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Research Article

A Modified Form of Inertial Viscosity Projection Methods for Variational Inequality and Fixed Point Problems

Watanjeet Singh and Sumit Chandok

Department of Mathematics, Thapar Institute of Engineering and Technology, Patiala 147004, India

Correspondence should be addressed to Sumit Chandok; sumit.chandok@thapar.edu

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This paper aims to introduce an iterative algorithm based on an inertial technique that uses the minimum number of projections onto a nonempty, closed, and convex set. We show that the algorithm generates a sequence that converges strongly to the common solution of a variational inequality involving inverse strongly monotone mapping and fixed point problems for a countable family of nonexpansive mappings in the setting of real Hilbert space. Numerical experiments are also presented to discuss the advantages of using our algorithm over earlier established algorithms. Moreover, we solve a real-life signal recovery problem via a minimization problem to demonstrate our algorithm's practicality.

1. Introduction

The theory of variational inequality established itself as an important field of study covering a broad class of results and emerged as an essential tool for solving various problems. This theory is a natural framework for recent numerical techniques developed to solve optimization problems.

Let C be a nonempty, closed, and convex subset of a real Hilbert space H. The classical variational inequality for a mapping $K: H \longrightarrow H$ is to find $a \in C$ such that

$$\langle Ka, b - a\rangle \geq 0, \tag{1}$$

for all $b \in C$. The solution set of variational inequality problem is denoted by $VI(C, K)$.

One of the simplest methods to solve (1) is the projection method which is the extension of the projected gradient method for optimization problems. The method worked with the assumption that K is L-Lipschitz continuity and strongly monotone. However, it was pointed out that the projection method may diverge if the strongly monotone assumption is replaced by monotonicity.

To overcome this, Korpelevich [1] proposed an extragradient method based on the computation of projection onto a feasible set twice in each iteration. In the extragradient method, one needs to calculate two projections onto C in each iteration. Since projections onto C are associated with the minimum distance problem, this might affect the efficiency and applicability of the algorithm if the mapping K or the feasible set C have complicated structures. So a natural question arises, can we create fast iterative algorithms that use the minimum number of projections onto C for solving variational inequality problems? To answer this, Tseng [2] introduced an extragradient algorithm for solving variational inequality involving a monotone and L-Lipschitz continuous mapping. In this method, only one projection is calculated onto C in each iteration followed by a standard gradient step using the projection onto C. In 2011, Censor et al. [3] modified the extragradient method of [1] by introducing the subgradient extragradient method where only one projection is calculated onto C, and the other projection onto C is replaced by a specific subgradient projection which can be calculated easily. In 2022, Anh [4] presented a novel convergence outcome for addressing the variational inequality problem characterized by strong monotonicity over the fixed point sets of nonexpansive mappings. Very recently, Anh [5] introduced an iterative methodology for solving the variational inequality problem by employing a recently devised proximal operator that converges to a unique solution.

One of the interesting problems in nonlinear analysis is dealing with the common elements of the set of solutions for variational inequality problems and fixed point problems. In 2003, Takahashi and Toyoda [6] introduced an iterative method that converges weakly to the common element for variational inequality involving τ-inverse strongly monotone mapping and fixed point problem involving nonexpansive mapping. Iiduka and Takahashi [7] obtained a strong convergence using an additional projection of the iterative sequence onto C by improving the iterative method of [6]. There are many methods in the literature that draw inspiration from [6] to obtain results based on finding the common element such as Iiduka and Takahashi's [8] strong convergence using Halpern's type iterative scheme, Ceng and Yao's [9] strong convergence result combining the extragradient method with Halpern's method, and Nadezhkina and Takahashi's [10] weak convergence using extragradient method.

Moudafi [11] introduced the viscosity approximation method for approximating fixed points of a nonexpansive mapping by the regularization procedure obtained using a suitable convex combination of the nonexpansive mapping. Marino and Xu [12] studied the viscosity approximation methods to discuss the optimality condition for the minimization problems. Chen et al. [13] incorporated viscosity approximation methods for finding the common elements to monotone and nonexpansive mappings. Numerous algorithms use viscosity approximation methods to find the common element of variational inequality problem and fixed point problem such as Ceng and Yao's [14] strong convergence result by combining the extragradient method and viscosity approximation method such that the two sequences generated by the algorithm converge strongly to the common element, a general three-step iterative process by Shang et al. [15] in which two projections are calculated onto C in first two steps, and in the third step, the third projection onto C is combined using viscosity approximation method, a generalized viscosity type extragradient method by Anh et al. [16] which uses a strongly positive linear bounded operator to converge to the common element for variational inequality problem, fixed point problem and equilibrium problem, and two-step extragradient-viscosity method by Hieu et al. [17] in which first step calculates three projections onto C and second step combines the projections using viscosity approximation method.

Anh and Phuong [18] in 2018 introduced, in their work, a robust convergence outcome for locating the common solution of a system encompassing unrelated variational inequalities and fixed-point problems. This addresses distinct feasible domains, adding versatility to the proposed solution methodology. Recently, Anh et al. [19] provided a weak convergence result using only one projection onto a closed convex set and combining using Mann-type iteration under some specific assumptions. In 2019, Thong and Hieu [20] introduced an extragradient viscosity algorithm with a step-size rule (VSEGM) which does not require the Lipschitz constant of the mapping. In 2022, Tan et al. [21] proposed a viscosity-type inertial subgradient extragradient algorithm (iVSEGM) which is a combination of VSEGM [20] with the inertial term. The use of inertial techniques helps to speed up the convergence. The most crucial aspect of algorithms based on the inertial term is that the next iteration depends upon combining the previous two iteration values. This improves the performance of the iterative algorithm to a great extent. For more literature on inertial techniques, we refer to [22] and references cited therein.

Motivated by the research going in this direction, we establish a new viscosity-type extragradient algorithm that uses a minimum number of projections onto C and converges strongly to the common solution of the variational inequality problem involving τ-inverse strongly monotone mapping and fixed point problem for a countable family of nonexpansive mappings in the setting of real Hilbert space. This new iterative algorithm is based on an inertial term combined with the viscosity type approximation method and a step-size selection rule enabling the algorithm to choose the step size value faster. The step-size choice plays an important role in determining the efficiency of the algorithm. We prove that under some suitable assumptions, the sequence generated by our algorithm converges strongly to the common element.

Some highlights of this paper are as follows:

(i) At each step, a single projection is calculated onto a closed and convex set.

(ii) We use a strongly positive linear bounded operator in our algorithm with a relaxed condition. The benefit of using this operator can be seen in our numerical experiments.

(iii) We provide a real-life application to our algorithm involving the recovery of signals.

We organize the rest of the paper as follows: Section 2 gives some preliminary results and definitions required to understand and prove the main results. Section 3 presents the main iterative algorithm and proves its strong convergence. In Sections 4 and 5, we provide numerical examples and applications, respectively, to support our results.

2. Preliminaries

In this section, we present several basic definitions and results that will be useful for proving the main result.

Suppose that C is a closed, convex subset of a real Hilbert space H. We denote the weak convergence and strong convergence of a sequence $\{a_n\}$ to a by $a_n \rightharpoonup a$ and $a_n \longrightarrow a$, respectively.

For each point $a \in H$, we have a unique point $P_C(a)$ in C such that $\|a - P_C(a)\| \le \|a - b\|$ for all $b \in C$. This $P_C(a)$ is called metric projection (see [23]) of H onto C and for all $a, b \in H$, P_C satisfies

$$\langle a-b, P_Ca - P_Cb\rangle \geq \|P_Ca - P_Cb\|^2. \quad (2)$$

From (2), we can write for all $a \in C, b \in H$,

$$\langle a-b, a - P_Cb\rangle \geq \|a - P_Cb\|^2. \quad (3)$$

Also, for all $b \in C$, we have

$$\langle a - P_Ca, b - P_Cb\rangle \leq 0. \quad (4)$$

Definition 1. Let K be a self-mapping on H. Then, K is said to be

(i) L-Lipschitz continuous with $L>0$ if for all $a, b \in H$

$$\|Ka - Kb\| \leq L\|a-b\|. \quad (5)$$

(ii) Contraction if for all $a, b \in H$, there exists a constant $t \in [0, 1)$ such that

$$\|Ka - Kb\| \leq t\|a-b\|. \quad (6)$$

(iii) Nonexpansive if for all $a, b \in H$

$$\|Ka - Kb\| \leq \|a-b\|. \quad (7)$$

(iv) Monotone if for all $a, b \in H$

$$\langle Ka - Kb, a-b\rangle \geq 0. \quad (8)$$

(v) τ-inverse strongly monotone (τ-ism) with $\tau > 0$ if for all $a, b \in H$

$$\langle a-b, Ka - Kb\rangle \geq \tau\|Ka - Kb\|^2. \quad (9)$$

(vi) Strongly positive linear bounded operator if there exists a constant $\bar{\rho} > 0$, such that for all $a \in H$,

$$\langle Ka, a\rangle \geq \bar{\rho}\|a\|^2. \quad (10)$$

A set-valued monotone mapping U from H to 2^H is considered to be maximal if the graph, Graph (U) of U is not properly contained in other monotone mapping's graph. Let K be τ-ism mapping of C into H, and N_Ca be the normal cone to C at $a \in C$, which is defined as $N_Ca = \{c \in H: \langle a-b, c\rangle \geq 0, \text{for all } b \in C\}$. Now, define

$$Ua = \begin{cases} Ka + N_Ca & \text{if } a \in C \\ \phi & \text{otherwise.} \end{cases} \quad (11)$$

Then, the map U is maximal monotone and $0 \in Ua$ if and only if $a \in \mathrm{VI}(C, K)$.

