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Research Article

Lie Symmetry Analysis for the Fractal Bond-Pricing Model of Mathematical Finance

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The classical bond-pricing models, as important financial tools, show strong vitality in bond pricing. However, these models also expose their theoretical defects, which leads to inconsistencies with the actual observation results and usually causes the theoretical prices of bonds to be lower than the actual market prices in the financial market. In order to change this situation, considering that the price change of the underlying is regarded as a fractal transmission system, the fractal derivative is introduced into the bond-pricing equation. In order to solve the fractal bond-pricing equation, we first convert it into an equivalent equation by using a fractal two-scale transform. Only in this case can we start to study it by means of the Lie symmetry analysis method. Then the geometric vector fields, the symmetry reductions, and the exact solution to the equations are obtained. Furthermore, the dynamic behaviors of the fractal bond-pricing equation are discussed. The results show that the fractal dimension bond-pricing formula can better explain price changes in the capital market than the classical one. That is to say, the classical bond-pricing equation is only a special case of the fractal-bond pricing equation, which makes up for the defect that the theoretical bond price given by the classical bond-pricing equation is often lower than the actual market price. The results of this paper provide a basis for bond pricing in the financial market in order to seek a more appropriate and real price.

1. Introduction

Sinkala et al. presented the group classification of the general bond-pricing equation as follows:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^{2\rho}\frac{\partial^2 V}{\partial S^2}+\left(\alpha+\mathrm{rS}-\lambda\sigma S^{\rho}\right)\frac{\partial V}{\partial S}-\mathrm{SV}=0, \tag{1}$$

where σ,ρ,α,r, and λ are real constants, t is time, S is the stock (share or equity) price or instantaneous short-term interest rate at current time t, and $V(S,t)$ is the current value of the option or bond (see [1–3] and references therein). The equation has an interesting characteristic: some corresponding classical financial mathematical models can be presented by taking different constants, such as the Vasicek model, the Longstaff model, and the Cox-Ingersoll-Ross model [4–6]. If $\rho=1,\alpha=\lambda=0$, then (1) reduces to the bond-pricing equation as follows:

$$\frac{\partial V}{\partial t}+\frac{1}{2}\sigma^2 S^2\frac{\partial^2 V}{\partial S^2}+\mathrm{rS}\frac{\partial V}{\partial S}-\mathrm{SV}=0, \tag{2}$$

which is rarely studied. Many classical bond-pricing models have been deeply studied and show a strong role in bond pricing. However, they still have some unsatisfactory aspects in theory, which leads to inconsistencies with the actual observed results, and usually causes the theoretical price of bonds being lower than the actual market price of the financial market. Then, how to improve it? Recently, based on the idea presented in [7] and the heuristic arguments established in [8, 9], fractional double barrier option models are investigated when the price change of the underlying is considered as a fractal transmission system [10, 11]. In view of this, we introduce fractal derivatives into equation (2) to study the fractal bond-pricing equation:

$$\begin{cases} \dfrac{\partial V}{\partial t^{\beta}} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - SV = 0, \\ V(S,T) = 1, \end{cases} \tag{3}$$

where σ and r are real constants, T is the terminal time, and $\partial V/\partial t^{\beta}$ is the fractal derivative defined as follows [12–14]:

$$\frac{\partial V}{\partial t^{\beta}} = \Gamma(1+\beta) \lim_{\substack{t-t_0=\Delta t \\ \Delta t \neq 0}} \frac{V(S,t) - V(S,t_0)}{(t-t_0)^{\beta}}, \tag{4}$$

where β is the fractal dimension, Δt is the smallest scales in time, and $\Delta t \neq 0$.

It is known that the Lie symmetry analysis is a systematic and powerful method for dealing with symmetries and exact solutions to nonlinear evolution equations (NLEEs) [15–22]. In recent years, the Lie symmetry method has been extended to solve $(2+1)$-dimensional and $(3+1)$-dimensional NLEEs [23, 24]. In the following, we start to study the fractal bond-pricing equation (3) by means of the lie symmetry analysis method. The results show that the fractal dimension bond-pricing formula can better explain price changes in the capital market than the classical bond-pricing formula, which makes up for the defect that the theoretical bond price given by the classical bond-pricing equation is often lower than the actual market price.

