$$E^{'}(t)=\frac{k\left(4k^2-3\right)}{2}\left|z_t(\alpha_k(t),t)\right|^2<0. \tag{27}$$

We can obtain that $E(t)$ is a monotonically decreasing function and

$$E(t)<E(0). \tag{28}$$

Lemma 11. *For any $(z^0,z^1)\in V(0,1)\times L^2(0,1)$ and $t\in[0,T]$, the corresponding solution z of (16) satisfies*

$$
\begin{aligned}
&\int_0^T\alpha_k(t)|z_t(\alpha_k(t),t)|^2dt\\
&=2\int_0^T E(t)dt+2\int_0^{\alpha_k(T)}xz_t(x,T)z_x(x,T)dx\\
&\quad-2\int_0^1 xz_t(x,0)z_x(x,0)dx.
\end{aligned}
\tag{29}
$$

Proof. Multiplying the first equation of (16) by $2xz_x(x,s)$ and integrating on $\widehat{Q}_T^k$, we have

$$
\begin{aligned}
0&=\int_0^T\int_0^{\alpha_k(t)}\left[z_{tt}(x,t)-z_{xx}(x,t)\right]2xz_x(x,t)dxdt\\
&=2\int_0^T\int_0^{\alpha_k(t)}\left(xz_t(x,t)z_x(x,t)\right)_tdxdt\\
&\quad-\int_0^T\int_0^{\alpha_k(t)}\left[x\left|z_t(x,t)\right|^2+x\left|z_x(x,t)\right|^2\right]_xdxdt\\
&\quad+\int_0^T\int_0^{\alpha_k(t)}\left[\left|z_t(x,t)\right|^2+\left|z_x(x,t)\right|^2\right]dxdt.
\end{aligned}
\tag{30}
$$

By the definition of $E(t)$ and $\alpha_{k,t}(t)=k$, we can deduce that

$$
\begin{aligned}
0&=2\int_0^T\frac{\partial}{\partial t}\int_0^{\alpha_k(t)}xz_t(x,t)z_x(x,t)dxdt\\
&\quad-2k\int_0^T\alpha_k(t)z_t(\alpha_k(t),t)z_x(\alpha_k(t),t)dt\\
&\quad-\int_0^T x\left[\left|z_t(x,t)\right|^2+\left|z_x(x,t)\right|^2\right]\Big|_0^{\alpha_k(t)}dt+2\int_0^T E(t)dt\\
&=2\int_0^{\alpha_k(T)}xz_t(x,T)z_x(x,T)dx\\
&\quad-2\int_0^1 xz_t(x,0)z_x(x,0)dx\\
&\quad-2k\int_0^T\alpha_k(t)z_t(\alpha_k(t),t)z_x(\alpha_k(t),t)dt\\
&\quad-\int_0^T\alpha_k(t)\left[\left|z_t(\alpha_k(t),t)\right|^2+\left|z_x(\alpha_k(t),t)\right|^2\right]dt\\
&\quad+2\int_0^T E(t)dt.
\end{aligned}
\tag{31}
$$

Furthermore, from $z_x(\alpha_k(t),t)=-2kz_t(\alpha_k(t),t)$, we get

$$
\begin{aligned}
&\int_0^T\alpha_k(t)|z_t(\alpha_k(t),t)|^2dt\\
&=2\int_0^T E(t)dt+2\int_0^{\alpha_k(T)}xz_t(x,T)z_x(x,T)dx\\
&\quad-2\int_0^1 xz_t(x,0)z_x(x,0)dx.
\end{aligned}
\tag{32}
$$

In the following, we will give the Proof of Theorem 6, which has two steps. □

Proof of Theorem 6

Step 1. In the following, we give the proof of the first inequality in (18).

From the Cauchy inequality, we obtain the estimate

$$\left|2\int_0^{\alpha_k(T)}xz_t(x,t)z_x(x,t)dx\right|\le 2\alpha_k(T)E(T), \tag{33}$$

$$\left|2\int_0^1 xz_t(x,0)z_x(x,0)dx\right|\le 2E(0). \tag{34}$$

Combining (28), (29), (33), and (34), it holds that

$$
\begin{aligned}
&\int_0^T\alpha_k(t)|z_t(\alpha_k(t),t)|^2dt\\
&=2\int_0^T E(t)dt+2\int_0^{\alpha_k(T)}xz_t(x,T)z_x(x,T)dx\\
&\quad-2\int_0^1 xz_t(x,0)z_x(x,0)dx\\
&\ge 2\int_0^T E(t)dt-2\alpha_k(T)E(T)-2E(0)\\
&\ge 2\int_0^T E(t)dt-2\alpha_k(T)E(0)-2E(0).
\end{aligned}
\tag{35}
$$