Lemma 2. *The following results hold in H.*

(1) $\|a+b\|^2 = \|a\|^2 + \|b\|^2 + 2\langle a, b\rangle$ for all $a, b \in H$

(2) $\|ta + (1-t)b\|^2 = t\|a\|^2 + (1-t)\|b\|^2 - t(1-t)\|a-b\|^2$ for all $a, b \in H$

Lemma 3 (see [24]). *Let $U: C \longrightarrow C$ be a nonexpansive mapping such that $Fix(U) \neq \phi$, where C is a closed convex subset of a real Hilbert space H. If a sequence $\{a_n\} \in C$ such that $a_n \rightharpoonup c$ and $a_n - Ua_n \longrightarrow 0$, then $c = Uc$.*

Lemma 4 (see [12]). *Assume that K is strongly positive linear bounded operator with coefficient $\bar{\gamma} > 0$ on a Hilbert space H such that $0 < \rho \leq \|K\|^{-1}$, then $\|I - \rho K\| \leq 1 - \rho\bar{\gamma}$.*

Lemma 5 (see [25]). *Assume that $\{a_n\}$ is a sequence of nonnegative real numbers such that*

$$a_{n+1} \leq (1-\tau_n)a_n + \delta_n, \quad n \geq 0, \quad (12)$$

where $\{\tau_n\} \subset (0, 1)$ and $\{\delta_n\}$ is a sequence in $\mathbb{R}$ such that

(1) $\sum_{n=1}^{\infty}\tau_n = \infty$

(2) $\limsup \delta_n/\tau_n \leq 0$ or $\sum_{n=1}^{\infty}|\delta_n| < \infty$

Then, $\lim_{n\longrightarrow\infty} a_n = 0$.

Before mentioning the next lemma, we discuss the AKTT-condition that is used to deal with the family of mappings. Let $\{U_n\}_{n=1}^{\infty}$ be a family of mappings on C such that $\cap_{n=1}^{\infty}\mathrm{Fix}(U_n) \neq \phi$. Then, $\{U_n\}$ satisfies the AKTT-condition if for each bounded subset C_1 of C, we have

$$\sum_{n=1}^{\infty} \sup\{\|U_{n+1}c - U_nc\|: c \in C_1\} < \infty. \quad (13)$$

To understand AKTT-condition through an example, we consider $U_nc = 1/2^{n-1}\sin c, n \in \mathbb{N}$. Then, it can be easily seen that $\{U_n\}_{n=1}^{\infty}$ is a family of nonexpansive mappings and $\cap_{n=1}^{\infty}\mathrm{Fix}(U_n) = 0$. Then, for each bounded subset C_1 of $\mathbb{R}$, we see that

$$\begin{aligned}\sum_{n=1}^{\infty} \sup\{\|U_{n+1}c - U_nc\|: c \in C_1\} &= \sum_{n=1}^{\infty} \sup\left\{\left\|\frac{1}{2^n}\sin c - \frac{1}{2^{n-1}}\sin c\right\|: c \in C_1\right\} \\ &\leq \sum_{n=1}^{\infty}\left|\frac{1}{2^n} - \frac{1}{2^{n-1}}\right| < \infty.\end{aligned} \quad (14)$$

Thus, we see that $\{U_n\}$ satisfies AKTT-condition.

Lemma 6 (see [26]). *Let C be a nonempty closed subset of Banach space B and $\{U_n\}$ be a family of self mappings onto C which satisfies the AKTT-condition. Then, $\{U_n a\}$ converges strongly to a point in C for each $a \in C$. Moreover,*

$$Ua = \lim_{n\longrightarrow\infty} U_n a, \quad \text{for all } a \in C. \tag{15}$$

Then, for every bounded subset C_1 of C,

$$\limsup_{n\longrightarrow\infty}\left\{\left\|U_{n+1}c - U_n c\right\|: c \in C_1\right\} = 0. \tag{16}$$

3. Main Results

This section presents our algorithm for the common element of solutions to variational inequality and common fixed points of a countable family of nonexpansive mappings.

Throughout this section, we denote C as a closed and convex subset of real Hilbert space H. We consider the following assumptions:

(A1) $\{U_n: H \longrightarrow H\}$ is a countable family of nonexpansive mappings

(A2) $K: H \longrightarrow H$ is τ-ism mapping

(A3) $\cap_{n=1}^{\infty} \mathrm{Fix}(U_n) \cap \mathrm{VI}(C, K) \neq \phi$

(A4) $G: H \longrightarrow H$ is strongly positive linear bounded operator with coefficient $0 < \bar{\rho} < 1$ and $0 < \|G\| \leq 1$

(A5) $g: C \longrightarrow C$ is t-contraction with $t \in (0, 1)$

(A6) $\{\varepsilon_n\}$ is a positive sequence such that $\lim_{n\longrightarrow\infty}\varepsilon_n/\omega_n = 0$, and $\omega_n \in (0, 1)$ satisfies $\lim_{n\longrightarrow\infty}\omega_n = 0$, $\sum_{n=1}^{\infty}\omega_n = \infty$ and $\sum_{n=1}^{\infty}|\omega_n - \omega_{n+1}| < \infty$

Remark 7. The sequence $\{\kappa_n\}$ generated by Algorithm 1 is non-increasing and $\lim_{n\longrightarrow\infty}\kappa_n$ exists (see [21]).

Remark 8. The iterative algorithm presented by Anh et al. [19] yields a weak convergence result by employing the Mann-type method and executing single projections onto a closed convex set. In contrast, our Algorithm 1 delivers a strong convergence result through a viscosity-type approximation, featuring a more relaxed condition on the strongly positive linear bounded operator.

Remark 9. In [16], Anh et al. achieve a strong convergence through a generalized viscosity-type approximation. This algorithm aims to identify a common element satisfying three distinct problems, with the norm of a strongly positive linear bounded operator constrained to be 1. In our method, we employ a generalized viscosity-type approximation with a more adaptable constraint on the norm of the strongly positive linear bounded operator G, permitting $0 < \|G\| \leq 1$. This adaptation is applied in the pursuit of a common solution to two specific problems.

Now, we state and prove our main result.

Theorem 10. *Under the assumptions (A1)-(A6) and if $\{\{U_n\}, U\}$ satisfies the AKTT-condition. Then, the sequence $\{a_n\}$ generated by algorithm converges strongly to $p \in \cap_{n=1}^{\infty} Fix(U_n) \cap VI(C, K)$.*

Proof. To begin with, we prove that the sequence $\{a_n\}$ is bounded. As K is τ-ism mapping, then for all $a, b \in C$, we have

$$\begin{aligned}\left\|(I - \kappa_n K)a - (I - \kappa_n K)b\right\|^2 &= \left\|a - b - \kappa_n (Ka - Kb)\right\|^2 \\ &= \|a - b\|^2 - 2\kappa_n \langle a - b, Ka - Kb\rangle + \kappa_n^2 \|Ka - Kb\|^2 \\ &\leq \|a - b\|^2 + \kappa_n (\kappa_n - 2\tau)\|Ka - Kb\|^2.\end{aligned} \tag{17}$$

As the sequence $\{\kappa_n\}$ is non-increasing, $\kappa_n \leq \kappa_1$, we get

$$\left\|(I - \kappa_n K)a - (I - \kappa_n K)b\right\|^2 \leq \|a - b\|^2 + \kappa_1 (\kappa_1 - 2\tau)\left\|Ka - Kb\right\|^2. \tag{18}$$

Initialization: Take $\gamma > 0, \kappa_1 \in (0, 2\tau)$, $0 < \rho < \bar{\rho}/t$ and $\nu \in (0, 1)$. Let $a_0, a_1 \in H$, then calculate a_{n+1} as:
Step 1: Set $c_n = a_n + \gamma_n (a_n - a_{n-1})$, where $\gamma_n = \begin{cases} \min\{\varepsilon_n/\|a_n - a_{n-1}\|, \gamma\} & \text{if } a_n \neq a_{n-1} \\ \gamma & \text{otherwise} \end{cases}$ and calculate $b_n = P_C(c_n - \kappa_n K c_n)$.
Step 2: Compute $a_{n+1} = \omega_n \rho g(a_n) + (I - \omega_n G) U_n b_n$ and update $\kappa_{n+1} = \begin{cases} \min\{\nu\|c_n - b_n\|/\|Kc_n - Kb_n\|, \kappa_n\} & \text{if } Kc_n - Kb_n \neq 0 \\ \kappa_n & \text{otherwise.} \end{cases}$
Set $n \leftarrow n+1$ and go to **Step 1**.

Algorithm 1: New inertial generalized viscosity-type approximation method.

