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Research Article

A Note on Weakly Semiprime Ideals and Their Relationship to Prime Radical in Noncommutative Rings

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In this paper, we introduce the concept of weakly semiprime ideals and weakly n-systems in noncommutative rings. We establish the equivalence between an ideal P being a weakly semiprime ideal and $R-P$ being a weakly n-system. We provide alternative definitions and explore the properties of weakly semiprime ideals. Additionally, we investigate scenarios where all ideals in a given ring are weakly semiprime and demonstrate that in Noetherian rings, where every ideal is weakly semiprime, the prime radical and the Jacobson radical coincide.

1. Introduction

The notion of weakly semiprime ideals in commutative rings was first introduced by Badawi [1] in 2016. In this framework, a proper ideal P of a commutative ring R with an identity is considered weakly semiprime if, for any $a \in R$, the condition $0 \neq a^2 \in P$ implies $a \in P$. Prior to this, Anderson and Smith [2] introduced the concept of weakly prime ideals in 2003. They defined an ideal P in a commutative ring R as weakly prime if, for any $a, b \in R$, the condition $0 \neq ab \in P$ implies either $a \in P$ or $b \in P$. Both weakly prime and weakly semiprime ideals can be viewed as generalizations of prime ideals.

Exploring the extension of mathematical concepts to noncommutative rings has garnered significant attention from researchers. Notably, works such as [3, 4] have delved into the generalization of weakly prime ideals in noncommutative rings. In the context of these generalizations, an ideal P of a ring R is considered weakly prime if, for any ideals J and K of R, the condition $0 \neq J.\, K \subseteq P$ implies either $J \subseteq P$ or $K \subseteq P$, as demonstrated in [4]. Furthermore, the concept of almost prime, right primary, and nilary ideals has been extended by the author and Fındık in [5–7].

Based on reference [8], let us consider a ring R. An m-system (n-system) in R is a nonempty subset $S \subseteq R$ such that for any, a, $b \in S$ ($a \in S$), there exists $r \in R$ satisfying $arb \in S$ ($ara \in S$). An ideal P of R is said to be prime (semiprime) if and only if the complement $R-P$ is an m-system (n-system). Furthermore, if an ideal P is maximally disjoint from any m-systems $S \subseteq R$, then P is a prime ideal.

In this article, we introduce the concepts of weakly semiprime ideals and weakly n-systems in a general ring R. A weakly semiprime ideal P in R is defined as an ideal that satisfies the condition: if A is an ideal of R with $0 \neq A^2 \subseteq P$, then $A \subseteq P$. On the other hand, a nonempty set $S \subseteq R$ is considered a weakly n-system if, for any ideal I of R with $I \cap S \neq \varnothing$ and $I^2 \neq 0$, we have $I^2 \cap S \neq \varnothing$.

We establish a fundamental result that characterizes weakly semiprime ideals: an ideal P in R is a weakly semiprime ideal if and only if its complement $R-P$ forms a weakly n-system. We also demonstrate the existence of minimal weakly semiprime ideals within any given weakly semiprime ideal (Theorem 7). Moreover, we provide several equivalent conditions for an ideal to be classified as a weakly semiprime ideal. Notably, Proposition 9 highlights that if an ideal P is a weakly semiprime ideal but not a semiprime ideal, then it is contained in the prime radical ($\mathrm{Nil}_*(R)$) of the ring R. In addition, we investigate the properties of images and inverse images of weakly semiprime ideals under ring homomorphisms. We provide characterizations for rings in which every ideal is a weakly semiprime ideal.

Specifically, we establish that in such rings, the square of the sum of all ideals whose squares are zero, denoted as $N(R)$, is contained in $\mathrm{Nil}_*(R)$. Furthermore, we determine the necessary and sufficient condition for a ring to be a fully weakly semiprime ring. Moreover, we prove that if every ideal of a Noetherian ring with identity is a weakly semiprime ideal, then we have $\mathrm{Nil}_*(R) = \mathrm{rad}(R)$ and $[N(R)]^2 = [\mathrm{Nil}_*(R)]^2 = [\mathrm{rad}(R)]^2 = 0$. Here, $\mathrm{rad}(R)$ refers to the Jacobson radical of the ring.

