Hindawi
Journal of Mathematics
Volume 2024, Article ID 9531298, 11 pages
https://doi.org/10.1155/2024/9531298

Research Article

On Partial Exact Controllability of Fractional Control Systems in Conformable Sense

Maher Jneid

Department of Mathematics and Computer Science, Faculty of Science, Beirut Arab University, Beirut, Lebanon

Correspondence should be addressed to Maher Jneid; m.jneid@bau.edu.lb

Received 1 November 2023; Revised 31 December 2023; Accepted 23 February 2024; Published 7 March 2024

Academic Editor: Valerii Obukhovskii

In this work, we investigate the partial exact controllability of fractional semilinear control systems in the sense of conformable derivatives. Initially, we establish the existence and uniqueness of the mild solution for this type of fractional control systems. Then, by employing a contraction mapping principle, we obtain sufficient conditions for the conformable fractional semilinear system to be partially exactly controllable, assuming that its associated linear part is partially exactly controllable. To demonstrate the efficacy of the theoretical findings, a typical example is provided at the end.

1. Introduction

Controllability concepts have been essential in a variety of disciplines, such as engineering, control theory, and applied mathematics. For instance, controllability is essential for the design of feedback controllers, which are used to regulate and control the behavior of systems. Following Kalman [1]'s definition, controllability is characterized by the capability to transition a control system's solution from a given initial state to a desirable state at a final time. Later, controllability has been split into two concepts: exact and approximate controllability. Exact controllability aligns with Kalman's definition, where the system can be directed from any initial state to any desired state within a finite time. On the other hand, approximate controllability implies that the system can be moved from any starting state to any desired state arbitrarily closely in a finite time. The distinction between these two concepts is crucial because some dynamical systems exhibit approximate controllability without achieving exact controllability (see Fattorini [2]). Several researchers have developed suitable controllability conditions for deterministic and stochastic control systems. For example, see [3–10].

Fractional differential equation (FDE) has emerged as an important attractive area of applied mathematics because of its powerful uses in the fields of engineering and physical sciences [11–13]. The utilization of FDEs has been showcased as a highly effective method for enhancing the modeling of various real-world problems and phenomena, such as heat transfer processes and dielectric polarization [14]. They provide a more accurate model of physical systems than traditional differential equations, enabling the resolution of problems that prove challenging under traditional modeling. Notably, in the domain of electrical circuits, fractional-order models have proven valuable for simulating electrical components and circuits, including resistors, domino ladders, capacitors, tree structures, and inductors [15].

In electrical circuit, there are some dissipative effects stemming from electrical resistance, ohmic friction, or temperature that standard theoretical calculations fail to consider. The ordinary derivative is insufficient to take into account these nonconservative features. Consequently, in order to place these dissipation effects on to a relevant theoretical basis, the fractional calculus emerges as a valuable mathematical tool in addressing this kind of electrical problems. For instance, the fractional derivatives allow capturing the nonlocal and hereditary properties neglected in integer-order models [16].

This novel type of calculus has attracted the attention of mathematicians, who have been working to develop new results and extend existing concepts to these fractional systems [17–19]. Researchers in this field have focused efforts on findings new theoretical results and expanding controllability notions to apply to fractional control systems. For example, Mahmudov [20] derived a collection of approximate controllability conditions for Sobolev-type equations with fractional derivatives. Sakthivel et al. [21] applied a fixed point theorem to obtain controllability conditions for nonlinear systems with fractional order. Jneid [22] used a compact semigroup operator and Schauder fixed point technique to derive a set of sufficient conditions for the approximate controllability of integrodifferential control systems of non-integer order. Dineshkumar et al. [23] utilized Bohnenblust-Karlin's fixed point theorem, cosine and sine functions of operators to acquire sufficient conditions for the approximate controllability of fractional stochastic differential inclusions with order $1 < r < 2$. Sivasankar et al. [24] studied the nonlocal controllability of stochastic control systems involving Hilfer fractional derivative by using almost sectorial operators with the help of the fixed point technique and measures of noncompactness.

The majority of previous researches concerning controllability issues of fractional control systems have employed the Riemann-Liouville, Caputo, and Hilfer fractional derivatives. Nevertheless, there has been limited investigation into the controllability problems associated with fractional systems using the conformable fractional derivative [25–28]. This represents a notable gap in the existing body of literature, considering that the conformable fractional derivative offers several advantages compared to the Riemann-Liouville, Caputo and Hilfer fractional derivatives, including its greater naturalness and geometric intuitiveness. Motivated by this observation, this current work focuses on addressing the controllability problems of semilinear control systems with conformable fractional derivatives in Hilbert spaces. Furthermore, we introduce and expand the partial controllability concepts to fractional differential systems. Roughly speaking, the study of partial controllability is an important part of controllability research overall. This significance arises from the fact that controllability theorems are often formulated for first-order differential equation systems. Nevertheless, many real-world systems, such as higher-order fractional differential equations and fractional wave equations, can only be written in first-order form by enlarging the state space dimension. As a consequence, the standard controllability conditions for these systems are too strong since they consider the expanded state space, while controllability notions need to focus on the original state space. To address this, we introduce an additional projection operator P that maps the enlarged state space back to the original state space. To illustrate the workings of partial controllability, we provide two typical examples in Section 2. To our knowledge, the specific research problem under study has not been previously investigated. We carried out a thorough investigation and literature review and failed to uncover any studies addressing partial exact controllability of fractional control systems. This underlines a gap in the existing literature that our research attempts to fill.

The rest of the paper is organized as follows: In Section 2, useful notations and definitions, a mathematical model of partial controllability notions, and beneficial preliminary results concerning the partial controllability of linear systems in conformable fractional sense are obtained. In Section 3, we obtain a set of sufficient conditions for the partial exact controllability of fractional semilinear systems, assuming partial exact controllability of its associated linear systems. In Section 4, we give an illustrative example to prove the applicability of the theoretical findings. In Section 5, we provide a brief discussion of the results that are shown in the illustrative example. Finally, in Section 6, a short conclusion is given to recap the obtained results.

2. Preliminaries

Throughout this paper, we will utilize the following:

(i) $(X, \|.\|)$ and $(U, \|.\|)$ are Hilbert spaces with the norms generated by convenient inner products as $\|x\|^2 = \langle x, x \rangle$.