Since $\kappa_1 \in (0, 2\tau)$, we get

$$\|(I - \kappa_n K)a - (I - \kappa_n K)b\|^2 \leq \|a - b\|^2. \tag{19}$$

So, $(I - \kappa_n K)$ is a nonexpansive mapping. Using (A3), assume that $p \in \cap_{n=1}^{\infty} \mathrm{Fix}(U_n) \cap VI(C, K)$. This means, $p = P_C(p - \kappa_n \mathrm{Kp})$. Consider

$$\begin{aligned} \|b_n - p\| &= \|P_C(c_n - \kappa_n K c_n) - p\| \\ &\leq \|c_n - \kappa_n K c_n - (p - \kappa_n K p)\| \\ &= \|(I - \kappa_n K)c_n - (I - \kappa_n K)p\| \\ &\leq \|c_n - p\| \\ &= \|a_n + \gamma_n (a_n - a_{n-1}) - p\| \\ &\leq \|a_n - p\| + \omega_n \frac{\gamma_n}{\omega_n} \|a_n - a_{n-1}\|. \end{aligned} \tag{20}$$

Using (A6) and assumptions of γ_n, we get $\gamma_n/\omega_n \|a_n - a_{n-1}\| \longrightarrow 0$ as $n \longrightarrow \infty$. Thus, there exists a constant $L_1 > 0$ such that $\gamma_n/\omega_n \|a_n - a_{n-1}\| \leq L_1$, for all $n \geq 1$. Hence, we get

$$\|b_n - p\| \leq \|c_n - p\| \leq \|a_n - p\| + \omega_n L_1, \quad \text{for all } n \geq 1. \tag{21}$$

Using Lemma 4, (A5) and (21), we have

$$\begin{aligned} \|a_{n+1} - p\| &= \|\omega_n \rho g(a_n) + (I - \omega_n G) U_n b_n - p\| \\ &= \|\omega_n (\rho g(a_n) - Gp) + (I - \omega_n G)(U_n b_n - p)\| \\ &\leq \omega_n \|\rho g(a_n) - Gp\| + (1 - \omega_n \bar{\rho}) \|U_n b_n - p\| \\ &\leq \omega_n t\rho \|a_n - p\| + \omega_n \|\rho g(p) - Gp\| + (1 - \omega_n \bar{\rho}) \|U_n b_n - p\| \\ &\leq \omega_n t\rho \|a_n - p\| + \omega_n \|\rho g(p) - Gp\| + (1 - \omega_n \bar{\rho}) \|b_n - p\| \\ &\leq \omega_n t\rho \|a_n - p\| + \omega_n \|\rho g(p) - Gp\| + (1 - \omega_n \bar{\rho})(\|a_n - p\| + \omega_n L_1) \\ &\leq (1 - \omega_n (\bar{\rho} - t\rho)) \|a_n - p\| + \omega_n (\bar{\rho} - t\rho) \frac{(\|\rho g(p) - Gp\| + L_1)}{\bar{\rho} - t\rho} \\ &\leq \max\left\{ \|a_n - p\|, \frac{\|\rho g(p) - Gp\| + L_1}{\bar{\rho} - t\rho} \right\} \\ &\;\;\vdots \\ &\leq \max\left\{ \|a_0 - p\|, \frac{\|\rho g(p) - Gp\| + L_1}{\bar{\rho} - t\rho} \right\}. \end{aligned} \tag{22}$$

It implies that the sequence $\{a_n\}$ is bounded. So, the sequences $\{c_n\}$, $\{b_n\}$, $\{U_n b_n\}$, $\{Ka_n\}$ and $\{g(a_n)\}$ are also bounded.

Now, we have to show that $\|a_{n+1} - a_n\| \longrightarrow 0$ and $\|b_n - U_n b_n\| \longrightarrow 0$ as $n \longrightarrow \infty$. Since P_C and $(I - \kappa_{n+1}K)$ are nonexpansive mappings, we have

$$
\begin{aligned}
\|b_{n+1} - b_n\| &= \|P_C(c_{n+1} - \kappa_{n+1}Kc_{n+1}) - P_C(c_n - \kappa_n Kc_n)\| \\
&\leq \|c_{n+1} - \kappa_{n+1}Kc_{n+1} - c_n + \kappa_n Kc_n\| \\
&= \|(c_{n+1} - \kappa_{n+1}Kc_{n+1}) - (c_n - \kappa_{n+1}Kc_n) + (\kappa_n - \kappa_{n+1})Kc_n\| \\
&\leq \|(I - \kappa_{n+1}K)c_{n+1} - (I - \kappa_{n+1}K)c_n\| + |\kappa_n - \kappa_{n+1}|\|Kc_n\| \\
&\leq \|c_{n+1} - c_n\| + |\kappa_n - \kappa_{n+1}|\|Kc_n\|.
\end{aligned}
\tag{23}
$$

Using (21) and (23), we get

$$
\begin{aligned}
\|a_{n+1} - a_n\| &= \|\omega_n \mathrm{pg}(a_n) + (I - \omega_n G)U_n b_n - \omega_{n-1} pg(a_{n-1}) - (I - \omega_{n-1}G)U_{n-1}b_{n-1}\| \\
&= \|\omega_n p(g(a_n) - g(a_n - 1)) + p(\omega_n - \omega_{n-1})g(a_n - 1) + (I - \omega_n G)(U_n b_n - U_{n-1}b_{n-1}) - (\omega_n - \omega_{n-1})\mathrm{GU}_{n-1}b_{n-1}\| \\
&\leq \omega_n p\|\mathrm{g}(a_n) - g(a_{n-1})\| + p|\omega_n - \omega_{n-1}|\|g(a_{n-1})\| + (1 - \omega_n \bar{p})\|U_n b_n - U_{n-1}b_{n-1}\| + |\omega_n - \omega_{n-1}|\|\mathrm{GU}_{n-1}b_{n-1}\| \\
&\leq \omega_n pt\|a_n - a_{n-1}\| + p|\omega_n - \omega_{n-1}|\|g(a_{n-1})\| + (1 - \omega_n \bar{p})\|U_n b_n - U_n b_{n-1} + U_n b_{n-1} - U_{n-1} - b_{n-1}\| \\
&\quad + |\omega_n - \omega_{n-1}|\|\mathrm{GU}_{n-1}b_{n-1}\| \\
&\leq \omega_n pt\|a_n - a_{n-1}\| + p|\omega_n - \omega_{n-1}|\|g(a_{n-1})\| + (1 - \omega_n \bar{p})\|U_n b_n - U_n b_{n-1}\| \\
&\quad + (1 - \omega_n \bar{p})\|U_n b_{n-1} - U_{n-1}b_{n-1}\| + |\omega_n - \omega_{n-1}|\|\mathrm{GU}_{n-1}b_{n-1}\| \\
&\leq \omega_n pt\|a_n - a_{n-1}\| + p|\omega_n - \omega_{n-1}|\|g(a_{n-1})\| + (1 - \omega_n \bar{p})\|b_n - b_{n-1}\| \\
&\quad + (1 - \omega_n \bar{p})\|U_n b_{n-1} - U_{n-1}b_{n-1}\| + |\omega_n - \omega_{n-1}|\|\mathrm{GU}_{n-1}b_{n-1}\| \\
&\leq \omega_n pt\|a_n - a_{n-1}\| + p|\omega_n - \omega_{n-1}|\|g(a_{n-1})\| + (1 - \omega_n \bar{p})(\|c_n - c_{n-1}\| + |k_{n-1} - k_n|\|kc_{n-1}\|) \\
&\quad + (1 - \omega_n \bar{p})\|U_n b_{n-1} - U_{n-1}b_{n-1}\| \\
&\quad + |\omega_n - \omega_{n-1}|\|\mathrm{GU}_{n-1}b_{n-1}\| \\
&= \omega_n pt\|a_n - a_{n-1}\| + p|\omega_n - \omega_{n-1}|\|g(a_{n-1})\| + (1 - \omega_n \bar{p})\|c_n - c_{n-1}\| \\
&\quad + (1 - \omega_n \bar{p})|k_{n-1} - k_n|\|kc_{n-1}\| + (1 - \omega_n \bar{p})\|U_n b_{n-1} - U_{n-1}b_{n-1}\| + |\omega_n - \omega_{n-1}|\|\mathrm{GU}_{n-1}b_{n-1}\|.
\end{aligned}
\tag{24}
$$

Using triangle inequality, we have

$$
\begin{aligned}
\|c_n - c_{n-1}\| &\leq \|a_n - a_{n-1}\| + \|\gamma_n(a_n - a_{n-1}) - \gamma_{n-1}(a_{n-1} - a_{n-2})\| \\
&\leq \|a_n - a_{n-1}\| + \gamma(\|a_n - a_{n-1}\| + \|a_{n-1} - a_{n-2}\|).
\end{aligned}
\tag{25}
$$

Choose $W = \sup_{n\geq 1}\{\gamma(\|a_n - a_{n-1}\| + \|a_{n-1} - a_{n-2}\|)\}$ and using (25) in (24), we get

$$\begin{aligned}\|a_{n+1} - a_n\| \leq{}& \omega_n\rho t\|a_n - a_{n-1}\| + \rho\|\omega_n - \omega_{n-1}\|\|g(a_{n-1})\| + (1-\omega_n\bar{\rho})\|a_n - a_{n-1}\| + (1-\omega_n\bar{\rho})W \\ &+ (1-\omega_n\bar{\rho})\|\kappa_{n-1} - \kappa_n\|\|Kc_{n-1}\| + (1-\omega_n\bar{\rho})\|U_n b_{n-1} - U_{n-1}b_{n-1}\| \\ &+ |\omega_n - \omega_{n-1}|\|GU_{n-1}b_{n-1}\|.\end{aligned} \tag{26}$$

