Hindawi
Journal of Mathematics
Volume 2024, Article ID 9932668, 19 pages
https://doi.org/10.1155/2024/9932668

Research Article

Some Conditions and Perturbation Theorem of Irregular Wavelet/Gabor Frames in Sobolev Space

Hui-Min Liu[1] and Yu Tian[2]

[1]*College of Basic Science, Zhengzhou University of Technology, Zhengzhou, Henan 450044, China*
[2]*Department of Mathematics and Information Science, Zhengzhou University of Light Industry, Zhengzhou, Henan 450002, China*

Correspondence should be addressed to Yu Tian; ytianzzuli@163.com

Received 12 October 2023; Revised 19 April 2024; Accepted 20 April 2024; Published 10 May 2024

Academic Editor: Ding-Xuan Zhou

Due to its potential applications in image restoration and deep convolutional neural networks, the study of irregular frames has interested some researchers. This paper addresses irregular wavelet systems (IWSs) and irregular Gabor systems (IGSs) in Sobolev space $H^s(\mathbb{R})$. We obtain the sufficient and necessary conditions for IWS and IGS to be frames. By applying these conditions, we also derive the characterizations of IWS and IGS to be frames. Finally, we discuss the perturbation theorem of irregular wavelet frames (IWFs) and irregular Gabor frames (IGFs). We also provided some examples to support our results.

1. Introduction

An at most countable sequence $\{e_i\}_{i\in\mathcal{J}}$ in a separable Hilbert space $\mathcal{H}$ is called a *Bessel sequence* in $\mathcal{H}$ if there exists $C>0$ such that

$$\sum_{i\in\mathcal{J}} |\langle f, e_i\rangle|^2 \le C\|f\|^2 \text{ for } f\in\mathcal{H}, \quad (1)$$

where C is called a Bessel bound; it is called a frame for $\mathcal{H}$ if there exists $0<C_1\le C_2<\infty$ such that

$$C_1\|f\|^2 \le \sum_{i\in\mathcal{J}} |\langle f, e_i\rangle|^2 \le C_2\|f\|^2 \text{ for } f\in\mathcal{H}, \quad (2)$$

where C_1 and C_2 are called frame bounds. The concept of frame was first proposed by Duffin and Schaeffer when studying the nonharmonic Fourier series in [1]. However, it did not attract people's attention at that time. Until 1986, Daubechies et al. in [2] noticed that frames can represent the functions in $L^2(\mathbb{R})$ in terms of series expansion. This expansion is very similar to the orthonormal basis expansion, but is more flexible than the orthonormal basis. Many scholars are beginning to realize the potential application of frame theory and frame theory is rapidly developing. So far, the frame theory is widely used in signal and image processing, biomedicine, applied mathematics, physical science, earth science, DCNNs, and many other fields. More details can be found in [2–18] and references therein.

Now the research on frame theory mainly focuses on regular wavelet frame (RWF) and regular Gabor frame (RGF) in $L^2(\mathbb{R})$ and Sobolev space $H^s(\mathbb{R})$. We recall that for $a>1, b>0$ and $\psi, g\in L^2(\mathbb{R})$, two sequences $\{a^{j/2}\psi(a^jx-kb)\colon j,k\in\mathbb{Z}\}$ and $\{e^{2\pi imbx}g(x-na)\colon j,k\in\mathbb{Z}\}$ are called RWS and RGS, if they form frames for $L^2(\mathbb{R})$, and we say that they are RWF and RGF for $L^2(\mathbb{R})$, respectively. For $s\in\mathbb{R}$, we denote by $H^s(\mathbb{R})$ the Sobolev space consisting of all tempered distributions f such that

$$\|f\|_s^2 = \int_{\mathbb{R}}\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2 d\eta < \infty. \quad (3)$$

It is easy to check that $H^s(\mathbb{R})$ is a Hilbert space under the inner product

$$\langle f, g\rangle_s = \int_{\mathbb{R}}\left(1+\eta^2\right)^s \widehat{f}(\eta)\overline{\widehat{g}(\eta)}d\eta, \quad f,g\in H^s(\mathbb{R}). \quad (4)$$

In particular, $H^0(\mathbb{R}) = L^2(\mathbb{R})$ by Plancherel theorem. Let $S(\mathbb{R})$ denote the Schwartz space and by [19], $S(\mathbb{R})$ satisfies the following property: a function $f \in S(\mathbb{R})$ if and only if $\widehat{f} \in S(\mathbb{R})$.

We first have an overview of RWF and RGF.

(i) RWF and RGF for $L^2(\mathbb{R})$ and its subspaces.

A core problem of wavelet/Gabor frame theory is what conditions we need to impose on the generator to make the wavelet/Gabor systems to be frames and dual frames. For relevant results about this, including the sufficient and necessary conditions for wavelet/Gabor systems to be frames and the characterizations of dual wavelet/Gabor frames, one can refer to [4, 20–23]. Li and Tian in [24] proposed the concept of partial Gabor systems (PGSs) and studied the conditions for PGS from Gabor frames. They also characterized the dual partial Gabor frames. For the latest research on wavelet/Gabor frame, see [25–28].