Throughout this paper, unless otherwise specified, we consider rings that are associative, noncommutative, and without identity. Additionally, when we refer to an ideal, we specifically mean a proper two-sided ideal.

2. Weakly Semiprime Ideals

Definition 1. Let P be an ideal of a ring R. We call P a weakly semiprime ideal if whenever A is an ideal of R with $0 \neq A^2 \subseteq P$, then $A \subseteq P$.

It is evident that every prime, weakly prime, and semiprime ideal is also a weakly semiprime ideal. The zero ideal is always a weakly semiprime ideal although it may not necessarily be a semiprime ideal in general.

Definition 2. Let R be a ring. A nonempty set $S \subseteq R$ is called a weakly n-system if whenever I is an ideal of R with $I \cap S \neq \phi$ and $I^2 \neq 0$, then $I^2 \cap S \neq \phi$.

Proposition 3. *An ideal P of a ring R is a weakly semiprime ideal if and only if $R-P$ is a weakly n-system.*

Proof. Suppose that P is a weakly semiprime ideal, and let I be an ideal such that $I \cap (R-P) \neq \varnothing$, and $I^2 \neq 0$. Assume that $I^2 \cap (R-P) = \varnothing$, then $0 \neq I^2 \subseteq P$, thus $I \subseteq P$. Contradiction. Thus, $I^2 \cap (R-P) \neq \varnothing$.

Conversely, suppose that $R-P$ is a weakly n-system, and let $0 \neq I^2 \subseteq P$, for any ideal I of R, and assume that $I \subseteq P$. Thus, $I \cap (R-P) \neq \varnothing$. Since $R-P$ is a weakly n-system, then $I^2 \cap (R-P) \neq \varnothing$, a contradiction because $I^2 \subseteq P$. Thus, $I \subseteq P$. □

Theorem 4. *For any ideal P of a ring R with identity, the following statements are equivalent:*

(1) P is a weakly semiprime ideal

(2) For $a \in R$, if $0 \neq \langle a \rangle^2 \subseteq P$, then $a \in P$

(3) For $a \in R$, if $0 \neq aRa \subseteq P$, then $a \in P$

(4) For any left ideal A of R, if $0 \neq A^2 \subseteq P$, then $A \subseteq P$

(5) For any right ideal A of R, if $0 \neq A^2 \subseteq P$, then $A \subseteq P$

Proof

(1) $\Longrightarrow$ (2) Clear by definition.

(2) $\Longrightarrow$ (3) Suppose that $0 \neq aRa \subseteq P$, for any $a \in R$, then $0 \neq RaRaR \subseteq P$, thus $0 \neq \langle a \rangle^2 \subseteq P$, hence by (2), $a \in P$.

(3) $\Longrightarrow$ (4) Let A be a left ideal such that $0 \neq A^2 \subseteq P$, and assume that $A \subseteq P$. Then, for any $a \in A\backslash P$ and any $b \in A \cap P$, we have $a+b \in A\backslash P$; thus, $(a+b)R(a+b) \subseteq A^2 \subseteq P$. By (3), we obtain $(a+b)R(a+b) = 0$, and hence, for $b=0$, we have $a^2 = ab = ba = b^2 = 0$. Thus, $A^2 = 0$, contradiction.

(4) $\Longrightarrow$ (1) The proof is immediate.

(3) $\Longrightarrow$ (5) Similar to (3) $\Longrightarrow$ (4).

(5) $\Longrightarrow$ (1) The proof is immediate. □

Theorem 5. *Let P be an ideal of a ring R with identity. Then, the following statements are equivalent:*

(1) P is a weakly semiprime ideal

(2) R-P is a weakly n-system

(3) For any right ideal I of R, if $I \cap (R-P) \neq \varnothing$ and $I^2 \neq 0$, then $I^2 \cap (R-P) \neq \varnothing$

(4) For any $a \in R-P$, if $\langle a \rangle^2 \neq 0$, then $\langle a \rangle^2 \cap (R-P) \neq \varnothing$

Proof

(1) $\Longleftrightarrow$ (2) follows from Proposition 3.