Take $R = \max\{\sup_{n\in\mathbb{N}} \rho\|g(a_{n-1})\|, \sup_{n\in\mathbb{N}}\|GU_{n-1}b_{n-1}\|\}$ and $S = \sup_{n\in\mathbb{N}}\{\|Kc_{n-1}\|\}$. So, we get

$$\begin{aligned}\|a_{n+1} - a_n\| \leq{}& (1-(\bar{\rho} - t\rho)\omega_n)\|a_n - a_{n-1}\| + 2R|\omega_n - \omega_{n-1}| + (1-\omega_n\bar{\rho})W \\ &+ (1-\omega_n\bar{\rho})|\kappa_{n-1} - \kappa_n|S + (1-\omega_n\bar{\rho})\sup_{b\in\{b_n\}}\|U_n b - U_{n-1}b\|.\end{aligned} \tag{27}$$

From Remark 7 we see that $\sum_{n=1}^{\infty}|\kappa_n - \kappa_{n-1}|$ is a telescoping series, which is convergent. Thus, we have $\sum_{n=1}^{\infty}|\kappa_n - \kappa_{n-1}| < \infty$. Also $\{U_n\}$ satisfies the AKTT-condition, $\sum_{n=1}^{\infty}|\omega_n - \omega_{n-1}| < \infty$, so from Lemma 5, we get

$$\|a_{n+1} - a_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{28}$$

Further, we consider

$$\begin{aligned}\|a_{n+1} - p\|^2 ={}& \|\omega_n\rho g(a_n) + (I - \omega_n G)U_n b_n - p\|^2 \\ ={}& \|\omega_n(\rho g(a_n) - Gp) + (I - \omega_n G)(U_n b_n - p)\|^2 \\ \leq{}& (\omega_n\|(\rho g(a_n) - Gp)\| + \|I - \omega_n G\|\|U_n b_n - p\|)^2 \\ \leq{}& (\omega_n\|\rho g(a_n) - Gp\| + (1-\omega_n\bar{\rho})\|b_n - p\|)^2 \\ \leq{}& \omega_n\|\rho g(a_n) - Gp\|^2 + (1-\omega_n\bar{\rho})\|b_n - p\|^2 + 2\omega_n(1-\omega_n\bar{\rho})\|\rho g(a_n) - Gp\|\|b_n - p\| \\ \leq{}& \omega_n\|\rho g(a_n) - Gp\|^2 + (1-\omega_n\bar{\rho})\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\|^2 \\ &+ 2\omega_n(1-\omega_n\bar{\rho})\|\rho g(a_n) - Gp\|\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\| \\ ={}& \omega_n\|\rho g(a_n) - Gp\|^2 + (1-\omega_n\bar{\rho})(\|c_n - p\|^2 - 2\kappa_n\langle c_n - p, Kc_n - Kp\rangle \\ &+ \kappa_n^2\|Kc_n - Kp\|^2) + 2\omega_n(1-\omega_n\bar{\rho})\|\rho g(a_n) - Gp\|\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\|.\end{aligned} \tag{29}$$

Using (A2) in the above inequality, we have

$$\begin{aligned}\|a_{n+1} - p\|^2 \leq{}& \omega_n\|\rho g(a_n) - Gp\|^2 + (1-\omega_n\bar{\rho})\left(\|c_n - p\|^2 - 2\kappa_n\tau\|Kc_n - Kp\|^2 + \kappa_n^2\|Kc_n - Kp\|^2\right) \\ &+ 2\omega_n(1-\omega_n\bar{\rho})\|\rho g(a_n) - Gp\|\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\| \\ ={}& \omega_n\|\rho g(a_n) - Gp\|^2 + (1-\omega_n\bar{\rho})(\|c_n - p\|^2 + \kappa_n(\kappa_n - 2\tau)\|Kc_n - Kp\|^2) \\ &+ 2\omega_n(1-\omega_n\bar{\rho})\|\rho g(a_n) - Gp\|\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\| \\ \leq{}& \omega_n\|\rho g(a_n) - Gp\|^2 + \|c_n - p\|^2 + (1-\omega_n\bar{\rho})\kappa_1(\kappa_1 - 2\tau)\|Kc_n - Kp\|^2 \\ &+ 2\omega_n(1-\omega_n\bar{\rho})\|\rho g(a_n) - Gp\|\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\|.\end{aligned} \tag{30}$$

From (21), we have $\|c_n - p\|^2 \le \|a_n - p\|^2 + \omega_n L_2$, for some $L_2 > 0$. Therefore, we get

$$\left\|a_{n+1} - p\right\|^2 \le \omega_n \left\|\rho g(a_n) - Gp\right\|^2 + \left\|a_n - p\right\|^2 + \omega_n L_2 + \left(1 - \omega_n \bar{\rho}\right)\kappa_1 (\kappa_1 - 2\tau)\left\|Kc_n - Kp\right\|^2 + 2\omega_n\left(1 - \omega_n\bar{\rho}\right)\left\|\rho g(a_n) - Gp\right\|\left\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\right\|. \tag{31}$$

Rearranging the terms, we get

$$\begin{aligned}
-\left(1 - \omega_n\bar{\rho}\right)\kappa_1(\kappa_1 - 2\tau)\left\|Kc_n - Kp\right\|^2 &\le \omega_n\left\|\rho g(a_n) - Gp\right\|^2 + \|a_n - p\|^2 - \left\|a_{n+1} - p\right\|^2 + \omega_n L_2 \\
&\quad + 2\omega_n\left(1 - \omega_n\bar{\rho}\right)\left\|\rho g(a_n) - Gp\right\|\left\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\right\| \\
&\le \omega_n\left\|\rho g(a_n) - Gp\right\|^2 \\
&\quad + \left\|a_{n+1} - a_n\right\|\left(\left\|a_n - p\right\| + \left\|a_{n+1} - p\right\|\right) + \omega_n L_2 \\
&\quad + 2\omega_n\left(1 - \omega_n\bar{\rho}\right)\left\|\rho g(a_n) - Gp\right\|\left\|(I - \kappa_n K)c_n - (I - \kappa_n K)p\right\|.
\end{aligned} \tag{32}$$

Using (28) and (A6), we get

$$\left\|Kc_n - Kp\right\| \longrightarrow 0 \text{ as } n \longrightarrow \infty. \tag{33}$$

Now, from the properties of projection mapping, we have

$$\begin{aligned}
\left\|b_n - p\right\|^2 &= \left\|P_C(c_n - \kappa_n Kc_n) - P_C(p - \kappa_n Kp)\right\|^2 \\
&\le \langle (c_n - \kappa_n Kc_n) - (p - \kappa_n Kp), b_n - p\rangle \\
&= \frac{1}{2}\Big(\left\|c_n - \kappa_n Kc_n - (p - \kappa_n Kp)\right\|^2 + \left\|b_n - p\right\|^2 \\
&\quad - \left\|c_n - \kappa_n Kc_n - (p - \kappa_n Kp) - (b_n - p)\right\|^2\Big) \\
&\le \frac{1}{2}\left(\left\|c_n - p\right\|^2 + \left\|b_n - p\right\|^2 + \kappa_1(\kappa_1 - 2\tau)\left\|Kc_n - Kp\right\|^2 - \left\|c_n - b_n - \kappa_n(Kc_n - Kp)\right\|^2\right) \\
&\le \left\|c_n - p\right\|^2 - \left\|c_n - b_n\right\|^2 + \kappa_1(\kappa_1 - 2\tau)\left\|Kc_n - Kp\right\|^2 + 2\kappa_n\langle c_n - b_n, Kc_n - Kp\rangle \\
&\quad - \kappa_n^2\left\|Kc_n - Kp\right\|^2 \\
&\le \left\|a_n - p\right\|^2 + \omega_n L_2 - \left\|c_n - b_n\right\|^2 + \kappa_1(\kappa_1 - 2\tau)\left\|Kc_n - Kp\right\|^2 \\
&\quad + 2\kappa_n\langle c_n - b_n, Kc_n - Kp\rangle - \kappa_n^2\left\|Kc_n - Kp\right\|^2.
\end{aligned} \tag{34}$$

Using (34), we have

$$\begin{aligned}\|a_{n+1}-p\|^2 &\leq \omega_n\|\rho g(a_n)-Gp\|^2+\left(1-\omega_n\bar{\rho}\right)\|b_n-p\|^2+2\omega_n\left(1-\omega_n\bar{\rho}\right)\|\rho g(a_n)-Gp\|\|b_n-p\|\\ &\leq \omega_n\|\rho g(a_n)-Gp\|^2+\left(1-\omega_n\bar{\rho}\right)\|a_n-p\|^2+\left(1-\omega_n\bar{\rho}\right)\omega_n L_2-\left(1-\omega_n\bar{\rho}\right)\|c_n-b_n\|^2\\ &\quad+\kappa_1(\kappa_1-2\tau)\|Kc_n-Kp\|^2\\ &\quad+2\kappa_n\left(1-\omega_n\bar{\rho}\right)\langle c_n-b_n,Kc_n-Kp\rangle-\left(1-\omega_n\bar{\rho}\right)\kappa_n^2\|Kc_n-Kp\|^2\\ &\quad+2\omega_n\left(1-\omega_n\bar{\rho}\right)\|\rho g(a_n)-Gp\|\|b_n-p\|.\end{aligned} \tag{35}$$