(ii) RWF for Sobolev spaces.

Ehler in [29] presented a method of constructing a pair of dual wavelet frames from any pair of multivariate refinable functions in a pair of Sobolev spaces. Han and Shen in [30] extended the mixed extension principle in $L^2(\mathbb{R}^d)$ to Sobolev Spaces. They in [31] also gave the characterization of the Sobolev spaces by using nonstationary tight wavelet frames for $L^2(\mathbb{R})$. Li and Zhang in [32] characterized the nonhomogeneous dual wavelet frames in Sobolev space and derived the mixed oblique extension principle. Li and Jia in [33] investigated the properties of weak nonhomogeneous wavelet biframes (WNWBF) in the reducing subspaces of a pair of dual Sobolev spaces and constructed the WNWBF. All the compactly supported m th-order derivative-orthogonal Riesz wavelets in Sobolev space are completely depicted by Han and Michelle in [34]. For other studies on frames in Sobolev spaces, one can refer to [35–37].

(iii) The perturbation of RWF and RGF.

Zhang in [38] presented the conditions for a suitable perturbation of a wavelet/Gabor frame which is still a wavelet/Gabor frame. Christensen in [39] studied the stability frames and applied them to the perturbations of a Gabor frame. Sun and Zhou in [40] also obtained some results about the stability of Gabor frames. Bownik and Christensen in [41] characterized the Gabor frames with rational parameters, and as an application, they obtained results concerning the stability of Gabor frames under perturbation of the generators.

In practice, the sampling points may be irregular and it is desirable to have wavelet and Gabor systems in some Sobolev space. This inspires us to study irregular wavelet and Gabor systems in Sobolev space. Given $L \in \mathbb{N}$ and $\{\lambda_p\colon p = 1, 2, \cdots\} \subset \mathbb{R}^+$, we assume that $\Psi = \{\psi_l\colon 1 \le l \le L\}$ and $G = \{g_l\colon 1 \le l \le L\}$ are two the subsets in $H^s(\mathbb{R})$. We define the IWS and IGS generated by Ψ and G as

$$X(\Psi) = \left\{\lambda_p^{(1/2-s)}\psi_l\left(\lambda_p x - kb\right)\colon\ 1 \le l \le L, p = 1, 2, \ldots, k \in \mathbb{Z}\right\}, \tag{5}$$

$$\mathscr{G}(G) = \left\{e^{2\pi i\lambda_p x} g_l(x - ak)\colon\ 1 \le l \le L, p = 1, 2, \ldots, k \in \mathbb{Z}\right\}. \tag{6}$$

So, we have

$$\mathscr{D} = \left\{f\colon\ f \in S(\mathbb{R}) \text{ and } \widehat{f} \text{ is compactly supported in } \mathbb{R}\backslash\{0\}\right\}. \tag{7}$$

Then, $\mathscr{D}$ is a dense subset of $H^s(\mathbb{R})$.

For the research of the IWS and IGS in $L^2(\mathbb{R})$, Sun and Zhou in [42] constructed the IWF and IGF and gave the sufficient conditions for an IWS and IGS to be a frame. They in [43] also studied the density of IWF. Christensen in [44] gave the different sufficient conditions for IWS and IGS to be frames. For other relevant results, see [45, 46] and the references therein.

Motivated by the existing results mentioned above, we naturally raise a few questions: Are there similar necessary and sufficient conditions for IWS and IGS to be IWF and IGF in Sobolev spaces? How to provide the perturbation characterizations of IWF and IGF in Sobolev spaces? Is it possible to construct some examples to support the relevant results? In this paper, we will address these issues. It is nontrivial due to the more flexibility of the dilation and modulation factor λ_p and the complexity of Sobolev spaces.

1.1. Plan of Work. This paper addresses the IWF and IGF of the form (5) and (6) in $H^s(\mathbb{R})$ and the rest of this paper is organized as follows. Section 2 is devoted to some lemmas for later use. In Section 3 and Section 4, we focus on the sufficient and necessary conditions of IWS and IGS to be frames. The characterization of IWS and IGS to be frames under certain restrictions is also obtained. In section 5, we present the perturbation theorem of IWF and IGF. Relevant examples are also presented. Finally, conclusions are drawn in Section 6.

2. Some Auxiliary Lemmas

We give some auxiliary lemmas in this section. The next lemma can be found in [4].

Lemma 1. *We assume that* $\{f_k\}_{k=1}^{\infty}$ *and* $\{g_k\}_{k=1}^{\infty} \subset \mathscr{H}$. *Then, let* $\{f_k\}_{k=1}^{\infty}$ *be a frame with bounds* A *and* B. *If there is a constant* $R < A$ *such that*

$$\sum_{k=1}^{\infty} \left|\left\langle f, f_k - g_k \right\rangle\right|^2 \le R\|f\|^2, \quad \forall f \in \mathscr{H}, \tag{8}$$

then $\{g_k\}_{k=1}^{\infty}$ *is a frame for* $\mathscr{H}$ *and*

$$A\left(1 - \sqrt{\frac{R}{A}}\right)^2, B\left(1 + \sqrt{\frac{R}{B}}\right)^2, \tag{9}$$

are the frame bounds.