The paper is organized as follows. In Section 2, we first convert the fractal bond-pricing equation into an equivalent one by using a fractal two-scale transform and then obtain all of the geometric vector fields of the equations on the basis of the arbitrary parameters by using the Lie symmetry analysis method. Section 3 presents symmetry reductions and generalized power series solution to the fractal bond-pricing equation and the proof of convergence of the generalized power series solutions. In Section 4, we discuss the dynamic behaviors of the fractal bond-pricing equation under the influence of different parameters. The final Section 5 gives the conclusion.

2. Lie Symmetry Analysis for the Fractal Bond-Pricing Equation (3)

First, in order to facilitate the solution and consider the financial significance of equation (3), according to the form of the fractal two-scale transform $T = t^{\alpha}$ given in references [12–14], we design an appropriate fractal two-scale transform:

$$\hat{t} = -(T-t)^{\beta}, \tag{5}$$

which is adopted to convert the fractal bond-pricing equation (3) to continuous one

$$\begin{cases} \dfrac{\partial V}{\partial \hat{t}} + \dfrac{1}{2}\sigma^2 S^2 \dfrac{\partial^2 V}{\partial S^2} + rS\dfrac{\partial V}{\partial S} - SV = 0, \\ V(S,T) = 1, \end{cases} \tag{6}$$

where σ and r are real constants. Then, we start to study it by means of the Lie symmetry analysis method. First, we will provide a list of all Lie symmetry algebras for the equation (6). Consider the geometric vector fields of equation (6) are as follows:

$$W = \zeta(S,\hat{t},V)\partial_S + \eta(S,\hat{t},V)\partial_{\hat{t}} + \psi(S,\hat{t},V)\partial_V, \tag{7}$$

where $\zeta(S,\hat{t},V), \eta(S,\hat{t},V)$, and $\psi(S,\hat{t},V)$ are the coefficient functions to be determined. The symmetry group of equation (6) will be derived from the vector field (7). Using the second prolongation $\mathrm{pr}^{(2)}W$ of W in equation (6) leads to the coefficient functions ζ, η, and ψ must meet the following condition:

$$\mathrm{pr}^{(2)}W(\triangle)\,|_{\triangle=0} = 0, \tag{8}$$

where $\triangle = \partial V/\partial\hat{t} + 1/2\sigma^2 S^2 \partial^2 V/\partial S^2 + rS\partial V/\partial S - SV$. Then, the Lie symmetry group calculation method gives rise to the following condition on the coefficient functions ζ, η and ψ:

$$\begin{aligned} &\zeta = \frac{1}{2}S\eta_{\hat{t}}\log S + S\varrho, \psi = \mu(S,\hat{t})V + \nu(S,\hat{t}), \\ &\mu = \frac{1}{4\sigma^2}\eta_{\hat{t}\hat{t}}\log^2 S + \left(\frac{1}{4} - \frac{r}{2\sigma^2}\right)\eta_{\hat{t}}\log S + \frac{1}{\sigma^2}\varrho_{\hat{t}}\log S + \kappa, \\ &\frac{1}{4\sigma^2}\eta_{\hat{t}\hat{t}\hat{t}}\log^2 S + \frac{1}{\sigma^2}\varrho_{\hat{t}\hat{t}}\log S + \frac{1}{4}\eta_{\hat{t}\hat{t}} - \left(\frac{1}{8}\sigma^2 + \frac{1}{2}r + \frac{r^2}{2\sigma^2}\right)\eta_{\hat{t}} - \left(\frac{1}{2} - \frac{r}{\sigma^2}\right)\varrho_{\hat{t}} + \kappa_{\hat{t}} = 0, \end{aligned} \tag{9}$$

where η, ϱ, and κ are the coefficient functions to be determined. By solving the equations, we get the vector field of equation (6) as follows:

$$W_1 = \partial_{\hat{t}}, W_2 = V\partial_V, W_{\nu} = \nu\partial_{\nu}, \tag{10}$$

where the function $\nu = \nu(S,\hat{t})$ satisfies equation (6). It is easy to find that $\{W_1, W_2, W_{\nu}\}$ is a basis of Lie algebra of equation (6). Furthermore, the one-parameter groups G_i generated by $W_i\,(i = 1, 2, \nu)$ are proposed as follows:

$$\begin{aligned} &G_1\colon\ (S,\hat{t},V) \longrightarrow (S,\hat{t}+\epsilon,V), \\ &G_2\colon\ (S,\hat{t},V) \longrightarrow (S,\hat{t},e^{\epsilon}V), \\ &G_{\nu}\colon\ (S,\hat{t},V) \longrightarrow (S,\hat{t},V+\epsilon\nu), \end{aligned} \tag{11}$$

where $\delta = e^{\sigma^2 \epsilon}$ and $\nu = \nu(S, \hat{t})$ is an arbitrary solution to equation (6).