Thus, we have

$$\begin{aligned}\left(1-\omega_n\bar{\rho}\right)\|c_n-b_n\|^2 &\leq \omega_n\|\rho g(a_n)-Gp\|^2+\left(\|a_n-p\|+\|a_{n+1}-p\|\right)\left(\|a_{n+1}-a_n\|\right)\\ &\quad+\left(1-\omega_n\bar{\rho}\right)\omega_n L_2+\kappa_1(\kappa_1-2\tau)\|Kc_n-Kp\|^2\\ &\quad+2\kappa_n\left(1-\omega_n\bar{\rho}\right)\langle c_n-b_n,Kc_n-Kp\rangle\\ &\quad-\left(1-\omega_n\bar{\rho}\right)\kappa_n^2\|Kc_n-Kp\|^2+2\omega_n\left(1-\omega_n\bar{\rho}\right)\|\rho g(a_n)-Gp\|\|b_n-p\|.\end{aligned} \tag{36}$$

Since $\omega_n\longrightarrow 0$, $\|Kc_n-Kp\|\longrightarrow 0$ and $\|a_{n+1}-a_n\|\longrightarrow 0$ as $n\longrightarrow\infty$, we get

$$\|c_n-b_n\|\longrightarrow 0. \tag{37}$$

Further, we consider

$$\begin{aligned}\|a_{n+1}-U_nb_n\| &= \|\omega_n\rho g(a_n)+(I-\omega_nG)U_nb_n-U_nb_n\|\\ &= \omega_n\|\rho g(a_n)-GU_nb_n\|.\end{aligned} \tag{38}$$

As $\omega_n\longrightarrow 0$ and since $\{g(a_n)\}$ and $\{GU_nb_n\}$ are bounded, we get

$$\|a_{n+1}-U_nb_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{39}$$

This means

$$\|a_n-U_nb_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{40}$$

Moreover

$$\begin{aligned}\|a_n-c_n\| &= \gamma_n\|a_n-a_{n-1}\|\\ &= \frac{\gamma_n}{\omega_n}\omega_n\|a_n-a_{n-1}\|,\end{aligned} \tag{41}$$

this implies

$$\|a_n-c_n\|\longrightarrow 0 \text{ as } n\longrightarrow\infty. \tag{42}$$

Further, it is easy to see that $\|a_n-b_n\|\longrightarrow 0$, $\|a_n-U_na_n\|\longrightarrow 0$, $\|b_n-U_nb_n\|\longrightarrow 0$, $\|Ua_n-a_n\|\longrightarrow 0$ as $n\longrightarrow\infty$.

Next, we'll show that $\{a_n\}$ converges to the common element. We observe that $P_{F_1}(\gamma g+I-G)$ is a contraction, where $F_1=\cap_{n=1}^{\infty}Fix(U_n)\cap VI(C,K)$. Since $\|I-G\|\leq 1-\bar{\rho}$ and $0<\rho<\bar{\rho}/t$, we get

$$\begin{aligned}\left\|P_{F_1}(\rho g+I-G)a-P_{F_1}(\rho g+I-G)b\right\| &\leq \|(\rho g+I-G)a-(\rho g+I-G)b\|\\ &= \|\rho g(a)+a-Ga-\rho g(b)-b+Gb\|\\ &\leq \rho\|g(a)-g(b)\|+\|I-G\|\|a-b\|\\ &\leq \rho t\|a-b\|+(1-\bar{\rho})\|a-b\|\\ &\leq (1-(\bar{\rho}-\rho t))\|a-b\|.\end{aligned} \tag{43}$$

Thus, from Banach's contraction principle we see that $P_{F_1}(\rho g + I - G)$ has a unique fixed point, say $p \in H$, such that $P_{F_1}(\rho g + I - G)p = p$. Thus, we have

$$\langle (\rho g - G)p, a - p\rangle \leq 0 \quad \text{for all } a \in F_1. \tag{44}$$

Let $\{b_{n_k}\}$ be a subsequence of $\{b_n\}$ such that

$$\limsup_{n\longrightarrow\infty}\langle (\rho g - G)p, U_n b_n - p\rangle = \lim_{k\longrightarrow\infty}\left\langle (\rho g - G)p, U_{n_k} b_{n_k} - p\right\rangle. \tag{45}$$

Since $\{b_{n_k}\}$ is a bounded sequence, thus a subsequence $\{b_{n_{k_i}}\}$ of $\{b_{n_k}\}$ converges weakly to u. We may assume that $b_{n_k} \rightharpoonup u$, without loss of generality. Since $\|b_n - U_n b_n\| \longrightarrow 0$, we obtain $U_{n_k} b_{n_k} \rightharpoonup u$. By Lemmas 3 and 6 and the fact that $\|a_n - Ua_n\| \longrightarrow 0$, $\|a_n - b_n\| \longrightarrow 0$, we have $u \in \cap_{n=1}^{\infty}\mathrm{Fix}(U_n)$. Let $\mathrm{Sa} = \begin{cases} \mathrm{Ka} + N_C a, & \text{if } a \in C \\ \phi, & \text{otherwise.}\end{cases}$ where $N_C(a)$ is the normal cone to C at $a \in C$, that is $N_C(a) = \{c \in H\colon \langle a - b, c\rangle \geq 0, \text{for all } b \in C\}$. Then S is maximal monotone. From the properties of projection mapping, we have

$$\langle a - b_n, b_n - (c_n - \kappa_n K c_n)\rangle \geq 0, \tag{46}$$

which implies

$$\langle a - b_n, (b_n - c_n)/\kappa_n + Kc_n\rangle \geq 0. \tag{47}$$

Let $(a, c) \in \mathrm{Graph(S)}$. Since $c - \mathrm{Ka} \in N_C(a)$ and $b_n \in C$, we get

$$\begin{aligned}
&\langle a - b_n, c - Ka\rangle \geq 0\\
&\left\langle a - b_{n_k}, c\right\rangle \geq \left\langle a - b_{n_k}, Ka\right\rangle\\
&\geq \left\langle a - b_{n_k}, Ka\right\rangle - \left\langle a - b_{n_k}, \frac{\left(b_{n_k} - c_{n_k}\right)}{\kappa_n + Kc_{n_k}}\right\rangle\\
&= \left\langle a - b_{n_k}, Ka - Kc_{n_k} - \frac{\left(b_{n_k} - c_{n_k}\right)}{\kappa_n}\right\rangle\\
&= \left\langle a - b_{n_k}, Ka - Kb_{n_k}\right\rangle + \left\langle a - b_{n_k}, Kb_{n_k} - Kc_{n_k}\right\rangle - \left\langle a - b_{n_k}, \frac{\left(b_{n_k} - c_{n_k}\right)}{\kappa_n}\right\rangle\\
&\geq \left\langle a - b_{n_k}, Kb_{n_k} - Kc_{n_k}\right\rangle - \left\langle a - b_{n_k}, \frac{\left(b_{n_k} - c_{n_k}\right)}{\kappa_n}\right\rangle.
\end{aligned} \tag{48}$$

This implies $\langle a - u, c\rangle \geq 0$, as $n \longrightarrow \infty$. Since S is maximal monotone, we have $u \in S^{-1}0$ and hence $u \in \mathrm{VI}(C, K)$. So, we obtain $p \in F_1 = \cap_{n=1}^{\infty}\mathrm{Fix}(U_n) \cap \mathrm{VI}(C, K)$. It follows that

$$\begin{aligned}
\limsup_{n\longrightarrow\infty}\langle (\rho g - G)p, U_n b_n - p\rangle &= \lim_{k\longrightarrow\infty}\left\langle (\rho g - G)p, U_{n_k} b_{n_k} - p\right\rangle\\
&= \langle (\rho g - G)p, u - p\rangle \leq 0.
\end{aligned} \tag{49}$$

Finally, we show$a_n \longrightarrow p$.