From (21), we find that

$$\begin{aligned}&\int_0^T \alpha_k(t)|z_t(\alpha_k(t),t)|^2dt\\&\geq 2\int_0^T E(0)+\frac{k(4k^2-3)}{2}\int_0^t |z_s(\alpha_k(s),s)|^2dsdt\\&\quad -2\alpha_k(T)E(0)-2E(0).\end{aligned}\tag{36}$$

From this, one concludes that

$$\begin{aligned}&\left[\alpha_k(T)-k\left(4k^2-3\right)T\right]\int_0^T \left|z_t(\alpha_k(t),t)\right|^2dt\\&\quad\geq 2(T-\alpha_k(T)-1)E(0).\end{aligned}\tag{37}$$

If $T>T_k^*$, we have $2[(T-\alpha_k(T))-1]>0$, and from this inequality and (37), it holds that

$$\begin{aligned}&\int_0^T \left|z_t(\alpha_k(t),t)\right|^2dt\\&\geq\left[\alpha_k(T)-k\left(4k^2-3\right)T\right]^{-1}[2(T-\alpha_k(T)-1)]E(0)\\&\geq\left[\alpha_k(T)-k\left(4k^2-3\right)T\right]^{-1}[2(T-\alpha_k(T)-1)]\left(\left|z^0\right|^2_{V(0,1)}\right|+\left|z^1\right|^2_{L^2(0,1)}\left|\right).\end{aligned}\tag{38}$$

This completes the proof of the first inequality in (18).

Step 2. In the following, we give the proof of the second inequality in (18).

From (28), (29), (33), and (34), one concludes that

$$\begin{aligned}&\int_0^T \alpha_k(t)|z_t(\alpha_k(t),t)|^2dt\\&=2\int_0^T E(t)dt+2\int_0^{\alpha_k(T)} xz_t(x,T)z_x(x,T)dx\\&\quad -2\int_0^1 xz_t(x,0)z_x(x,0)dx\\&\leq 2\int_0^T E(t)dt+2\alpha_k(T)E(T)+2E(0)\\&\leq 2(T+\alpha_k(T)+1)E(0).\end{aligned}\tag{39}$$

This implies that one can find a positive constant C such that

$$\begin{aligned}&\int_0^T |z_t(\alpha_k(t),t)|^2dt\\&\leq C[2(T+\alpha_k(T)+1)]E\\&\leq C[2(T+\alpha_k(T)+1)]\left(\left|z^0\right|^2_{V(0,1)}+\left|z^1\right|^2_{L^2(0,1)}\right).\end{aligned}\tag{40}$$

This completes the proof of the second inequality in (18).

By (38) and (40), we get the desired results in Theorem 6. □

4. Controllability: Proof of Theorem 5

In this section, we prove the exact null controllability for wave (4) in the noncylindrical domain $\widehat{Q}_T^k$ (Theorem 5) for $k\in(0,\sqrt{3}/2)$ by the Hilbert uniqueness method.

Proof of Theorem 5. We divide the Proof of Theorem 5 into three steps.

Step 1. We define the linear operator $\Gamma: V(0,1)\times L^2(0,1)\longrightarrow[V(0,1)]'\times L^2(0,1)$.

For any $(z^0,z^1)\in V(0,1)\times L^2(0,1)$, we denote by z the corresponding solution of (16). Consider the wave equation

$$\begin{cases}\xi_{tt}-\xi_{xx}=0, & \text{in }\widehat{Q}_T^k,\\ \xi_x(0,t)=0,\ \xi_x(\alpha_k(t),t)=G_{z(\alpha_k(t),t)}, & \text{on }(0,T),\\ \xi(x,T)=0,\ \xi_t(x,T)=0, & \text{in }(0,1).\end{cases}\tag{41}$$

It is worth noting that here $G_{z(\alpha_k(t),t)}$ is defined as follows:

$$\begin{aligned}&\left\langle G_{z(\alpha_k(t),t)},\phi\right\rangle_{((H^1(0,T))',H^1(0,T))}\\&\quad=\int_0^T z_t(\alpha_k(t),t)\phi_t(t)dt,\quad\text{for any }\phi\in H^1(0,T).\end{aligned}\tag{42}$$

By [1], we know that (41) has a unique weak solution ξ in the sense of a transposition. We set

$$\left(\xi^0,\xi^1\right)\triangleq(\xi(x,0),\xi_t(x,0))\in L^2(0,1)\times[V(0,1)]'.\tag{43}$$

Now, we define the operator

$$\begin{aligned}\Gamma\colon V(0,1)\times L^2(0,1) &\longrightarrow L^2(0,1)\times [V(0,1)]',\\ \left(z^0,z^1\right) &\longrightarrow \left(\xi^0,-\xi^1\right).\end{aligned} \tag{44}$$