(1) $\Longrightarrow$ (3) Suppose that $I \cap (R-P) \neq \varnothing$ and $I^2 \neq 0$ for any right ideal I of R. Then, $I \subseteq P$. Thus, by Theorem 4, $I^2 \subseteq P$, and hence, $I^2 \cap (R-P) \neq \varnothing$.

(3) $\Longrightarrow$ (4) Suppose that $\langle a \rangle^2 \neq 0$, for some $a \in R-P$. Then, $(a\rangle \cap (R-P) \neq \varnothing$. If $(a\rangle^2 = 0$, then $\langle a \rangle^2 = R(a\rangle R(a\rangle = 0$, contradiction. So $(a\rangle^2 \neq 0$, thus by (3), $(a\rangle^2 \cap (R-P) \neq \varnothing$, hence $\langle a \rangle^2 \cap (R-P) \neq \varnothing$.

(4) $\Longrightarrow$ (1) Suppose that $0 \neq aRa \subseteq P$. Assume that $a \notin P$. Clearly $\langle a \rangle^2 \neq 0$, and thus, by (4), we obtain $\langle a \rangle^2 \cap (R-P) \neq \varnothing$, so there is an element $x \notin P$ such that $x \in \langle a \rangle^2 \subseteq RPR \subseteq P$, contradiction. Hence, $a \in P$, so by Theorem 4, P is a weakly semiprime ideal of R. □

Theorem 6. *Let $S \subseteq R$ be a weakly n-system of a ring R. Let P be an ideal of R maximal with respect to the property that P is disjoint from S. Then, P is a weakly semiprime ideal.*

Proof. Suppose that $0 \neq I^2 \subseteq P$, for any ideal I of R. Assume that $I \subseteq P$, then by the maximal property of P, we obtain $(I+P) \cap S \neq \varnothing$. Since $0 \neq I^2 \subseteq (I+P)^2$, $(I+P)^2 \cap S \neq \varnothing$, which means that $(I+P)^2 \subseteq P$, and hence, $I^2 \subseteq P$, contradiction. Thus, $I \subseteq P$, and P is a weakly semiprime ideal. □

Theorem 7. *Given a weakly semiprime ideal A in a ring R, there exists a maximal weakly n-system S_1 that is disjoint from A. Furthermore, the complement $R-S_1$ forms a minimal weakly semiprime ideal with respect to A.*

Proof. Let A be a weakly semiprime ideal. By Proposition 3, we know that $R-A$ is a *weakly* n-system. Let $\Omega = \{N\colon N \text{ is a weakly } n-\text{system}, A \cap N = \varnothing\}$. It follows that $R-A \in \Omega$ and $\Omega \neq \varnothing$. By Zorn's lemma, Ω has a maximal element S_1. (Note that $R-S_1 \subseteq A$.) Now, let $\Gamma = \{\text{I}\colon I \text{ is an ideal disjoint from } S_1\}$. It follows that $A \in \Gamma$

and $\Gamma \neq \varnothing$. By Zorn's lemma, Γ has a maximal element K. Thus, $S_1 \subseteq R - K$. However, by Theorem 6, we know that K is a weakly semiprime ideal, and by Proposition 3, $R - K$ is a weakly n-system. Considering the maximal property of S_1, we conclude that $S_1 = R - K$, which implies $R - S_1 = K$. If P_1 is a weakly semiprime ideal such that $P_1 \subseteq K$ and $P_1 \cap S_1 = \varnothing$, then $R - K \subseteq R - P_1$, which implies $R - K = R - P_1$, and therefore, $K = P_1$.