(ii) $U_{ad} = C(0, \tau; U)$, where $C(0, \tau; U)$ is defined as the vector space of all $U-$ valued continuous functions on $[0, \tau]$ endowed with the sup-norm as follows:

$$\|u\|_{C(0,\tau;U)} = \sup_{0 \le t \le \tau} \{\|u(t)\|\}. \tag{1}$$

(iii) $C(0, \tau; X) \times U_{ad}$ is the product space of two Banach spaces which is also a Banach space equipped with the following norm:

$$\|(.,.)\|_{C(0,\tau;X)\times U_{ad}} = \|.\|_{C(0,\tau;X)} + \|.\|_{U_{ad}}, \tag{2}$$

(iv) $L^2(a, b)$ represents the space of square-integrable functions on (a, b).

(v) A is an infinitesimal generator of C_0-semigroup $\Theta(t), t \ge 0$, on X.

(vi) B is a linear bounded operator from U to X.

(vii) H is a closed subspace of X.

(viii) $x(t)$ and $u(t)$ are denoted as x_t u_t, respectively.

Now, let us review some important concepts and findings about the conformable fractional derivative and the controllability of linear systems. We also establish the necessary assumptions, which will be required in the upcoming sections.

Definition 1 (see [29]). The conformable fractional derivative (CFD) of h: $[0, \infty) \longrightarrow \mathbb{R}^m$ at $x > 0$ of order $q \in (0, 1]$ is defined by

$$C_0^q h(x) = \left(\lim_{\delta \longrightarrow 0} \frac{h_1\left(x + \delta x^{1-q}\right) - h_1(x)}{\delta}, \ldots, \lim_{\delta \longrightarrow 0} \frac{h_m\left(x + \delta x^{1-q}\right) - h_m(x)}{\delta} \right), \tag{3}$$

on condition that the expression on the right side is exists as a finite value.

Consequently,

$$\begin{aligned} C_0^{2q} h(x) &= C_0^q \left(C_0^q h\right)(x), \ldots, \text{and} \\ C_0^{nq} h(x) &= C_0^q \left(C_0^{(n-1)q} h\right)(x), \end{aligned} \tag{4}$$

$C_0^{nq} h(x)$ is called n−times conformable fractional differentiable of order nq.

For every constant c and $r \in \mathbb{R}$, the following properties hold.

(a) $C_0^q(c) = 0$
(b) $C_0^q(x^r) = r x^{r-q}$
(c) $C_0^q(e^{x^q/q}) = e^{x^q/q}$

Definition 2 (see [29]). The CF integral of h: $[0,\infty) \longrightarrow \mathbb{R}$, is defined by

$$I_0^q h(t) = \int_0^t x^{q-1} h(x) dx, \quad q \in (0, 1], \tag{5}$$

on condition that the improper integral on the right side is a finite value.

Definition 3 (see [30]). The CF Laplace transform of h is defined by

$$\begin{aligned} \mathcal{T}_0^q\{h(t)\}(s) \&9; &= H_0^q(s) \\ \&9; &= \int_0^\infty t^{q-1} e^{-st^q/q} h(t) \mathrm{d}t, \end{aligned} \tag{6}$$

It is easy to show that

$$\mathcal{T}_0^q\{C_0^q h(t)\}(s) = s H_0^q(s) - h(0). \tag{7}$$

Let the CF− linear system be given as follows:

$$\begin{cases} C_0^q x_t = A x_t + f(t), 0 < t \leq \tau, \\ x_0 = \psi \in X, \end{cases} \tag{8}$$

where C_0^q is the CFD−operator, $x \in C(0, \tau; X)$, $f \in C(0, \tau; X), 0 < q \leq 1$ and A is as defined above. By using CF Laplace transform, we obtain

$$X_0^q(s) - \psi = \mathrm{A} X_0^q(s) + F_0^q(s), \tag{9}$$

which clearly gives that

$$X_0^q(s) = (sI - A)^{-1} \psi + (sI - A)^{-1} F_0^q(s), \tag{10}$$

where I is the identity operator.

Now, applying CF inverse Laplace transform and relevant properties from [30], we can derive the mild solution of the system (8) as follows:

$$\begin{aligned} x_t &= \left(\mathcal{T}_0^q\right)^{-1} \left[(sI - A)^{-1} \psi + (sI - A)^{-1} F_0^q(s) \right] \\ &= \left(\mathcal{T}_0^q\right)^{-1} \left[(sI - A)^{-1} \psi \right] + \left(\mathcal{T}_0^q\right)^{-1} \left[(sI - A)^{-1} F_0^q(s) \right] \\ &= \Theta\left(\frac{t^q}{q}\right) \psi + \int_0^t r^{q-1} \Theta\left(\frac{t^q}{q} - \frac{r^q}{q}\right) f(r) \mathrm{d}r, \end{aligned} \tag{11}$$

where $\Theta\ (t^q/q)$ is called CF− semigroup generated by A, and $(sI - A)^{-1}$ is the inverse operator of $(sI - A)$.

Partial controllability is a useful concept for control systems that can be modeled as first-order differential equations by augmenting the original state space. This is because the partial controllability concepts are more suited for such systems than the traditional notions of controllability. The projection operator P can be used to map the expanded state space to the original state space, which makes it easier to analyze and design controllers for the system. Partial controllability has several advantages, which will be illustrated in the following examples.