$$\begin{aligned}
\left\|a_{n+1}-p\right\|^{2} &= \left\|\omega_{n}\rho g\left(a_{n}\right)+\left(I-\omega_{n}G\right)U_{n}b_{n}-p\right\|^{2} \\
&= \left\|\omega_{n}\left(\rho g\left(a_{n}\right)-GP\right)+\left(I-\omega_{n}G\right)\left(U_{n}b_{n}-p\right)\right\|^{2} \\
&\leq \left\|\left(I-\omega_{n}G\right)\left(U_{n}b_{n}-p\right)\right\|^{2}+\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2} \\
&\quad +2\omega_{n}\left\langle\left(I-\omega_{n}G\right)\left(U_{n}b_{n}-p\right),\rho g\left(a_{n}\right)-G(p)\right\rangle \\
&\leq \left(1-\omega_{n}\bar{p}\right)^{2}\left\|b_{n}-p\right\|^{2}+\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2} \\
&\quad +2\omega_{n}\left\langle U_{n}b_{n}-p,\rho g\left(a_{n}\right)-G(p)\right\rangle-2\omega_{n}^{2}\left\langle G\left(U_{n}b_{n}-p\right),\rho g\left(a_{n}\right)-G(p)\right\rangle \\
&\leq \left(1-\omega_{n}\bar{p}\right)^{2}\left(\left\|a_{n}-p\right\|^{2}+\omega_{n}L_{2}\right)+\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2} \\
&\quad +2\omega_{n}\left\langle U_{n}b_{n}-p,pg\left(a_{n}\right)-pg(p)\right\rangle+2\omega_{n}\left\langle U_{n}b_{n}-p,pg(p)-G(p)\right\rangle \\
&\quad -2\omega_{n}^{2}\left\langle G\left(U_{n}b_{n}-p\right),pg\left(a_{n}\right)-G(p)\right\rangle \\
&\leq \left(1-\omega_{n}\bar{p}\right)^{2}\left\|a_{n}-p\right\|^{2}+\left(1-\omega_{n}\bar{p}\right)^{2}\omega_{n}L_{2}+\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2} \\
&\quad +2\omega_{n}p\left\|U_{n}b_{n}-p\right\|\left\|g\left(a_{n}\right)-g(p)\right\|+2\omega_{n}\left\langle U_{n}b_{n}-p,\rho g(p)-G(p)\right\rangle \\
&\quad -2\omega_{n}^{2}\left\langle G\left(U_{n}b_{n}-p\right),\rho g\left(a_{n}\right)-G(p)\right\rangle \\
&\leq \left(1-\omega_{n}\bar{p}\right)^{2}\left\|a_{n}-p\right\|^{2}+\left(1-\omega_{n}\bar{p}\right)^{2}\omega_{n}L_{2}+\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2} \\
&\quad +2\omega_{n}pt\left\|b_{n}-p\right\|\left\|a_{n}-p\right\|+2\omega_{n}\left\langle U_{n}b_{n}-p,\rho g(p)-G(p)\right\rangle \\
&\quad -2\omega_{n}^{2}\left\langle G\left(U_{n}b_{n}-p\right),\rho g\left(a_{n}\right)-G(p)\right\rangle \\
&\leq \left(1-\omega_{n}\bar{p}\right)^{2}\left\|a_{n}-p\right\|^{2}+\left(1-\omega_{n}\bar{p}\right)^{2}\omega_{n}L_{2}+\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2} \\
&\quad +2\omega_{n}pt\left(\left\|a_{n}-p\right\|+\omega_{n}L_{1}\right)\left\|a_{n}-p\right\|+2\omega_{n}\left\langle U_{n}b_{n}-p,\rho g(p)-G(p)\right\rangle \\
&\quad -2\omega_{n}^{2}\left\langle G\left(U_{n}b_{n}-p\right),\rho g\left(a_{n}\right)-G(p)\right\rangle \\
&\leq \left(1-\omega_{n}\bar{p}\right)^{2}\left\|a_{n}-p\right\|^{2}+\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2}+\left(1-\omega_{n}\bar{p}\right)^{2}\omega_{n}L_{2} \\
&\quad +2\omega_{n}pt\left\|a_{n}-p\right\|^{2}+2\omega_{n}^{2}pt\,L_{1}\left\|a_{n}-p\right\|+2\omega_{n}\left\langle U_{n}b_{n}-p,\rho g(p)-G(p)\right\rangle \\
&\quad -2\omega_{n}^{2}\left\langle G\left(U_{n}b_{n}-p\right),\rho g\left(a_{n}\right)-G(p)\right\rangle \\
&\leq \left(1-2(\bar{p}-tp)\omega_{n}\right)\left\|a_{n}-p\right\|^{2}+2\omega_{n}\left\langle U_{n}b_{n}-p,\rho g(p)-G(p)\right\rangle \\
&\quad +\omega_{n}^{2}\left\|\rho g\left(a_{n}\right)-G(p)\right\|^{2}-2\omega_{n}^{2}\left\langle G\left(U_{n}b_{n}-p\right),\rho g\left(a_{n}\right)-G(p))\right\rangle \\
&\quad +2\omega_{n}^{2}ptL_{1}\left\|a_{n}-p\right\|+\left(1-\omega_{n}\bar{p}\right)^{2}\omega_{n}L_{2}.
\end{aligned} \tag{50}$$

As $\limsup_{n\longrightarrow\infty}\langle U_n b_n - p, \rho g(p) - Gp\rangle \leq 0$, then by using Lemma 5 along with the assumption $\lim_{n\longrightarrow\infty}\omega_n = 0$, we have $a_n \longrightarrow p$. This completes our proof. □

Remark 11

(1) Many researchers have calculated projections onto C followed by projections onto the half-space. In our Algorithm 1, we calculate only one projection per iteration with a self-adaptive step size rule and inertial extrapolation step.

(2) The inertial extrapolation step introduced in our algorithm is a combination of the previous two values of the iteration along with the inertia γ_n attaining any value greater than zero. This speeds up the convergence of the sequence converging to the common element.

4. Numerical Examples

In this section, we discuss some numerical examples to validate our theorem. The performance of our Algorithm 1 is compared with other well-established algorithms such as iVSEGM [21], MSEGM [27] and VSEGM [20]. We denote the error sequence as $E_n = \|a_n - p\|$ and study the behavior of this sequence. The convergence of $\{E_n\} \longrightarrow 0$ implies that the sequence $\{a_n\} \longrightarrow p$. All the programs are carried out in MATLAB 2018a on Intel(R) Core(TM) i3-10110U CPU @ 2.10 GHz computer with RAM 8.00 GB.

We consider $\omega_n = 1/n + 1$ in each of the example discussed. For a fair comparison with the earlier established algorithms, we consider $\nu = 0.6$, $g(a) = 0.5a$, $\gamma = 0.1$ and $\varepsilon_n = 100/(n+1)^2$. For MSEGM, we consider $\alpha_n = 1/n + 1$ and $\tau_n = n/2n + 1$.

Example 1. Consider a nonlinear operator $K\colon \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$K(a) = a - \sin a, \tag{51}$$

and $C = [0, 1]$ the closed, convex subset of $\mathbb{R}$. Next, we prove that K is ism. So

$$\begin{aligned}\|K(a) - K(b)\|^2 &= \langle K(a) - K(b), K(a) - K(b)\rangle \\ &= \langle a - \sin(a) - b + \sin(b), a - \sin(a) - b + \sin(b)\rangle \\ &= \|a - b\|^2 + \|\sin(a) - \sin(b)\|^2 - 2\langle \sin(a) - \sin(b), a - b\rangle.\end{aligned} \tag{52}$$

This implies

$$\begin{aligned}2\langle \sin(a) - \sin(b), a - b\rangle &= \|a - b\|^2 + \|\sin(a) - \sin(b)\|^2 - \|K(a) - K(b)\|^2 \\ 2\langle a - a + \sin(a) - b + b - \sin(b), a - b\rangle &= \|a - b\|^2 + \|\sin(a) - \sin(b)\|^2 - \|K(a) - K(b)\|^2 \\ 2\langle a - K(a) - b + K(b), a - b\rangle &= \|a - b\|^2 + \|\sin(a) - \sin(b)\|^2 - \|K(a) - K(b)\|^2 \\ 2\|a - b\|^2 - 2\langle K(a) - K(b), a - b\rangle &\le \|a - b\|^2 + \|a - b\|^2 - \|K(a) - K(b)\|^2.\end{aligned} \tag{53}$$

Finally, we get

$$\langle K(a) - K(b), a - b\rangle \ge \frac{1}{2}\|K(a) - K(b)\|^2. \tag{54}$$

Therefore, $K$is 1/2-ism mapping. Assume that $\kappa_1 = 0.4$ and let $\{U_n\}$ be the family of self mappings on $\mathbb{R}$ be

$$U_n(a) = \frac{1}{2}(a + \sin a). \tag{55}$$

Observe that the mapping U_n is nonexpansive for each n and satisfies the AKTT-condition. Assume the strongly positive linear bounded operator G on H to be $G(a) = 1/2a$, with constants $\bar{\rho}$ and ρ equal to 1/2. The initial values considered are $a_0 = 1$ and $a_1 = 1.1$. Since each and every assumption of Theorem 10 is satisfied, so the sequence $\{a_n\}$ generated by Algorithm 1 converges to $0 \in \cap_{n=1}^{\infty}\mathrm{Fix}(U_n) \cap VI(C, K)$. Moreover, we also see that the error sequence E_n converges to 0 much faster and more efficiently than the well-known schemes given in the literature (see Figure 1).

Example 2. Consider a problem in infinite-dimensional Hilbert space $H = L^2([0, 1])$ equipped with inner product $\langle a, b\rangle = \int_0^1 a(t)b(t)dt$ and norm $\|a\| = (\int_0^1 |a(t)|^2)dt)^{1/2}$, for all $a, b \in H$. We define the feasible set as the unit ball $C = \{a \in H\colon \|a\| \le 1\}$. Now, consider the operator

$$K(a(t)) = \max\{0, a(t) - h(a(t))\}, \tag{56}$$

where $t \in [0, 1]$ and $h(a) = \sin(a)$. It can be easily shown that K is 1/2-ism mapping and the proof is on similar lines as of Example 1. The projection on C is explicitly defined as

$$P_C(a) = \begin{cases} \dfrac{a}{\|a\|}, & \text{if } \|a\| > 1, \\ a, & \text{if } \|a\| \le 1. \end{cases} \tag{57}$$

Let $\{U_n\}$ be the family of self mappings on $L^2([0, 1])$ be

$$U_n(a(t)) = \frac{1}{2}\sin(a(t)). \tag{58}$$

Assume that $\tau_1 = 0.4$, the mapping $\{U_n\}$ is nonexpansive for each n and satisfies the AKTT-condition. Assume the strongly positive linear bounded operator G on H to be I, the identity operator with constants $\bar{\rho}$ and ρ equal to 1/2. The initial values considered are $a_0 = 1$ and $a_1 = 1$. Since each and every assumption of Theorem 10 is satisfied, so the sequence $\{a_n\}$ generated by Algorithm 1 converges to $0 \in \cap_{n=1}^{\infty}\mathrm{Fix}(U_n) \cap VI(C, K)$. Moreover, we also see that the error sequence E_n converges to 0 much faster and more efficiently than the well-known schemes given in the literature (see Figure 2).