3. Symmetry Reductions and Generalized Power Series Solution to the Fractal Bond-Pricing Equation (3)

In Section 2, we presented the symmetry and symmetry groups of equation (6). Now, let us consider similarity reduction of (6), but we find, in fact, for this equation, we have only one nontrivial case: $W_1 = W_1 + \gamma W_2\,(\gamma \neq 0)$ of (6), then the corresponding similarity transformation is given in the following equation:

$$\zeta = S, \theta = \log V - \gamma\hat{t}, \tag{12}$$

and the similarity solution is $\theta = f(\zeta)$, that is,

$$V = \exp[f(S) + \gamma\hat{t}]. \tag{13}$$

Substituting (13) into (6), one has the following ODE:

$$\frac{1}{2}\sigma^2\zeta^2 f'' + \frac{1}{2}\sigma^2\zeta^2 f'^2 + r\zeta f' + \zeta + \gamma = 0, \tag{14}$$

where $f' = \mathrm{d}f/\mathrm{d}\zeta$. This is a nonlinear second-order ODE; we will deal with it by the special transformation technique and generalized power series method.

First, let $f' = y$, then equation (14) becomes the following Riccati type of equation:

$$\frac{1}{2}\sigma^2\zeta^2 y' + \frac{1}{2}\sigma^2\zeta^2 y^2 + r\zeta y + \zeta + \gamma = 0. \tag{15}$$

We construct a solution of (15) in a generalized power series of the following form:

$$y = a\zeta^{-1} + \sum_{n=0}^{\infty} c_n \zeta^n, \tag{16}$$

where the parameters a and $c_n\,(n = 0, 1, 2, \ldots)$ are constants to be determined.

Inserting (16) into (15), we have

$$\begin{aligned}
&-\frac{1}{2}\sigma^2 a + \frac{1}{2}\sigma^2 c_1\zeta^2 + \frac{1}{2}\sigma^2 \sum_{n=1}^{\infty}(n+1)c_{n+1}\zeta^{n+2} + \frac{1}{2}\sigma^2 a^2 + \frac{1}{2}\sigma^2 c_0^2\zeta^2 \\
&+ \sigma^2 a c_0\zeta + \sigma^2 a c_1\zeta^2 + \frac{1}{2}\sigma^2 \sum_{n=1}^{\infty}\left(\sum_{k=0}^{n} c_k c_{n-k}\right)\zeta^{n+2} + \sigma^2 a \sum_{n=1}^{\infty} c_{n+1}\zeta^{n+2} \\
&+ ra + rc_0\zeta + rc_1\zeta^2 + r\sum_{n=1}^{\infty} c_{n+1}\zeta^{n+2} + \zeta + \gamma = 0.
\end{aligned} \tag{17}$$

Comparing coefficients of (17) gives

$$\frac{1}{2}\sigma^2 a^2 - \left(\frac{1}{2}\sigma^2 a^2 - r\right)a + \gamma = 0, \tag{18}$$

$$\sigma^2 \mathrm{a} c_0 + \gamma = 0, \tag{19}$$

and

$$\left(\frac{1}{2}\sigma^2 + \sigma^2 a + r\right)c_1 + \frac{1}{2}\sigma^2 c_0^2 = 0. \tag{20}$$

Generally, for all $n \geq 1, n \in N$, the following recurrence formula can be obtained by (17):

$$c_{n+1} = -\frac{\sigma^2}{(n+1)\sigma^2 + 2\sigma^2 a + 2r}\sum_{k=0}^{n} c_k c_{n-k}. \tag{21}$$

Then, by solving equation (18), we have

$$a = \frac{1}{2} - \frac{r}{\sigma^2} \pm \frac{1}{2\sigma^2}\sqrt{\left(\sigma^2 - 2r\right)^2 - 8\sigma^2\gamma}. \tag{22}$$

From (19) and (20), one has

$$\begin{aligned}
c_0 &= \frac{-1}{a\sigma^2 + r}, \\
c_1 &= \frac{-\sigma^2 c_0^2}{\sigma^2 + 2a\sigma^2 + 2r}.
\end{aligned} \tag{23}$$

Furthermore, (21) gives rise to

$$\begin{aligned}
c_2 &= \frac{-\sigma^2 c_0 c_1}{\sigma^2 + a\sigma^2 + r}, \\
c_3 &= \frac{-\sigma^2\left(2c_0c_2 + c_1^2\right)}{3\sigma^2 + 2a\sigma^2 + 2r}\ldots.
\end{aligned} \tag{24}$$

Therefore, all the remaining items of the sequence $\{c_n\}_{n=2}^{\infty}$ can be uniquely obtained sequentially by recurrence formula (21). That is to say, for equation (15), there is a generalized power series solution (16), and its coefficients are uniquely determined by the expressions (21)–(24). Furthermore, we can show the convergence of the generalized power series solution (16) [25, 26].