Example 1. Let the n-times conformable fractional differential system be

$$C_0^{nq} y_t = g\left(t, y_t, C_0^q y_t, \ldots, C_0^{(n-1)q} y_t, u_t\right). \tag{12}$$

Consider $\mathbb{R}$ as the state space for the system (12). By definition, controllability concepts for this system revolve around whether the relevant reachable set is either equal to or densely spread across the real space $\mathbb{R}$. Expressing this system in the form of a first-order differential equation is straightforward as follows:

$$C_0^q x_t = A x_t + G(t, x_t, u_t), \tag{13}$$

if

$$\begin{aligned} x_t &= \begin{pmatrix} y_t \\ y_t^q \\ \vdots \\ y_t^{(n-2)q} \\ y_t^{(n-1)q} \end{pmatrix}, \\ A &= \begin{pmatrix} 0 & 1 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 1 \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}, \\ G(t, x, u) &= \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ g\left(t, y_t, y_t^q, \ldots, y_t^{(n-1)q}, u_t\right) \end{pmatrix}. \end{aligned} \tag{14}$$

The fractional control system (13) is defined within an n-dimensional Euclidean space, denoted by $\mathbb{R}^n$, and as a result, its reachable set shall be a subset in $\mathbb{R}^n$. Consequently, the controllability criteria for the system (13) are more stringent compared to those of the system (12). Nonetheless, these criteria can be facilitated through the use of the projection operator P, which can be defined as follows:

$$P = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \end{bmatrix} : \mathbb{R}^n \longrightarrow \mathbb{R}. \tag{15}$$

This operator makes the conditions for partial controllability of system (13) the same as the conditions for controllability of system (12).

Example 2. Given a time CF-control system of wave equation as follows:

$$\partial_c^{2q} y_t(t,r) = \frac{\partial^2 y(t,r)}{\partial r^2} + k u_t + g\left(t, y(t,r), \partial_c^q y_t(t,r), u_t\right), \tag{16}$$

where $0 < q \leq 1$, y is a real valued function defined on $[0,\infty) \times (0, 1)$ and ∂_c^q is a partial conformable fractional operator of order q. The state space of this system is $L^2(0, 1)$. By enlarging the state space, we can rewrite this system in the first-order CF-control system as follows:

$$C_0^q x_t = A x_t + B u_t + G\left(t, x_t, u_t\right), \tag{17}$$

if

$$\begin{aligned} x_t &= \begin{bmatrix} x^1 \\ x^2 \end{bmatrix} = \begin{bmatrix} y(t,r) \\ \partial_c^q y_t(t,r) \end{bmatrix}, \\ A &= \begin{bmatrix} 0 & 1 \\ d^2/dr^2 & 0 \end{bmatrix}, \\ B &= \begin{bmatrix} k \\ 0 \end{bmatrix}, \\ G(t,x,u) &= \begin{bmatrix} 0 \\ g\left(t, x^1, x^2, u\right) \end{bmatrix}, \end{aligned} \tag{18}$$

where $x \in L^2(0, 1) \times L^2(0, 1)$. The ordinary controllability concept for the system (20) are too strong comparable with the same for the system (17). However, we can reduce this difficulty by defining the projection operator P as

$$P = \begin{bmatrix} I & 0 \end{bmatrix} : L^2(0, 1) \times L^2(0, 1) \longrightarrow L^2(0, 1), \tag{19}$$

which makes the studying the partial CF controllability of the system (20) the same as the studying ordinary CF controllability of the system (17).

Consider the abstract fractional semilinear system with conformable derivatives as follows:

$$\begin{cases} C_0^q x_t = A x_t + B u_t + f\left(t, x_t, u_t\right), 0 < t \leq \tau, 0 < q \leq 1, \\ x_0 = \psi, \end{cases} \tag{20}$$

where x and u are state and control values, respectively.

Now, let impose the following assumptions

(A1) The continuous map $f\colon [0,\tau] \times X \times U \longrightarrow X$ satisfies

(i) $\exists L > 0 \, \| f(t,x,u) \| \leq L \, \forall (t,z,u) \in [0,\tau] \times X \times U$

(ii) $\forall t \in [0,\tau]$, $u, v \in U$ and $y, x \in X$, $\exists N > 0$ so that

$$\| f(t,y,u) - f(t,x,v) \| \leq N \left(\| y - x \| + \| u - v \| \right). \tag{21}$$

(A2) P is a projection operator from X to H.

Under the above assumptions, the (22) has a unique mild solution $x \in C(0,\tau; X)$ for every $u \in U_{ad}$ and $x_0 \in X$ (see, Jaiswal and Bahuguna [31]), and this solution can be written as follows:

$$x_t = \Theta\left(\frac{t^q}{q}\right)\psi + \int_0^t s^{q-1} \Theta\left(\frac{t^q}{q} - \frac{s^q}{q}\right)\left(B u_s + f\left(s, x_s, u_s\right)\right) \mathrm{d}s. \tag{22}$$

Define the set

$$R(\tau,\psi) = \left\{ h \in H : \exists u \in U_{ad} \text{ so that } h = P x_\tau \right\}, \tag{23}$$

that stands for the attainable set of control system (22) at a finite time τ.

Definition 4. The fractional system (22) is said to be partially exactly controllable on U_{ad} if $R(\tau,\psi) = H$ for every $\psi \in X$.

Let the fractional controllability operator with a conformable fractional derivative Π_r^t be given as:

$$\Pi_r^t = \int_r^t s^{q-1} \left[\Theta\left(\frac{t^q}{q} - \frac{s^q}{q}\right) B \right] \left[\Theta\left(\frac{t^q}{q} - \frac{s^q}{q}\right) B \right]^* \mathrm{d}s, \quad 0 \leq r \leq t \leq \tau. \tag{24}$$

where $*$ indicates the adjoint operator.

(A3) Let $\widetilde{\Pi}_0^t = P \Pi_0^t P^*$. Assume $\widetilde{\Pi}_0^\tau$ is coercive, i.e., $\exists \mu > 0$ so that $\left\langle \widetilde{\Pi}_0^x, x \right\rangle \geq \mu \| x \|^2$, $\forall x \in H$. Then, $(\widetilde{\Pi}_0^\tau)^{-1}$ exists and $\| (\widetilde{\Pi}_0^\tau)^{-1} \| \leq 1/\mu$.

Now, consider the conformable fractional linear control system

$$\begin{cases} C_0^q x_t = A x_t + B u_t, 0 < t \leq \tau, \\ x_0 = \psi \in X, \end{cases} \tag{25}$$

where x, u, X, U, A, and B are as defined above.

Partial controllability of linear control systems is similar to ordinary controllability in many ways. In particular, if we replace the controllability operator with its partial version, most of the results on ordinary controllability can be applied to partial controllability as well. This is possible by imposing certain conditions on the partial controllability operator $\widetilde{\Pi}_0^T$. In the following theorem, we give necessary and sufficient conditions for partial controllability of conformable fractional linear systems.