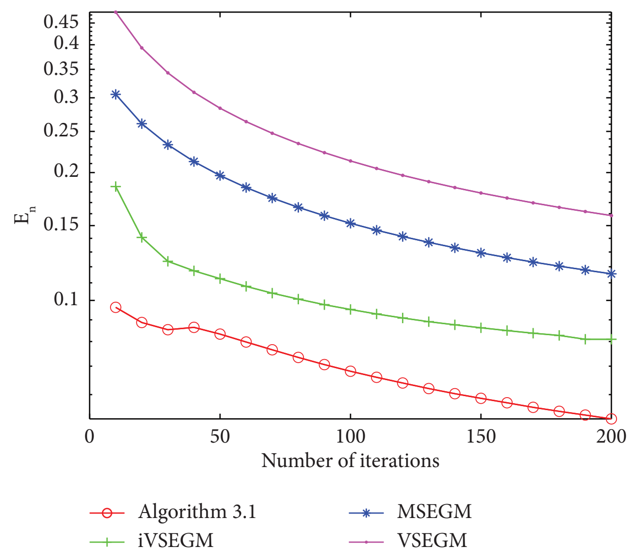

Figure 1: Graphical representation of iterative algorithm converging to 0 in Example 1.

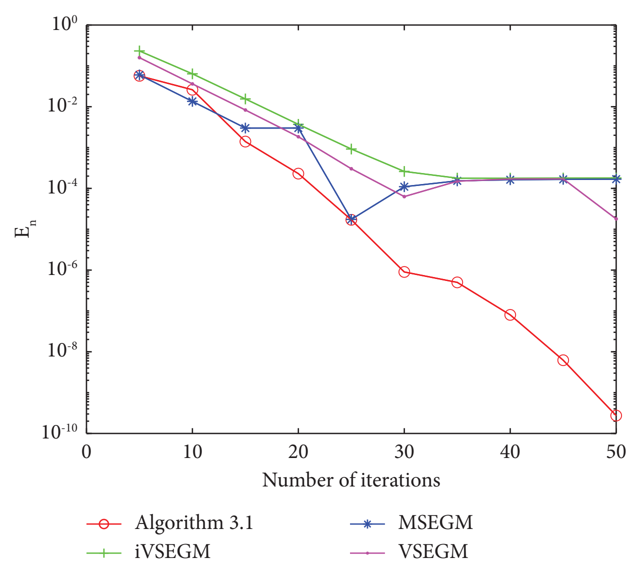

Figure 2: Graphical representation of iterative algorithm converging to 0 in Example 2.

Example 3. Consider an operator $K: \mathbb{R} \longrightarrow \mathbb{R}$ defined by

$$K(a) = \begin{cases} a - \dfrac{1}{2}, & \text{if } a \in (-\infty, 0], \\ a - \dfrac{1}{4}, & \text{if } a \in (0, \infty), \end{cases} \tag{59}$$

and $C = [0, 1]$ the closed, convex subset of $\mathbb{R}$. Now, we prove that K is ism mapping.

Case 1. When $a, b \in (-\infty, 0]$, we have

$$\begin{aligned} \langle K(a) - K(b), a - b\rangle &= \left\langle a - \frac{1}{2} - b + \frac{1}{2}, a - b\right\rangle \\ &= \langle a - b, a - b\rangle \\ &= \|a - b\|^2 \\ &\geq \|K(a) - K(b)\|^2. \end{aligned} \tag{60}$$

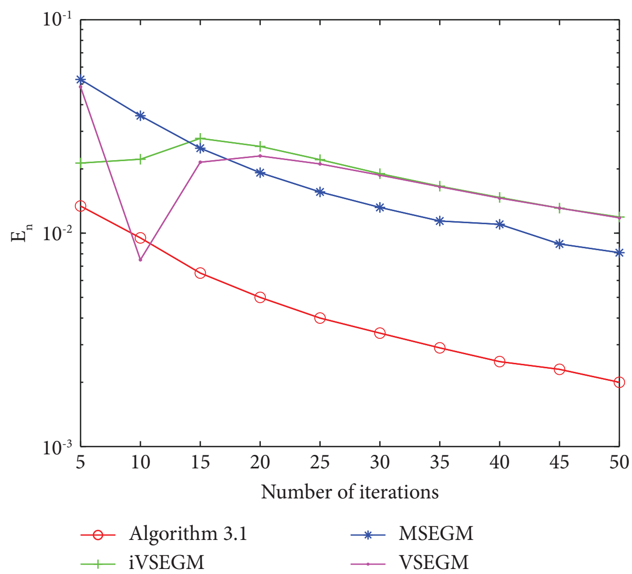

FIGURE 3: Graphical representation of iterative algorithm converging to 0 in Example 3.

TABLE 1: Numerical results for Example 1 after 200 iterations.

Algorithm	CPU time (sec.)	E_n
Algorithm 1	0.67	0.0526
iVSEGM	1.79	0.0810
MSEGM	1.66	0.1155
VSEGM	1.60	0.1584

TABLE 2: Numerical results for Example 2 after 50 iterations.

Algorithm	CPU time (sec.)	E_n
Algorithm 1	2.31	$2.74e-10$
iVSEGM	4.12	$1.795e-04$
MSEGM	2.88	$1.684e-04$
VSEGM	2.33	$1.792e-04$

TABLE 3: Numerical results for Example 3 after 50 iterations.

Algorithm	CPU time (sec.)	E_n
Algorithm 1	0.16	0.0020
iVSEGM	0.52	0.0119
MSEGM	0.49	0.0081
VSEGM	0.45	0.0118

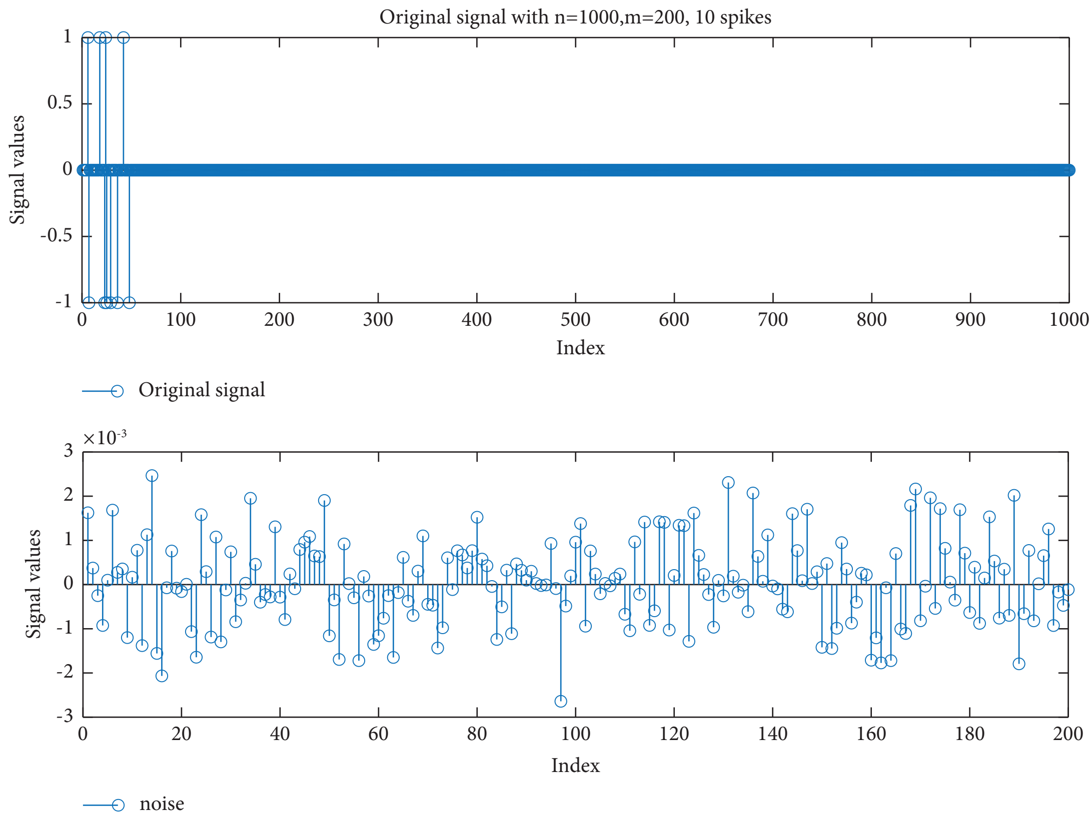

Figure 4: Original signal and the noisy signal.