Now, we present several results that provide insights into the characteristics of weakly semiprime ideals. □

Proposition 8. *If P is a weakly semiprime ideal of a ring R that is not a semiprime ideal, then there exists an ideal A such that the sum $A + B$ is nilpotent for every ideal B of R that is contained in P.*

Proof. Suppose that P is a weakly semiprime ideal but not a semiprime. Then, there exists an ideal A of R such that $A^2 = 0$ and $A \subseteq P$. For any ideal B of R contained in P, since $A + B \subseteq P$, $(A + B)^2 = AB + BA + B^2 \subseteq P$ and P is a weakly semiprime ideal. Then, $(A + B)^2 = 0$. □

Proposition 9. *Let P be a weakly semiprime ideal of a ring R with identity. If P is not a semiprime ideal, then $P \subseteq Nil_*(R)$.*

Proof. Suppose that P is a weakly semiprime ideal but not a semiprime. There exists an ideal A of R such that $A^2 = 0$ and $A \subseteq P$. Let $a \in P$, then by Proposition 8, we obtain $(A + \langle a \rangle)^2 = 0$. Thus, $(A + \langle a \rangle)^2 \subseteq \mathrm{Nil}_*(R)$, and hence, $A + \langle a \rangle \subseteq \mathrm{Nil}_*(R)$, because $\mathrm{Nil}_*(R)$ is a semiprime ideal. Then, $A \subseteq \mathrm{Nil}_*(R)$; thus, $a \in \langle a \rangle \subseteq \mathrm{Nil}_*(R)$, and hence, $P \subseteq \mathrm{Nil}_*(R)$. □

Lemma 10. *Let P be a weakly semiprime ideal of a ring R. If P is not a semiprime ideal, then $P^2 = 0$.*

Proof. Suppose that P is a weakly semiprime ideal but not a semiprime. Then, there exists an ideal A of R such that $A^2 = 0$ and $A \subseteq P$. Now, assume that $P^2 \neq 0$, then $0 \neq P^2 \subseteq (A + P)^2 \subseteq P$, thus $A + P \subseteq P$, and hence, $A \subseteq P$, a contradiction; thus, $P^2 = 0$.

In the following example, we show that an ideal P with $P^2 = 0$ needs not to be weakly semiprime ideal. □

Example 1. Let $R = \mathbb{M}_2(\mathbb{Z}_{16})$, and let $P = \mathbb{M}_2(\langle 4 \rangle)$. Then, $P^2 = 0$. Now, $0 \neq \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} R \begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \subseteq P$. However, $\begin{pmatrix} 2 & 4 \\ 0 & 2 \end{pmatrix} \notin P$. Thus, P is not weakly semiprime ideal, by (3) of Theorem 4.

Corollary 11. *Let P be an ideal of a ring R. If $P^2 \neq 0$, then the semiprime and weakly semiprime ideals coincide.*

Proof. Clear by Lemma 10. □

Remark 12. Let P be an ideal of a semiprime ring R. If P is semiprime, then clearly P is weakly semiprime. For the converse, suppose that P is weakly semiprime. If $P = 0$, then P is trivially semiprime. If P is not zero, then it must be semiprime; otherwise, it would be contained in the prime radical of R according to Proposition 9, which is zero.

Therefore, in semiprime rings, the concepts of semiprime ideals and weakly semiprime ideals coincide.

Theorem 13. *Let R be a ring. Let $P \subset R$ be a weaky semiprime ideal that is not semiprime. Then, we have we have the following:*

(1) Whenever $I^2 \subseteq P$, then $I^2 = 0$, for any ideal I of R.
(2) Every ideal of R contained in P is a weakly semiprime ideal.
(3) IP (or PI) is a weakly semiprime ideal for any ideal I of R. Particularly, $Nil_*(R)P$ (or $Nil_*(R)$) is a weakly semiprime ideal.

Proof. Since P is a weakly semiprime ideal but not a semiprime, then $P^2 = 0$ by Lemma 10.