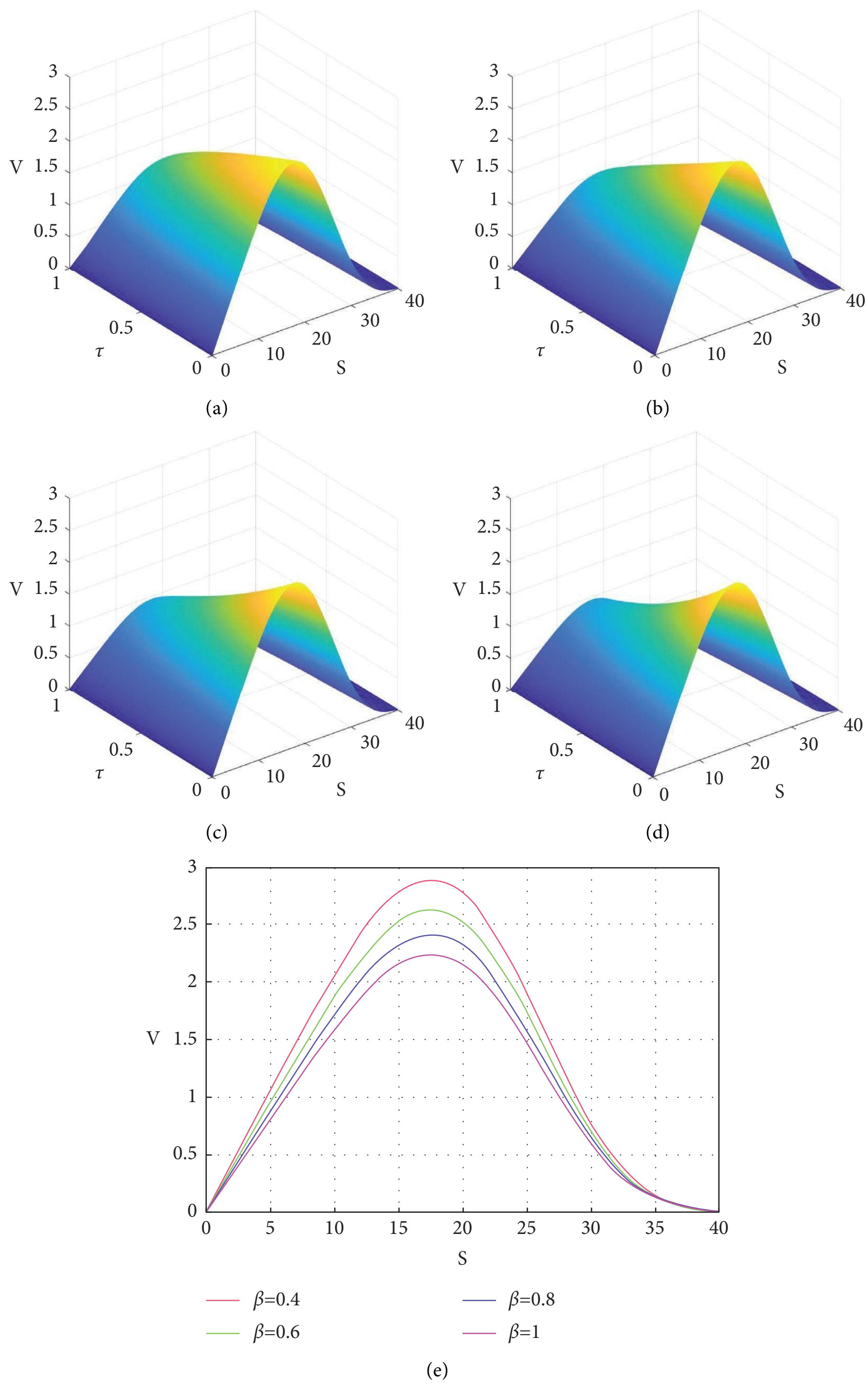

Figure 1: Dynamic behaviors of solution (33) with different fractal dimensions of β for $c_0 = c_1 = c_2 = c_3 = -0.00001, a = 1, c = 0.07, \gamma = -1$ and $T = 1$. (a) $\beta = 0.4$. (b) $\beta = 0.6$. (c) $\beta = 0.8$. (d) $\beta = 1$. (e) $\tau = 0.5$.