Theorem 5. *Under the above assumptions, the following assertions are equivalent:a*

(i) The linear system (32) is partially exactly controllable
(ii) $\widetilde{\Pi}_0^\tau$ *is coercive*
(iii) $(\epsilon I, -\widetilde{\Pi}_0^\tau)^{-1}$ *is uniformly convergent as* $\epsilon \longrightarrow 0^+$
(iv) $(\epsilon I, -\widetilde{\Pi}_0^\tau)^{-1}$ *is strongly convergent as* $\epsilon \longrightarrow 0^+$
(v) $(\epsilon I, -\widetilde{\Pi}_0^\tau)^{-1}$ *is weakly convergent as* $\epsilon \longrightarrow 0^+$
(vi) $\epsilon(\epsilon I, -\widetilde{\Pi}_0^\tau)^{-1} \longrightarrow 0$ *uniformly as* $\epsilon \longrightarrow 0^+$

Proof. The proof of this theorem closely follows the proofs of similar theorems presented in many papers, for example, see the works of Mahmudov [32] and Jneid [33]. So we are not going to repeat the proof here. □

3. Partial Exact Controllability

In this section, we provide a sufficient condition set of partial exact controllability for the semi-linear fractional control system in conformable sense by using a contraction mapping principle.

Lemma 6. *Assume that the assumptions (A1)–(A3) hold true. Then, for every* $0 \le t \le \tau$, *the following inequalities valid*

$$\left\|\Pi_0^t\right\| \le \left\|\Pi_0^\tau\right\| \text{ and } \left\|\widetilde{\Pi}_0^t\right\| \le \left\|\widetilde{\Pi}_0^\tau\right\|. \tag{26}$$

Proof. It is clear that for every $0 \le t \le \tau$, $\widetilde{\Pi}_0^t = (\widetilde{\Pi}_0^t)^*$ and $x \in X$

$$\left\langle \Pi_0^t x, x \right\rangle \ge 0. \tag{27}$$

Hence,

$$\left\|\Pi_0^t\right\| = \sup_{\|x\| \le 1} \left|\left\langle \Pi_0^t x, x \right\rangle\right|. \tag{28}$$

Therefore,

$$\begin{aligned}
\left\langle \Pi_0^\tau x, x \right\rangle &= \left\langle \int_0^\tau s^{q-1}\left[\Theta\left(\frac{s^q}{q}\right)B\right]\left[\Theta\left(\frac{s^q}{q}\right)B\right]^* \mathrm{d}s\, x, x \right\rangle \\
&= \left\langle \int_0^t s^{q-1}\left[\Theta\left(\frac{s^q}{q}\right)B\right]\left[\Theta\left(\frac{s^q}{q}\right)B\right]^* \mathrm{d}s\, x, x \right\rangle + \left\langle \int_t^\tau s^{q-1}\left[\Theta\left(\frac{s^q}{q}\right)B\right]\left[\Theta\left(\frac{s^q}{q}\right)B\right]^* \mathrm{d}s\, x, x \right\rangle \\
&= \left\langle \Pi_0^t x, x \right\rangle + \left\langle \int_t^\tau s^{q-1}\left[\Theta\left(\frac{s^q}{q}\right)B\right]\left[\Theta\left(\frac{s^q}{q}\right)B\right]^* \mathrm{d}s\, x, x \right\rangle,
\end{aligned} \tag{29}$$

where

$$\begin{aligned}
\left\langle \int_t^\tau s^{q-1}\left[\Theta\left(\frac{s^q}{q}\right)B\right]\left[\Theta\left(\frac{s^q}{q}\right)B\right]^* \mathrm{d}s\, x, x \right\rangle &= \int_t^\tau s^{q-1}\left\langle \left[\Theta\left(\frac{s^q}{q}\right)B\right]^* x, \left[\Theta\left(\frac{s^q}{q}\right)B\right]^* x \right\rangle \mathrm{d}s \\
&= \int_t^\tau s^{q-1}\left\|\left[\Theta\left(\frac{s^q}{q}\right)B\right]^* x\right\| \mathrm{d}s \ge 0.
\end{aligned} \tag{30}$$

Then, $\langle \Pi_0^t x, x\rangle \le \langle \Pi_0^\tau x, x\rangle$, and consequently $\|\Pi_0^t\| \le \|\Pi_0^\tau\|$. For $\|\widetilde{\Pi}_0^t\| \le \|\widetilde{\Pi}_0^\tau\|$ $\forall 0 \le t \le \tau$, it follows directly from this equality $\left\langle \widetilde{\Pi}_0^t x, x \right\rangle = \left\langle \widetilde{\Pi}_0^t P^* x, P^* x \right\rangle$. □

Lemma 7. *Suppose that (A1)–(A3) hold, and* $h \in H$. *Then, for the operator Q that maps* $C(0, \tau; X) \times U_{ad}$ *into itself and is defined by*

$$Q(y,v)=(Y,V), \tag{31}$$

where, for every $0\le t\le\tau$,

$$Y(t)=-\Pi_0^t\,\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1}P\int_t^\tau s^{q-1}\,\Theta\left(\frac{\tau^q}{q}-\frac{s^q}{q}\right)f(s,y_s,v_s)\mathrm{d}s+\int_0^t s^{q-1}\,\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)f(s,y_s,v_s)\mathrm{d}s, \tag{32}$$

$$V(t)=B^*\,\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1}\left(h-P\,\Theta^*\left(\frac{\tau^q}{q}\right)\psi-P\int_t^\tau s^{q-1}\,\Theta\left(\frac{\tau^q}{q}-\frac{s^q}{q}\right)f(s,y_s,v_s)\mathrm{d}s\right). \tag{33}$$

The following estimate is true:

$$\|Q(y,v)-Q(x,w)\|\le L_q(\|y-x\|+\|v-w\|), \tag{34}$$

where

$$\begin{aligned}L_q&=\frac{\tau^q}{q}\left(1+\frac{\|\Pi_0^\tau|M}{\mu}+\frac{\|B\|M}{\mu}\right)\mathrm{MN},\\ M&=\sup_{0\le t\le\tau}\left\|\Theta\left(\frac{t^q}{q}\right)\right\|.\end{aligned} \tag{35}$$