(1) Let I be an ideal such that $I^2 \subseteq P$, then the square of I must be zero since if it is not, then $I \subseteq P$, and hence $I^2 \subseteq P^2 = 0$, a contradiction.
(2) Let I be an ideal of R contained in P. Suppose that J is an ideal of R such that $0 \neq J^2 \subseteq I$, then $0 \neq J^2 \subseteq P$, thus $J \subseteq P$, and hence $J^2 \subseteq P^2 = 0$, contradiction.
(3) Since $\mathrm{IP} \subseteq P$, by (2), we are done.

The pseudo radical of an ideal A in a ring R, denoted as $\sqrt{A}$, is defined as the sum of all ideals W of R such that $W^n \subseteq A$ for some positive integer n. On the other hand, the prime radical of an ideal A, denoted as $\mathrm{rad}(A)$, is the intersection of all prime ideals that contain A. Furthermore, $\sqrt{A} \subseteq \mathrm{rad}(A)$; see [9]. □

Corollary 14. *The pseudo radical of any ideal P of a Noetherian ring R is a weakly semiprime ideal.*

Proof. Let $0 \neq A^2 \subseteq \sqrt{P}$ for any ideal A of R. Then, $A^{2n} = [A^n]^2 \subseteq P$ for some positive integer n, by Lemma 1.2 of [9], and hence $A \subseteq \sqrt{P}$. Thus, $\sqrt{P}$ is a weakly semiprime ideal. □

Example 2. Let F be a field. Consider the ring $R = F[x, y]$. Then, R is Noetherian. Thus, by Corollary 14, $\sqrt{P}$ is a weakly semiprime ideal for any ideal P of R.

Proposition 15. *Let R, S be any rings with identities. Then, we have the following:*

(1) If $I \times S$ is a weakly semiprime ideal of $R \times S$, then I is a weakly semiprime ideal, for any ideal I of R
(2) If I is a weakly semiprime ideal of R, and R is a semiprime ring, then $I \times S$ is a weakly semiprime ideal of $R \times S$

Proof

(1) Suppose that $0 \neq A^2 \subseteq I$ for any ideal A of R. Then, $0 \times 0 \neq A^2 \times 0 = (A \times 0)^2 \subseteq I \times S$. Since $I \times S$ is a weakly semiprime ideal, $A \times 0 \subseteq I \times S$, and hence, $A \subseteq I$.

(2) Suppose that $0 \times 0 \neq (A \times B)^2 \subseteq I \times S$, for any ideal A, B of R, S, respectively. Then, $A^2 \subseteq I$. If $A = 0$, then $A \subseteq I$. If $A \neq 0$, then since R is a semiprime ring, we obtain $0 \neq A^2 \subseteq I$, and since I is a weakly semiprime ideal of R, we have $A \subseteq I$. In both cases, we see that $A \times B \subseteq I \times S$. □

Theorem 16. *Let I be an ideal of a ring R. Let P be an ideal of R such that $I \subseteq P$. Then, we have the following:*

(1) If P is a weakly semiprime ideal of R, then P/I is a weakly semiprime ideal of R/I

(2) If I and P/I are weakly semiprime ideals of R and R/I, respectively, then P is a weakly semiprime ideal

Proof

(1) Suppose that P is a weakly semiprime ideal of R and let $\overline{0} \neq (\overline{A})^2 \subseteq P/I$ for any ideal $\overline{A}$ of R/I. Then, $\overline{A} = A/I$ where $A \supseteq I$. So, $\overline{0} \neq (A^2 + I)/I \subseteq P/I$; thus, $0 \neq A^2 \subseteq P$. So by assumption, $A \subseteq P$, which implies $\overline{A} \subseteq P/I$. Hence, P/I is a weakly semiprime ideal of R/I.