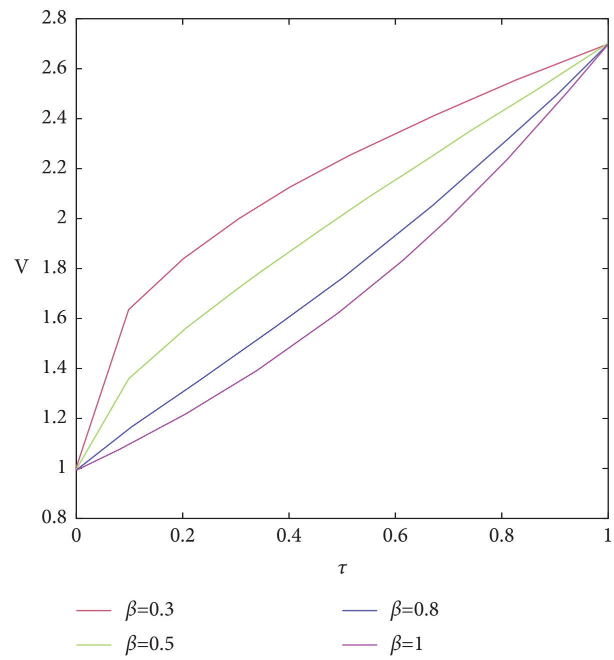

Figure 2: The influence of the fractal dimension β on solution (32) for $S = 20, \sigma = 0.8, r = 0.70026, \gamma = -1, \bar{c} = 153$, and $T = 1$.

In fact, from (21), we have

$$|c_{n+1}| \le M \sum_{k=0}^{n} |c_k||c_{n-k}|, \quad n = 1, 2, 3, \cdots, \tag{25}$$

where $M = |\, \sigma^2/2\sigma^2 + 2\sigma^2 a + 2r \,|$ $(a \ge -1/2, r \ge 0)$.

Now, we define a power series

$$\varsigma = P(\zeta) = \sum_{n=0}^{\infty} p_n \zeta^n, \tag{26}$$

along with

$$\begin{aligned} p_0 &= |c_0|, \\ p_n &= M \sum_{k=0}^{n} p_k p_{n-k}, \quad n = 1, 2, 3, \cdots. \end{aligned} \tag{27}$$

Then, it is easy to find that

$$|c_n| \le |p_n|, \quad n = 1, 2, 3, \cdots \tag{28}$$

That is to say, the series (26) is a majorant series of (16). Next, we show that series (26) has a positive radius of convergence. Indeed, by formal calculation, we have

$$\begin{aligned} P(\zeta) &= p_0 + p_1\zeta + \sum_{n=2}^{\infty} p_n \zeta^n = p_0 + p_1\zeta \\ &\quad + M\left[P^2(\zeta) - 2p_0 P(\zeta) + p_0^2\right]. \end{aligned} \tag{29}$$

Consider now the following implicit functional equation:

$$F(\zeta, \varsigma) = \varsigma - p_0 - p_1\zeta - M(\varsigma - p_0)^2 = 0. \tag{30}$$

Because $F(\zeta, \varsigma)$ is analytic on the plane $\{\zeta, \varsigma\}$, $F(0, p_0) = 0$, $F'_\varsigma(0, p_0) = 1 \ne 0$, in terms of the implicit function theorem, we see that $\varsigma = P(\zeta)$ is analytic in a neighborhood of the point $(0, p_0)$ of the plane and with a positive radius. This implies that the power series (16) converges in a neighborhood of the point $(0, p_0)$. This completes the proof.