Proof. Let $(y,v),(x,w)\in C(0,\tau;X)\times U_{ad}$ with $Q(y,v)=(Y,V)$ and $Q(x,w)=(X,W)$. Then,

$$\|Q(y,v)-Q(x,w)\|_{C(0,\tau;X)\times U_{ad}}=\|Y-X\|_{C(0,\tau;X)}+\|V-W\|_{U_{ad}}. \tag{36}$$

Starting with the first norm $\|Y-X\|_{C(0,\tau;X)}$, we obtain

$$\begin{aligned}\|Y-X\|&=\sup_{t\in[0,\tau]}\left\|\Pi_0^t\,\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1}P\int_0^\tau s^{q-1}\,\Theta\left(\frac{\tau^q}{q}-\frac{s^q}{q}\right)(f(s,y_s,v_s)-f(s,x_s,w_s))\mathrm{d}s\right.\\&\quad\left.+\int_0^t s^{q-1}\,\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)(f(s,y_s,v_s)-f(s,x_s,w_s))\mathrm{d}s\right\|\le\sup_{t\in[0,\tau]}\left(M+M^2\left\|\Pi_0^t\right\|.\left\|\left[\widetilde{\Pi}_0^\tau\right]^{-1}\right\|\right)\\&\quad\cdot\int_0^\tau s^{q-1}\left\|f(s,y_s,v_s)-f(s,x_s,w_s)\right\|\mathrm{d}s\le\left(1+\frac{\left\|\Pi_0^\tau\right\|M}{\mu}\right)\mathrm{MN}\int_0^\tau s^{q-1}\left(\left\|y_s-x_s\right\|+\left\|v_s-w_s\right\|\right)\\&\quad\cdot\mathrm{d}s\le\frac{\tau^q}{q}\left(1+\frac{\left\|\Pi_0^\tau\right\|M}{\mu}\right)\mathrm{MN}(\|y-x\|+\|v-w\|).\end{aligned} \tag{37}$$

In a similar way, for the second norm $\|V-W\|_{U_{ad}}$, one can obtain:

$$\begin{aligned}\|V-W\|&=\sup_{t\in[0,\tau]}\left\|B^*\,\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1}P\int_0^\tau s^{q-1}\,\Theta\left(\frac{\tau^q}{q}-\frac{s^q}{q}\right)(f(s,y_s,v_s)-f(s,x_s,w_s))\mathrm{d}s\right\|\\&\le\sup_{t\in[0,\tau]}\left(M^2\|B\|.\left\|\left[\widetilde{\Pi}_0^\tau\right]^{-1}\right\|\right)\int_0^\tau s^{q-1}\left\|f(s,y_s,v_s)-f(s,x_s,w_s)\right\|\mathrm{d}s\le\frac{1}{\mu}M^2\|B\|\\&\quad\cdot N\int_0^\tau s^{q-1}\left(\left\|y_s-x_s\right\|+\left\|v_s-w_s\right\|\right)\mathrm{d}s\le\frac{\tau^q}{q}\frac{\|B\|}{\gamma}M^2N(\|y-x\|+\|v-w\|).\end{aligned} \tag{38}$$

By combining the inequalities (25) and (32), the proof is done. □

Lemma 8. *Assume that (A1)–(A3) hold. If the inequality*

$$L_q = \frac{\tau^q}{q}\left(1 + \frac{\|\Pi_0^\tau\|M}{\mu} + \frac{\|B\|M}{\mu}\right)\mathrm{MN} < 1, \tag{39}$$

holds, then the non-linear operator Q takes $C(0,\tau;X)\times U_{ad}$ into itself and admits a single fixed point $(x,\mathrm{u}) \in C(0,\tau;X)\times U_{ad}$.

Proof. Due to Lemma 7 and the inequality (40), it is clear that Q is a contraction mapping on $C(0,\tau;X)\times U_{ad}$. Therefore, Thank to the well-known Banach fixed point theorem Q admits a single fixed point. □

Theorem 9. *Assume (A1)–(A3) and (39) are fulfilled. Then, the fractional control system (11) is partially exactly controllable.*

Proof. Let $h \in H$. We would prove that there is a control state $u \in U_{ad}$ such that $h = Px_\tau$. To do this, define u by

$$u_t = B^*\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}P\left(h - P\Theta^*\left(\frac{\tau^q}{q}\right)\psi - P\int_0^\tau s^{q-1}\Theta\left(\frac{\tau^q}{q}-\frac{s^q}{q}\right)f(s,x_s,u_s)\mathrm{d}s\right). \tag{40}$$

By inserting (40) into (22), one can obtain:

$$\begin{aligned}
x_t = {} & \Theta\left(\frac{\tau^q}{q}\right)\psi + \int_0^\tau s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)BB^*\Theta^*\left(\frac{t^q}{q}-\frac{s^q}{q}\right)\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right) \\
& \cdot P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}\left(h - P\Theta\left(\frac{\tau^q}{q}\right)\psi\right)\mathrm{d}s - \int_0^t s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)BB^*\Theta^*\left(\frac{t^q}{q}-\frac{s^q}{q}\right)\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}P \\
& \times\left(\int_0^\tau s^{q-1}\Theta\left(\frac{\tau^q}{q}-\frac{r^q}{q}\right)f(r,x_r,u_r)dr\right)\mathrm{d}s + \int_0^t s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)f(s,x_s,u_s)\mathrm{d}s \\
= {} & \Theta\left(\frac{t^q}{q}\right)\psi + \Pi_0^t\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P*\left[\tilde{\Pi}_0^\tau\right]^{-1}\left(h - P\Theta\left(\frac{\tau^q}{q}\right)\psi\right) + \int_0^t s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)f(s,x_s,u_s)\mathrm{d}s \\
& - \int_0^\tau\int_0^t s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)BB^*\Theta^*\left(\frac{t^q}{q}-\frac{s^q}{q}\right)\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}P\Theta\left(\frac{\tau^q}{q}-\frac{r^q}{q}\right)f(r,x_r,u_r)\mathrm{d}sr^{q-1}\mathrm{d}r \\
= {} & \Theta\left(\frac{t^q}{q}\right)\psi + \Pi_0^t\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}\left(h - P\Theta\left(\frac{\tau^q}{q}\right)\psi\right) + \int_0^t s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)f(s,x_s,u_s)\mathrm{d}s \\
& - \int_0^\tau r^{q-1}\Pi_0^t\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}P\Theta\left(\frac{\tau^q}{q}-\frac{r^q}{q}\right)f(r,x_r,u_r)\mathrm{d}r \\
= {} & \Theta\left(\frac{t^q}{q}\right)\psi + \Pi_0^t\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}\left(h - P\Theta\left(\frac{\tau^q}{q}\right)\psi\right) + \int_0^t s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)f(s,x_s,u_s)\mathrm{d}s \\
& - \Pi_0^t\Theta^*\left(\frac{\tau^q}{q}-\frac{t^q}{q}\right)P^*\left[\tilde{\Pi}_0^\tau\right]^{-1}P\int_0^\tau r^{q-1}\Theta\left(\frac{\tau^q}{q}-\frac{r^q}{q}\right)f(r,x_r,u_r)\mathrm{d}r.
\end{aligned} \tag{41}$$