(2) Suppose that $0 \neq A^2 \subseteq P$ for any ideal A of R. If $0 \neq A^2 \subseteq I$, then $A \subseteq P$. If $A^2 \nsubseteq I$, then $\overline{0} \neq [(A + I)/I]^2 \subseteq P/I$. Thus, $[(A + I)/I] \subseteq P/I$. Hence, $A \subseteq P$. □

Theorem 17. *Let R be a ring. Then, we have the following:*

(1) The sum of a finite number of weakly semiprime ideals of a ring R is a weakly semiprime ideal

(2) The intersection of a finite number of weakly semiprime ideals of a ring R is a weakly semiprime ideal

Proof. It is sufficient to prove that the sum and intersection of two weakly semiprime ideals is weakly semiprime, and then, the proof follows by induction. Suppose that P and Q are weakly semiprime ideals of R.

(1) Since Q is weakly semiprime ideal of R, $Q/(P \cap Q)$ is a weakly semiprime ideal of $R/(P \cap Q)$ by (1) of Theorem 16. Since $(P + Q)/Q \cong Q/(P \cap Q)$, $(P + Q)/Q$ is a weakly semiprime ideal of R/Q. Thus, by (2) of Theorem 16, $P + Q$ is a weakly semiprime ideal of R.

(2) One can easily show this.

The following theorem indicates that the property of being a weakly semiprime ideal is preserved when passing to the image ring via a ring epimorphism f. □

Theorem 18. *Let $f: R_1 \longrightarrow R_2$ be a ring epimorphism. Let P be an ideal of R_1 such that $\mathrm{Ker}(f) \subseteq P$. Then, we have the following:*

(1) If P is a weakly semiprime ideal of R_1, then $f(P)$ is a weakly semiprime ideal of R_2.

(2) If P is a weakly semiprime ideal of R_2 and $\mathrm{Ker}(f)$ is a weakly semiprime ideal of R_1, then $f^{-1}(P)$ is a weakly semiprime ideal of R_1.

Proof

(1) Suppose that P is a weakly semiprime ideal of R_1. Since $\mathrm{Ker}(f) \subseteq P$, we obtain that $P/\mathrm{Ker}(f)$ is a weakly semiprime ideal of $R_1/\mathrm{Ker}(f)$, by (1) of Theorem 16. Hence, since $R_1/\mathrm{Ker}(f) \cong R_2$, $f(P) \cong P/\mathrm{Ker}(f)$ is a weakly semiprime ideal of R_2.

(2) Suppose that P is a weakly semiprime ideal of R_2. Then, $\mathrm{Ker}(f) \subseteq f^{-1}(P)$. Hence, since $R_1/\mathrm{Ker}(f) \cong R_2$, $f^{-1}(P)/\mathrm{Ker}(f) \cong P$ is a weakly semiprime ideal of $R_1/\mathrm{Ker}(f)$. Thus, by (2) of Theorem 16, the ideal $f^{-1}(P)$ is a weakly semiprime ideal of R_1. □

Remark 19. We can also prove part (1) of Theorem 18 as the following. Suppose that $0 \neq B^2 \subseteq f(P)$ for any ideal B of R_2. Then, $\mathrm{Ker}(f) \subseteq f^{-1}(B) = A$ for some ideal A of R_1. Hence, $f(A) = B$ because f is an epimorphism. Then, we have

$$0 \neq B^2 = [f(A)]^2 = f\left(A^2\right) \subseteq f(P). \quad (1)$$

Thus,

$$0 \neq A^2 \subseteq f^{-1}\left(f\left(A^2\right)\right) \subseteq f^{-1}(f(P)) = P. \quad (2)$$

Now by assumption, $A \subseteq P$, i.e., $B \subseteq f(P)$.

3. Fully Weakly Semiprime Rings

Definition 20. A ring R is called fully weakly semiprime (right) ring, if every (right) ideal of R is a weakly semiprime ideal.

If $R^2 = 0$, then clearly R is a fully weakly semiprime (right) ring. Thus, we can conclude the following corollary.

Corollary 21. *Let P be a weakly semiprime ideal of a ring R. If P is not a semiprime ideal, then P is a fully weakly semiprime ring.*

Proof. By Lemma 10, $P^2 = 0$; hence, every ideal of P is a weakly semiprime ideal.