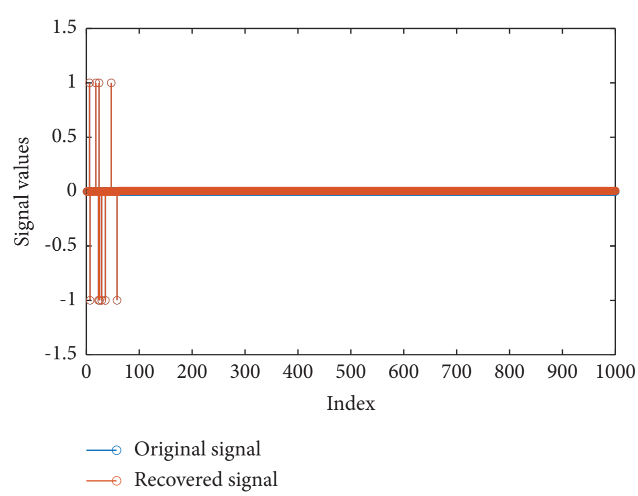

Figure 5: Graphical representation of recovered signal along with the original signal.

Case 2. When $a, b \in (0, \infty)$, we have

$$\begin{aligned}\langle K(a) - K(b), a - b\rangle &= \left\langle a - \frac{1}{4} - b + \frac{1}{4}, a - b\right\rangle \\ &= \|a - b\|^2 \\ &\geq \|K(a) - K(b)\|^2.\end{aligned} \tag{61}$$

Case 3. When $a \in (-\infty, 0]$ and $b \in (0, \infty)$, we have

$$\begin{aligned}\langle K(a) - K(b), a - b\rangle &= \left\langle a - \frac{1}{2} - b + \frac{1}{4}, a - b\right\rangle \\ &= \left(a - b - \frac{1}{4}\right)(a - b) \\ &\geq \left(a - b - \frac{1}{4}\right)\left(a - b - \frac{1}{4}\right) \\ &= \left\|a - b - \frac{1}{4}\right\|^2 \\ &= \|K(a) - K(b)\|^2.\end{aligned} \tag{62}$$

Thus, we see that K is 1-ism. Assume that $\kappa_1 = 0.4$ and let $\{U_n\}$ be the family of self mappings on $\mathbb{R}$ be

$$U_n(a) = \frac{1}{4} + \sin\left(a - \frac{1}{4}\right). \tag{63}$$

Observe that the mapping U_n is nonexpansive for each n and satisfies the AKTT-condition. Assume the strongly positive linear bounded operator G on H to be $G(a) = 1/2a$ with constants $\bar{\rho}$ and ρ equal to 1/2. The initial values considered are $a_0 = 1$ and $a_1 = 2$. Since each and every assumption of Theorem 10 is satisfied, so the sequence $\{a_n\}$ generated by Algorithm 1 converges to $0 \in \cap_{n=1}^{\infty} Fix(U_n) \cap VI(C, K)$. Moreover, we also see that the error sequence E_n converges to 0 much faster and more efficiently than the well-known schemes given in the literature (see Figure 3).

Remark 12

(1) We can see from Tables 1–3 that our algorithm outperforms earlier established algorithms both in terms of speed and accuracy. Moreover, it is easy to implement.

(2) Our algorithm performs well in both finite and infinite dimensional Hilbert space.

5. Applications

In this section, we give some applications that can be solved through our main result.

Let H be a real Hilbert space with $\langle\cdot,\cdot\rangle$, $\|\cdot\|$ being its inner product and norm, respectively. Let C be a closed, convex subset of H and $K\colon H \longrightarrow H$ be a nonlinear mapping.

5.1. Application to Convex Minimization Problems. Let $h\colon C \longrightarrow \mathbb{R}$ be a convex mapping. We consider the following minimization problem

$$\min_{a\in C} h(a). \tag{64}$$

Suppose that the mapping h is Frechet differentiable. Then our optimization problem (64) has a solution a^* if and only if the *variational* inequality below satisfies:

$$a^* \in C, \langle \nabla h a^*, a - a^*\rangle \geq 0, \quad \text{for all } a \in C, \tag{65}$$

that is, $a^* \in \mathrm{VI}(C, \nabla h)$.

Suppose we take $U_n = I$ for each $n \in \mathbb{N}$ and $K = \nabla h$ in our algorithm. Then, we have the following theorem.

Theorem 13. *Suppose that $h\colon C \longrightarrow \mathbb{R}$ is a convex mapping such that its gradient ∇h is L-Lipschitz continuous mapping and $g\colon C \longrightarrow C$ is a t-contraction with constant $t \in [0, 1)$. Also, consider $G\colon H \longrightarrow H$ is a strongly positive linear bounded operator with coefficient $0 < \bar{\rho} < 1$ such that $0 < \|G\| \leq 1$ and $0 < \rho < \bar{\rho}/t$. Assume that $\{\varepsilon_n\}$ is a positive sequence such that $\lim_{n\longrightarrow\infty} \varepsilon_n/\omega_n = 0$. where $\omega_n \in [0, 1]$ satisfies $\lim_{n\longrightarrow\infty} \omega_n = 0, \sum_{n=1}^{\infty} \omega_n = \infty$ and $\sum_{n=1}^{\infty} |\omega_n - \omega_{n+1}| < \infty$. If $VI(C, \nabla h) \neq 0$, then for any $a_0, a_1 \in \mathbb{R}$, the Algorithm 1 converges to $a^* \in VI(C, \nabla h)$.*

Proof. Put $K = \nabla h$ in our main algorithm since ∇h is L-Lipschitz continuous. This means that ∇h is $1/L$-inverse strongly monotone mapping. Observe that $U_n = I$ is nonexpansive for each n. Therefore, by Theorem we obtain $a^* \in \mathrm{VI}(C, \nabla g) \cap \mathrm{Fix}(I) = \mathrm{VI}(C, \nabla g)$. This means a^* is a solution to the variational inequality problem. Hence, the result. □

5.2. Application to Signal Processing Problems. Since communications in the actual world can experience interference during transmission, the signal recovery problem deals with the recovery of the original clean signals from noisy signals. The model for signal processing problems is described as

$$b = \mathrm{P}a + h, \tag{66}$$

where $a \in \mathbb{R}^n$ has t non zero elements as the original signal, $b \in \mathbb{R}^m$ is our observed noisy signal, $P\colon \mathbb{R}^{m\times n}$ is a linear operator which is bounded and the noise observation is $h \in \mathbb{R}^m$. This model works with the assumption that the signal a is sparse, which means that the number of non-zero elements in the signal a is much less compared to the dimension of a. This model (66) can be solved using the Least Absolute Shrinkage and Selection Operator(LASSO) model. This model is expressed as:

$$\begin{aligned}&\min_{a\in\mathbb{R}^n} f(a) = \frac{1}{2}\|Pa - b\|_2^2 \\ &\text{s.t.} \quad \|a\|_1 \leq l, \quad l > 0.\end{aligned} \tag{67}$$

Here $\|\cdot\|_2$ and $\|\cdot\|_1$ represents 2-norm and 1-norm respectively. (67) is further equivalent to solving a variational inequality problem of finding $a^* \in C$ such that

$$\nabla f(a^*)^T (a - a^*) \geq 0, \tag{68}$$

for all $a \in C$. The gradient of the function f is known to be $\nabla f(a) = P^T(Pa - b)$. We set $K(a) = P^T(Pa - b)$, $C = \{a \in \mathbb{R}^n \colon \|a\|_1 \leq l\}$ and $U_n = I$ for each $n \in \mathbb{N}$ in our proposed algorithm. Notice that K is monotone and $\|P^T P\|$-Lipschitz continuous. It is easy to show that f is a convex function. Thus, we get K to be $1/\|P^T P\|$-ism. To verify numerically, we set $\omega_n = 1/n + 1$, $\nu = 0.1$, $g(a) = 0.5a$, $\gamma = 0.2$, $\kappa_1 = 0.02$, $\bar{\rho} = \rho = 1/2$ and $\varepsilon_n = 100/(n+1)^2$. Assume that the original signal $a \in \mathbb{R}^n$ contains t randomly generated ± 1 spikes, which are very less as compared to the dimension of a. The matrix P and the noisy observation h is generated by randn(m, n), 10^{-3} randn(m, 1) respectively in the Matlab. Thus, the observation b is obtained using (63). We apply our algorithm when $n = 1000$, $m = 200$, initial points $a_0 = a_1 = 0$ and the randomly generated spikes $t = 10$. Thus, applying our Algorithm 1 by choosing $l = t$, we have a sequence that converges to the point, which minimizes the function f and thus the noisy observation (see Figures 4 and 5).

6. Conclusion

This paper discussed an iterative algorithm based on inertial term combined with the viscosity type approximation method, and some numerical computations of the proposed algorithm both in finite and infinite dimensional Hilbert space, are also presented to show the efficiency of the proposed algorithm. We concluded the discussion by giving applications of the proposed algorithm through the convex minimization problem and signal processing problem. For future work, we ask the following question: Is it possible to modify Algorithm 1 to deal with variational inequality problems involving much weaker forms than τ-inverse monotonicity?.

Data Availability

No underlying data were collected or produced in this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contributions

All authors contributed equally.

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