Now, setting $t = \tau$ in (41), we acquire

$$
\begin{aligned}
Px_T &= P\left(\Theta\left(\frac{\tau^q}{q}\right)\psi + \Pi_0^\tau P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1}\left(h - P\Theta\left(\frac{\tau^q}{q}\right)\psi\right) + \int_0^\tau s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)f(s,x_s,u_s)\mathrm{d}s - \Pi_0^\tau P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1}\right. \\
&\quad \left.\cdot P\int_0^\tau r^{q-1}\Theta\left(\frac{\tau^q}{q}-\frac{r^q}{q}\right)f(r,x_r,u_r)\mathrm{d}r\right) \\
&= P\Theta\left(\frac{\tau^q}{q}\right)\psi + P\Pi_0^\tau P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1}\left(h - P\Theta\left(\frac{\tau^q}{q}\right)\psi\right) + P\int_0^\tau s^{q-1}\Theta\left(\frac{t^q}{q}-\frac{s^q}{q}\right)f(s,x_s,u_s)\mathrm{d}s - P\Pi_0^\tau P^*\left[\widetilde{\Pi}_0^\tau\right]^{-1} \\
&\quad \cdot P\int_0^\tau r^{q-1}\Theta\left(\frac{\tau^q}{q}-\frac{r^q}{q}\right)f(r,x_r,u_r)dr\right) \\
&= h.
\end{aligned}
\tag{42}
$$

Therefore, we observe that the control $u \in U_{ad}$ steers the control system (20) from ψ at initial time to x_τ at terminal time τ, such that the partial state $Px_\tau = h$ is accomplished. Therefore, the given control system (20) in conformable sense is partially exactly controllable on U_{ad} for the terminal time τ. □

4. Illustrative Example

Example 3. Given a control system of fractional equations in conformable sense

$$
\begin{cases}
C_0^q y_t = z_t + u_t + \dfrac{t^2}{40+3t^2}\cos\left(z_t + y_t + u_t\right), y_0 \in \mathbb{R}, \\
C_0^q z_t = \dfrac{1}{50+t^2}\sqrt{y_t^2 + u_t^2 + 7}, z_0 \in \mathbb{R},
\end{cases}
\tag{43}
$$

where $u \in C(0,\tau;\mathbb{R})$, $(y,z) \in \mathbb{R}\times\mathbb{R}$ and $t \in [0, 1]$. The concept of controllability is explained in $\mathbb{R}\times\mathbb{R}$ as

$$
\left\{(y,z) \in \mathbb{R}^2 : \exists u \in U_{ad} \text{ so that } \left(y_\tau, z_\tau\right) = (y,z)\right\} = \mathbb{R}^2,
\tag{44}
$$

While, the concept of partial controllability is understood in $\mathbb{R}$ as

$$
\left\{y \in \mathbb{R} : \exists u \in U_{ad} \text{ so that } y_\tau = y\right\} = \mathbb{R}.
\tag{45}
$$

The control system (43) can be interpreted in $\mathbb{R}^2$ as

$$
C_0^q x_t = Ax_t + Bu_t + G\left(t, x_t, u_t\right),
\tag{46}
$$

where

$$
\begin{aligned}
x_t &= \begin{bmatrix} y_t \\ z_t \end{bmatrix}, \\
A &= \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \\
B &= \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \\
G\left(t, x_t, u_t\right) &= \begin{bmatrix} \dfrac{t^2}{40+3t^2}\cos\left(y_t + z_t + u_t\right) \\ \dfrac{1}{50+t^2}\sqrt{y_t^2 + u_t^2 + 7} \end{bmatrix}.
\end{aligned}
\tag{47}
$$

Since the operator A is a matrix, the CF− semigroup $\Theta\,(t^q/q)$ can be simply calculated as follows:

$$
\begin{aligned}
\Theta\left(\frac{t^q}{q}\right) &= e^{A\left(t^q/q\right)} \\
&= \begin{bmatrix} 1 & \dfrac{t^q}{q} \\ 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} 0 & \dfrac{t^q}{q} \\ 0 & 0 \end{bmatrix}.
\end{aligned}
\tag{48}
$$

Hence, we can compute M as follows:

$$M = \sup_{0\le t\le 1}\left\| \Theta\left(\frac{t^q}{q}\right)\right\| = \left\| e^{At^q/q}\right\| = 1 + \frac{1}{q}. \tag{49}$$

By carrying out elementary calculations, the CF− controllability operator (24) of the system(46) can be computed as follows:

$$\begin{aligned}\Pi_0^1 &= \int_0^1 s^{q-1} e^{As^q/q} BB^* e^{A^* s^q/q}\mathrm{d}s \\ &= \int_0^1 s^{q-1}\begin{bmatrix} 1 & \dfrac{s^q}{q} \\ 0 & 1\end{bmatrix}\begin{bmatrix} 1 \\ 0\end{bmatrix}\begin{bmatrix} 1 & 0\end{bmatrix}\begin{bmatrix} 1 & 0 \\ \dfrac{s^q}{q} & 1\end{bmatrix}\mathrm{d}s \\ &= \int_0^1 s^{q-1}\begin{bmatrix} 1 & 0 \\ 0 & 1\end{bmatrix}\mathrm{d}s \\ &= \frac{1}{q}\begin{bmatrix} 1 & 0 \\ 0 & 0\end{bmatrix}.\end{aligned} \tag{50}$$

Clearly, Π_0^1 is not coercive. Therefore, the control system (43) is not exactly controllable. However, the system (43) can be exactly controllable under the appropriate conditions of the function G if we analyze the partial exact controllability considering only the first component y_t of the state vector x_t. This can be done by defining the projector operator P as $P = [1\ \ 0]$. Therefore,

$$\widetilde{\Pi}_0^1 = P\,\Pi_0^1 P^* = \frac{1}{q} > 0. \tag{51}$$

This shows that the corresponding linear part of the semilinear system (46) is partially exactly controllable.