As an example, consider a local ring R with a unique maximal ideal M. If $M^2 = 0$, then R is a fully almost prime ring (as shown in [6]) and a fully almost right primary ring (as shown in [7]). Furthermore, R is a fully weakly semiprime ring. If the local ring R satisfies the property $aR = Ra$ for all $a \in R$, and every principal ideal is almost prime, then by Theorem 2.16 of [6], we have $M^2 = 0$, and therefore, R is a fully weakly semiprime ring. □

Proposition 22. *A ring R is a fully weakly semiprime ring if and only if, for any ideal P of R, either $P^2 = 0$ or P is idempotent.*

Proof. Suppose that the ring R is fully weakly semiprime ring, and let P be an ideal of R. If $P^2 \neq 0$, then $0 \neq P^2 \subseteq P^2$. Since P^2 is a weakly semiprime ideal, $P \subseteq P^2$. Hence $P = P^2$.

Conversely, for any ideal P of R, let A be an ideal of R such that $0 \neq A^2 \subseteq P$. Then, $A = A^2 \subseteq P$. □

Example 3. For the set $R = \{0, r_1, r_2, r_3\}$, we define the following binary operations:

$+$	0	r_1	r_2	r_3
0	0	r_1	r_2	r_3
r_1	r_1	0	r_3	r_2
r_2	r_2	r_3	0	r_1
r_3	r_3	r_2	r_1	0

$.$	0	r_1	r_2	r_3
0	0	0	0	0
r_1	0	r_1	r_1	0
r_2	0	r_2	r_2	0
r_3	0	r_3	r_3	0

Then, R is noncommutative ring. The right ideals of the ring R are $I = \{0, r_2\}$, $J = \{0, r_3\}$, $K = \{0, r_1\}$, and 0. Moreover, $I^2 = I$, $J^2 = 0$, and $K^2 = K$.

The proofs of the right ideal versions of Lemma 10 and Proposition 22 can be set analogously of their proofs. Hence, R is a fully weakly semiprime right ring.

It can be observed that in a fully weakly semiprime ring R, every nonzero idempotent ideal is a semiprime ideal. This follows from Lemma 10 and Proposition 22. Therefore, a fully weakly semiprime ring that does not contain a nonzero nilpotent ideal is a fully idempotent ring, and hence a fully semiprime ring. In such cases, it can be noted that the center of R is regular, as stated in Proposition 1.5 of [10].

Theorem 23. *Let R be a fully weakly semiprime ring. Then, we have the following:*

(1) $[N(R)]^2 = 0$

(2) $N(R) \subseteq Nil_*(R)$

Proof

(1) Suppose $N(R) \neq 0$. Let $L = \{I_1, \ldots, I_n\}$ be a finite set of nonzero ideals whose squares are zero for $n \geqslant 1$. Then, each term of $(I_1 + \cdots + I_n)^{n+1}$ contains at least one repeated ideal I_i with $1 \leqslant i \leqslant n$. As a result, each term of $(I_1 + \cdots + I_n)^{n+1}$ is contained in some I_i^2, implying that $(I_1 + \cdots + I_n)^{n+1} = 0$. Now, since $N(R)$ is not an idempotent ideal, Proposition 22 shows that $(I_1 + \cdots + I_n)^2 = 0$, thus leading to $[N(R)]^2 = 0$.

(2) Since, by (1), $[N(R)]^2 = 0$, $[N(R)]^2 \subseteq \mathrm{Nil}_*(R)$, and thus, $N(R) \subseteq \mathrm{Nil}_*(R)$. □

Theorem 24. *Let R be a fully weakly semiprime ring with identity. If R does not contain any nonzero idempotent ideal, then we have the following:*

(1) $N(R)$ *and* $Nil_*(R)$ *are prime ideals*

(2) $N(R) = Nil_*(R) = rad(R)$ *and* $[N(R)]^2 = [Nil_*(R)]^2 = [rad(R)]^2 = 0$

Proof

(1) Let P be any ideal of R. Then, by Proposition 22, $P^2 = 0$, thus $P^2 \subseteq \mathrm{Nil}_*(R)$, hence $P \subseteq \mathrm{Nil}_*(R)$. Thus, $\mathrm{Nil}_*(R)$ is a prime ideal. On the other hand, since $P^2 = 0$, $P \subseteq N(R)$, and hence, $N(R)$ is also a prime ideal.