Thus, the generalized power series solution (16) is the exact analytic solution, and the exact generalized power series solution of (14) can be presented as follows:

$$f(\zeta) = \bar{c} + a\log|\zeta| + c_0\zeta + \frac{1}{2}c_1\zeta^2 + \sum_{n=1}^{\infty} \frac{1}{n+2} c_{n+1}\zeta^{n+2}. \tag{31}$$

Substituting (31) into (13), we have the exact analytic solution to (3) as follows:

$$V(S, t) = cS^a \exp\left[c_0 S + \frac{1}{2}c_1 S^2 + \sum_{n=1}^{\infty} \frac{1}{n+2} c_{n+1} S^{n+2} - \gamma (T - t)^\beta\right], \tag{32}$$

where $c = e^{\bar{c}}$ is an arbitrary constant and a and $c_n\,(n = 0, 1, 2, \ldots)$ are shown by (21)–(24), respectively. The free parameters in the general solution are then chosen suitably so that the solution satisfies the auxiliary condition $V(S, T) = 1$.

4. Discussion on Dynamic Characteristics of the Fractal Bond-Pricing Equation (3)

It is necessary to illustrate the characteristics of the exact analytic solution to (3). First, for its practical application, we consider the following approximate solution formula derived from (32):

$$V(S, t) = cS^a \exp\left[c_0 S + \frac{1}{2}c_1 S^2 + \frac{1}{3}c_2 S^3 + \frac{1}{4}c_3 S^4 - \gamma (T - t)^\beta\right]. \tag{33}$$

In the following, we focus on analyzing the dynamic characteristics in terms of (33). In Figure 1, some parameters are chosen as $c_0 = c_1 = c_2 = c_3 = -0.00001, a = 1, c = 0.07, \gamma = -1$ and $T = 1$. Figures 1(a)–1(d) depict the dynamic behaviors of the solution (33) for the fractal bond-pricing equation (3) with different fractal dimensions $\beta = 0.4, 0.6, 0.8$ and 1.0, respectively. It can be seen from Figure 1(e) that within a certain range of stock price S, the lower the fractal dimension β, the higher the price V given by the corresponding bond equation, which indicates that the classical bond-pricing equation often underestimates the bond price. The fractal dimension bond-pricing formula can better explain the price changes in the capital market than the classical bond-pricing formula.

Figure 2 presents the influence of the fractal dimension β on the bond price for $S = 20, \sigma = 0.8, r = 0.70026, \gamma = -1, \bar{c} = 153$, and $T = 1$. It can be clearly observed from the development trend that with the increase of the β value of the time fractal derivative, the corresponding curve gradually approaches that of the integer dimension $\beta = 1$. The dynamic characteristics of the fractal dimension exact solution tend to be consistent with those of integer dimension exact solution as the expiration time approaches. That is to say, the classical bond-pricing equation (2) is a special case of the fractal bond-pricing equation (3), which shows the rationality of this paper. On the other hand, as the expiration time approaches, the smaller β, the higher the corresponding bond price, which makes up for the defect that the theoretical bond price given by the classical bond-pricing equation(2) is often lower than the actual market price.

It is clear that, as a generalization of the classical bond-pricing model, the fractal-bond pricing equation provides a basis for bond pricing in the financial market to seek a more appropriate and real price and will be of great interest to researchers in further work.

5. Conclusion

The main task of this work is to study and analyze the exact solution and dynamic behaviors of the fractal bond-pricing equation. In recent years, the classical bond-pricing models, as important financial tools, have been deeply studied and shown strong vitality in bond pricing. However, they still have theoretical defects, which are inconsistent with the actual observation results and usually cause the theoretical price of bonds to be lower than the actual market price in the financial market. In order to change this situation, considering that the price change of the underlying is regarded as a fractal transmission system, the fractal derivative is introduced into the bond-pricing equation to try to achieve the ideal expectation of market justice. As we all know, the fractal bond-pricing equation is a fractal partial differential equation with variable coefficients. How to obtain its exact solution is still a challenging problem. In order to overcome it, we first convert it into an equivalent equation by using a fractal two-scale transform. Only in this case can we start to study it by using the Lie symmetry analysis method. Then the geometric vector fields, the symmetry reductions and generalized power series solution to the fractal bond-pricing equation and the proof of convergence of the generalized power series solutions are obtained. Furthermore, the dynamic behaviors of the fractal bond-pricing equation are discussed. The results show that the fractal dimension bond-pricing formula can better explain the price changes in the capital market than the classical bond-pricing formula. That is to say, the classical bond-pricing equation is only a special case of the fractal-bond pricing equation, which makes up for the defect that the theoretical bond price given by the classical bond-pricing equation is often lower than the actual market price. The results of this paper provide a basis for bond pricing in the financial market in order to seek a more appropriate and real price. In the next research, we will further explore the application of this method in the accurate estimation of hedging ratios and risk management, so as to better hedge the price fluctuation and improve the efficiency of risk management.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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