We now compute the Lipschitz constant N for the function $G = (G_1, G_2)^T$ as follows: First, we find the Lipschitz constant for $G_1(t, y, z, u) = t^2/40 + 3t^2 \cos(y_t + z_t + u_t)$. To find the Lipschitz constants in y denoted by N_y^1, in z denoted by N_z^1 and in u denoted by N_u^1, we take the partial derivatives in terms of y, z and u respectively. Then calculate the supremum of the absolute values of the partial, we obtain $N_y^1 = 1/40 = N_z^1 = N_u^1$. We select $N_1 = \max\{N_y^1, N_z^1, N_u^1\} = 1/40$.

Similarly, for N_2 the Lipschitz constant of $G_2(t, y, z, u) = 1/50 + t^2\sqrt{y_t^2 + u_t^2 + 7}$, following the same procedure with elementary computation, we can obtain as follows:$N_y^2 = 1/50 = N_u^2$, $N_z^2 = 0$ and we take $N_2 = \max\{N_y^2, N_z^2, N_u^2\} = 1/50$.

For N, we have

$$N = \sqrt{(N_1)^2 + (N_2)^2} = \sqrt{0.000625 + 0.0004} = \sqrt{0.0001025} \le 0.0323. \tag{52}$$

Now, evaluating the expression L_q, defined previously by (39) we acquire

$$L_q = \left(3 + \frac{1}{q} + q\right)\left(1 + \frac{1}{q}\right)\frac{1}{q}N. \tag{53}$$

Let $q = 2/3$. Substitute this value into (53), we obtain as follows:

$$L_q = \left(3 + \frac{3}{2} + \frac{2}{3}\right)\left(1 + \frac{3}{2}\right)\frac{3}{2}N \le 0.6263 < 1, \tag{54}$$

which guaranties that the condition (39) holds.

Let $q = 5/7$. Substitute this value into (53), we obtain as follows:

$$L_q = \left(3 + \frac{7}{5} + \frac{5}{7}\right)\left(1 + \frac{7}{5}\right)\frac{7}{5}N \le 0.5552 < 1, \tag{55}$$

which guaranties that the condition (39) holds.

Let $q = 9/10$. Substitute this value into (53), we obtain as follows:

$$\begin{aligned}L_q &= \left(3 + \frac{10}{9} + \frac{10}{9}\right)\left(1 + \frac{10}{9}\right)\frac{10}{9}N \\ &\le (3 + 1.1112 + 0.9)(1 + 1.1112)(1.1112)(0.0323) \\ &\le 0.3798 < 1,\end{aligned} \tag{56}$$

which guaranties that the condition (39) holds.

Let $q = 1$. Substitute this value into (53), we obtain as follows:

$$L_q = (3 + 1 + 1)(1 + 1)N \le 0.323 < 1, \tag{57}$$

which guaranties that the condition (39) holds.

Hence, regarding the above four values of q, all the assumptions of Theorem 9 are fulfilled. Consequently, the CF control system described in (43) is partially exactly controllability over the interval $[0, 1]$. Given a control system of fractional equations in conformable sense

5. Discussion

As observed in the variation of the parameter q, it becomes clear that each choice of the fractional-order q leads to a distinct fractional control system that can be partially exactly controllable. This emphasizes the idea that fractional calculus introduces new opportunities and dimensions to control theory, while still preserving known findings from ordinary calculus when $q = 1$. Furthermore, the order q becomes a powerful modeling parameter that can be optimized and precisely calibrated to suit the control requirements. This underscores the potential of fractional calculus to open new unexplored boundaries in control theory.

6. Conclusion

In this paper, the notion of partial exact controllability for conformable fractional control systems is introduced and a sufficient condition set for them is obtained. These conditions are obtained for semilinear control system, given that its associated linear part is also partially exactly controllable. The method employed for this type of system closely resembles the one used for nonpartial systems, with a small adjustment. The effectiveness of this approach has been demonstrated through an illustrative example.

Future research studies will focus on the partial controllability of stochastic fractional control systems with infinite and finite delay, fractional systems with non-instantaneous impulses, and partially observable stochastic control systems with non-integer orders.

Data Availability

This study neither generated nor analyzed any new data.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] R. E. Kalman, "A new approach to linear filtering and prediction problems," *Journal of Basic Engineering D, Transactions of ASME*, vol. 82, pp. 35–45, 1960.

[2] H. O. Fattorini, "Some remarks on complete controllability," *SIAM Journal on Control*, vol. 4, pp. 686–694, 1966.

[3] A. E. Bashirov and M. Jneid, "On partial complete controllability of semilinear systems," *Abstract and Applied Analysis*, vol. 2013, Article ID 521052, 8 pages, 2013.

[4] A. E. Bashirov and M. Jneid, "Partial complete controllability of deterministic semilinear systems," *TWMS Journal of Applied and Engineering Mathematics*, vol. 4, pp. 216–225, 2014.

[5] C. Guevara and H. Leiva, "Controllability of the impulsive semilinear heat equation with memory and delay," *Journal of Dynamical and Control Systems*, vol. 24, pp. 1–11, 2016.

[6] H. Leiva, "Approximate controllability of semilinear impulsive evolution equations," *Abstract and Applied Analysis*, vol. 2015, Article ID 797439, 7 pages, 2015.