(2) Since $N(R)$ is prime ideal by (1), $\mathrm{Nil}_*(R) \subseteq N(R)$, and by Theorem 23, $N(R) \subseteq \mathrm{Nil}_*(R)$. Hence, $N(R) = \mathrm{Nil}_*(R)$. Furthermore, $[N(R)]^2 = [\mathrm{Nil}_*(R)]^2 = 0$. In addition, $[\mathrm{rad}(R)]^2 = 0 \subseteq \mathrm{Nil}_*(R)$, and thus, $\mathrm{rad}(R) \subseteq \mathrm{Nil}_*(R) \subseteq \mathrm{rad}(R)$.

Recall that the ring $R/\mathrm{Nil}_*(R)$ is always a semiprime ring. Theorem 24 provides further insight by stating that if R is a fully weakly semiprime ring with identity and does not contain any nonzero idempotent ideal, then the quotient ring $R/\mathrm{Nil}_*(R)$ is a prime ring. □

Proposition 25. *Let R be a fully weakly semiprime ring with identity. If R is Noetherian, then $Nil_*(R) = rad(R)$ and $[N(R)]^2 = [Nil_*(R)]^2 = [rad(R)]^2 = 0$.*

Proof. In a fully weakly semiprime ring R, it holds that for any ideal P of R, either $P^2 = 0$ or $P^2 = P$, as stated in Proposition 22. Consequently, if $[\mathrm{rad}(R)]^2 = 0$, it follows that $\mathrm{Nil}_*(R) = \mathrm{rad}(R)$. Similarly, if $[\mathrm{rad}(R)]^2 = \mathrm{rad}(R)$, considering that $\mathrm{rad}(R)$ is finitely generated, Nakayama's lemma implies $\mathrm{rad}(R) = 0$, which in turn leads to $\mathrm{Nil}_*(R) = \mathrm{rad}(R)$. □

Proposition 26. *If every nonzero ideal of a ring R with identity is a weakly semiprime but not a semiprime, then R is simple, i.e., a semiprime ring.*

Proof. Let P be any ideal of R. Then, by Proposition 9, $P \subseteq \mathrm{Nil}_*(R)$. However, $\mathrm{Nil}_*(R) = 0$, and thus, $P = 0$. □

Theorem 27. *Let R be a ring, and I be an ideal of R. If R is a fully weakly semiprime ring, so is R/I.*

Proof. Suppose that $\overline{P}$ is an ideal of R/I. Then, there exists an ideal $P \supseteq I$ of R such that $\overline{P} = P/I$. Clearly, P is a weakly semiprime ideal of R. Hence, by (1) of Theorem 16, $\overline{P}$ is a weakly semiprime ideal of R/I. □

Theorem 28. *Let $f\colon R \longrightarrow S$ be a ring epimorphism. If R is a fully weakly semiprime ring, so is S.*

Proof. Let P be an ideal of S. Then, $f^{-1}(P)$ is a weakly semiprime ideal of R. Thus, by (1) of Theorem 18, we obtain that $f(f^{-1}(P)) = P$ is a weakly semiprime ideal of S. □

4. Conclusions

We have characterized weakly semiprime ideals and established their key properties. We have shown that an ideal P is weakly semiprime if and only if its complement forms a weakly n-system. We have also explored minimal weakly semiprime ideals, equivalent conditions for weakly semiprime ideals, and their relationship to the prime radical of the ring. Additionally, we have investigated the behavior of weakly semiprime ideals under ring homomorphisms and studied the conditions for a ring to be fully weakly semiprime. Our findings provide valuable insights into the structure and properties of weakly semiprime ideals in ring theory.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The author declares that he has no conflicts of interest.

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