Therefore,

$$\left\langle \Gamma\left(z^0,z^1\right),\left(z^0,z^1\right)\right\rangle = \int_0^1 z^1\xi^0 - z^0\xi^1 dx. \tag{45}$$

Step 2. Multiplying the first equation of (41) by $z(x,t)$ and integrating on $\widehat{Q}_T^k$, we obtain

$$\begin{aligned}0 &= \int_0^T\int_0^{\alpha_k(t)} \left[\xi_{tt}(x,t) - \xi_{xx}(x,t)\right] z(x,t)dxdt\\ &= \int_0^T\int_0^{\alpha_k(t)} \left[\xi_t(x,t)z(x,t) - \xi(x,t)z_t(x,t)\right]_t dxdt\\ &\quad - \int_0^T\int_0^{\alpha_k(t)} \left[\xi_x(x,t)z(x,t) - \xi(x,t)z_x(x,t)\right]_x dxdt.\end{aligned} \tag{46}$$

Since $\alpha_{k,t}(t) = k$, it follows from (46) that

$$\begin{aligned}0 &= \left[\left[\int_0^{\alpha_k(t)} \xi_t(x,t)z(x,t) - \xi(x,t)z_t(x,t)dx\right]\right|_0^T\\ &\quad - k\int_0^T \left[\xi_t(\alpha_k(t),t)z(\alpha_k(t),t) - \xi(\alpha_k(t),t)z_t(\alpha_k(t),t)\right]dt\\ &\quad - \int_0^T \left[\xi_x(x,t)z(x,t) - \xi(x,t)z_x(x,t)\right]\Big|_0^{\alpha_k(t)}dt\\ &= \int_0^{\alpha_k(T)} \xi_t(x,T)z(x,T) - \xi(x,T)z_t(x,T)dx\\ &\quad - \int_0^1 \xi_t(x,0)z(x,0) - \xi(x,0)z_t(x,0)dx\\ &\quad - k\int_0^T \left[\xi_t(\alpha_k(t),t)z(\alpha_k(t),t) - \xi(\alpha_k(t),t)z_t(\alpha_k(t),t)\right]dt\\ &\quad - \int_0^T \left[\xi_x(\alpha_k(t),t)z(\alpha_k(t),t) - \xi(\alpha_k(t),t)z_x(\alpha_k(t),t)\right]dt\\ &\quad + \int_0^T \left[\xi_x(0,t)z(0,t) - \xi(0,t)z_x(0,t)\right]dt.\end{aligned} \tag{47}$$

Using the following conditions,

$$\begin{aligned}\xi_t(T) = \xi(T) &= z_x(0,t)\\ &= \xi_x(0,t) = 0,\\ z_x(\alpha_k(t),t) + 2kz_t(\alpha_k(t),t) &= 0,\end{aligned} \tag{48}$$

equation (47) can conclude that

$$\int_0^T G_{z(\alpha_k(t),t)} z(\alpha_k(t),t)dt = \int_0^1 z^1\xi^0 - z^0\xi^1 dx. \tag{49}$$

From (42), it holds that

$$\int_0^T \left|z_t(\alpha_k(t),t)\right|^2 dt = \int_0^1 z^1\xi^0 - z^0\xi^1 dx. \tag{50}$$

By Theorem 6, Γ is bounded and coercive. Hence, by the Lax–Milgram theorem, we can conclude that Γ is an isomorphism.

Step 3. We proved exact null controllability of (4). Indeed, for any given initial value

$$\left(u^0,u^1\right) \in L^2(0,1)\times [V(0,1)]', \tag{51}$$

we choose

$$v(\cdot) = G_{z(\cdot,t)} \in \left(H^1(0,T)\right)', \tag{52}$$

where z is the solution of (16) associated to $\Gamma(z^0,z^1) = (u^0,-u^1)$.

From the definition of (14), we conclude that

$$\Gamma\left(z^0,z^1\right) = \left(\xi^0,-\xi^1\right), \tag{53}$$

where ξ is the solution of (41). Then, ξ satisfies

$$\left(\xi^0,-\xi^1\right) = \left(u^0,-u^1\right). \tag{54}$$

By the uniqueness of (41), u satisfies

$$\left(u(x,T),u_t(x,T)\right) = (0,0). \tag{55}$$

Hence, we obtain exact null controllability of (4). □

5. Conclusion

In this paper, we focus on exact null controllability of a one-dimensional wave equation in noncylindrical domains. Both the fixed endpoint and the moving endpoint are Neumann-type boundary conditions. When the speed of the moving endpoint is less than the characteristic speed, we obtain exact null controllability of this equation by using the Hilbert uniqueness method. In the future, we will consider the case of more complex wave equations, such as variable coefficient wave equations.

Data Availability

No data were used to support the findings of this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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