[7] C. Dineshkumar, R. Udhayakumar, V. Vijayakumar, and K. Sooppy Nisar, "Results on approximate controllability of neutral integro-differential stochastic system with state-dependent delay," *Numerical Methods for Partial Differential Equations*, vol. 40, no. 1, 2020.

[8] S. Sivasankar and R. Udhayakumar, "A note on approximate controllability of second-order neutral stochastic delay integro-differential evolution inclusions with impulses," *Mathematical Methods in the Applied Sciences*, vol. 45, no. 11, pp. 6650–6676, 2022.

[9] S. Kumar and S. M. Abdal, "Approximate controllability of nonautonomous second-order nonlocal measure driven systems with state-dependent delay," *International Journal of Control*, vol. 96, no. 4, pp. 1014–1025, 2023.

[10] S. Kumar and S. Yadav, "Approximate controllability of stochastic delay differential systems driven by Poisson jumps with instantaneous and noninstantaneous impulses," *Asian Journal of Control*, vol. 25, no. 5, pp. 4039–4057, 2023.

[11] A. Loverro, *Fractional calculus: History, Definitions and Applications for the Engineer*, ResearchGate, Berlin, Germany, 2004.

[12] A. Alsaedi, J. J. Nieto, and V. Venktesh, "Fractional electrical circuits," *Advances in Mechanical Engineering*, vol. 7, no. 12, Article ID 168781401561812, 2015.

[13] K. S. Miller and B. Ross, *An Introduction to the Fractional Calculus and Fractional Differential Equations*, John Wiley and Sons, New York, NY, USA, 1993.

[14] P. Arena, R. Caponetto, L. Fortuna, and D. Porto, "Nonlinear non integer order circuits and systems: an introduction," *Vol. 38 of World Scientific Series on Nonlinear Science Series A*, World Scientific Publishing, Singapore, 2000.

[15] J. F. Gómez-Aguilar, H. Yépez-Martínez, R. F. Escobar-Jiménez, C. M. Astorga-Zaragoza, and J. Reyes-Reyes, "Analytical and numerical solutions of electrical circuits described by fractional derivatives," *Applied Mathematical Modelling*, vol. 40, no. 21-22, pp. 9079–9094, 2016.

[16] H. Ertik, A. E. Calik, H. Sirin, M. Sen, and B. Der, "Investigations of electrical RC circuit within the framework of fractional calculus," *Revista Mexicana de Física*, vol. 61, pp. 58–63, 2015.

[17] A. Chaouk and M. Jneid, "Analytic solution for systems of two-dimensional time conformable fractional PDEs by using CFRDTM," *International Journal of Mathematics and Mathematical Sciences*, vol. 2019, Article ID 7869516, 7 pages, 2019.

[18] K. Abuasbeh, M. Awadalla, and M. Jneid, "Nonlinear Hadamard fractional boundary value problems with different orders," *Rocky Mountain Journal of Mathematics*, vol. 51, no. 1, pp. 17–29, 2021.

[19] M. Jneid and A. Chaouk, "The conformable reduced differential transform method for solving Newell-Whitehead-Segel Equation with non-integer order," *Journal of Analysis and Applications*, vol. 18, pp. 35–51, 2020.

[20] N. I. Mahmudov, "Approximate controllability of fractional Sobolev-type evolution equations in Banach spaces," *Abstract and Applied Analysis*, vol. 2013, Article ID 502839, 9 pages, 2013.

[21] R. Sakthivel, N. I. Mahmudov, and J. J. Nieto, "Controllability for a class of fractional-order neutral evolution control systems," *Applied Mathematics and Computation*, vol. 218, no. 20, pp. 10334–10340, 2012.

[22] M. Jneid, "Approximate controllability of semilinear integro-differential fractional control systems with nonlocal conditions," *Applied Mathematical Sciences*, vol. 11, no. 29, pp. 1441–1453, 2017.

[23] C. Dineshkumar, K. S. Nisar, R. Udhayakumar, and V. Vijayakumar, "New discussion about the approximate controllability of fractional stochastic differential inclusions with order 1<r<2," *Asian Journal of Control*, vol. 24, no. 5, pp. 2519–2533, 2022.

[24] S. Sivasankar, R. Udhayakumar, M. Hari Kishor, S. E. Alhazmi, and S. Al-Omari, "A New result concerning

nonlocal controllability of Hilfer fractional stochastic differential equations via almost sectorial operators," *Mathematics*, vol. 11, no. 1, p. 159, 2022.

[25] M. Jneid and M. Awadalla, "On the controllability of conformable fractional deterministic control systems in finite dimensional spaces," *International Journal of Mathematics and Mathematical Sciences*, vol. 2020, Article ID 9026973, 7 pages, 2020.

[26] M. Jneid, "Exact controllability of semilinear control systems involving conformable fractional derivatives," in *AIP Conference Proceedings*, vol 2159, AIP Publishing LLC, 2019.

[27] T. Ennouari, B. Abouzaid, and M. E. Achhab, "Controllability of infinite-dimensional conformable linear and semilinear systems," *International Journal of Dynamics and Control*, vol. 11, no. 3, pp. 1265–1275, 2023.

[28] M. Jneid, "Results on partial approximate controllability of fractional control systems in Hilbert spaces with conformable derivatives," *AIP Advances 14*, Article ID 025128, 2024.

[29] R. Khalil, M. Al Horani, A. Yousef, and M. Sababheh, "A new definition of fractional derivative," *Journal of Computational and Applied Mathematics*, vol. 264, pp. 65–70, 2014.

[30] T. Abdeljawad, "On conformable fractional calculus," *Journal of Computational and Applied Mathematics*, vol. 279, pp. 57–66, 2015.

[31] A. Jaiswal and D. Bahuguna, "Semilinear conformable fractional differential equations in banach spaces," *Differential Equations and Dynamical Systems*, vol. 27, p. 313, 2019.

[32] N. I. Mahmudov, "Controllability of semilinear stochastic systems in Hilbert spaces," *Jornal of Mathematical Analysis and Applications*, vol. 288, 2001.

[33] M. Jneid, "Patial complete controllability of semilinear control systems," Dissertation, Eastern Mediterranean University, İsmet İnönü Bulvarı, Gazimağusa, 2014.