Lemma 2. *Let* $s \in \mathbb{R}$, $b > 0$, $\{\lambda_p\colon p = 1, 2, \cdots\} \subset \mathbb{R}^+$, *and* $\psi \in H^s(\mathbb{R})$. *Then, we have*

$$\begin{aligned} I &= \sum_{p=1}^{\infty} \sum_{k\in\mathbb{Z}} \left|\left\langle f(\cdot), \lambda_p^{(1/2-s)} \psi\left(\lambda_p \cdot - kb\right)\right\rangle_s\right|^2 \\ &= \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \int_{\mathbb{R}} \left(1 + \eta^2\right)^{2s} \left|\hat{f}(\eta)\right|^2 \left|\hat{\psi}\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta \\ &\quad + \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \int_{\mathbb{R}} \sum_{0\ne k\in\mathbb{Z}} \left(1 + \eta^2\right)^s \left(1 + \left(\eta + \frac{\lambda_p}{b}k\right)^2\right)^s \overline{\hat{f}(\eta)} \hat{f}\left(\eta + \frac{\lambda_p}{b}k\right) \hat{\psi}\left(\frac{\eta}{\lambda_p}\right) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p} + \frac{k}{b}\right)} d\eta, \end{aligned} \tag{10}$$

for $f \in \mathscr{D}$.

Proof. For arbitrary $f \in \mathscr{D}$, we have

$$\begin{aligned} I &= \sum_{p=1}^{\infty} \sum_{k\in\mathbb{Z}} \left|\left\langle f(\cdot), \lambda_p^{(1/2-s)} \psi\left(\lambda_p \cdot - kb\right)\right\rangle_s\right|^2 \\ &= \sum_{p=1}^{\infty} \lambda_p^{-2s} \sum_{k\in\mathbb{Z}} \left|\int_{\mathbb{R}} \left(1 + \eta^2\right)^s \hat{f}(\eta) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p}\right)} \frac{1}{\sqrt{\lambda_p}} e^{2\pi i \left(kb/\lambda_p\right)\eta} d\eta\right|^2. \end{aligned} \tag{11}$$

By the periodic process, we have

$$I = \sum_{p=1}^{\infty} \lambda_p^{-2s} \sum_{k\in\mathbb{Z}} \left|\sum_{m\in\mathbb{Z}} \int_{\lambda_p/b m}^{\lambda_p/b(m+1)} \left(1 + \eta^2\right)^s \hat{f}(\eta) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p}\right)} \frac{1}{\sqrt{\lambda_p}} e^{2\pi i \left(kb/\lambda_p\right)\eta} d\eta\right|^2. \tag{12}$$

Using variable substitution of $\eta \longrightarrow \eta - (\lambda_p/b)m$, we get

$$I = \sum_{p=1}^{\infty} \lambda_p^{-2s} \sum_{k\in\mathbb{Z}} \left| \sum_{m\in\mathbb{Z}} \int_0^{\lambda_p/b} \left(1 + \left(\eta + \frac{\lambda_p}{b} m\right)^2\right)^s \hat{f}\left(\eta + \frac{\lambda_p}{b} m\right) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p} + \frac{m}{b}\right)} \frac{1}{\sqrt{\lambda_p}} e^{2\pi i (kb/\lambda_p)\eta} d\eta \right|^2$$
$$= \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \sum_{k\in\mathbb{Z}} \left| \int_0^{\lambda_p/b} \sum_{m\in\mathbb{Z}} \left(1 + \left(\eta + \frac{\lambda_p}{b} m\right)^2\right)^s \hat{f}\left(\eta + \frac{\lambda_p}{b} m\right) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p} + \frac{m}{b}\right)} \sqrt{\frac{b}{\lambda_p}} e^{2\pi i (kb/\lambda_p)\eta} d\eta \right|^2. \tag{13}$$

After a simple calculation, we have

$$\sum_{m\in\mathbb{Z}} \left(1 + \left(\eta + \frac{\lambda_p}{b} m\right)^2\right)^s \hat{f}\left(\eta + \frac{\lambda_p}{b} m\right) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p} + \frac{m}{b}\right)}, \tag{14}$$

which belongs to $L^2[0, (\lambda_p/b)]$, and $\left\{\sqrt{(b/\lambda_p)}\, e^{2\pi i (kb/\lambda_p)\eta}\right\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0, (\lambda_p/b)]$, and we have

$$I = \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \int_0^{\lambda_p/b} \left| \sum_{k\in\mathbb{Z}} \left(1 + \left(\eta + \frac{\lambda_p}{b} k\right)^2\right)^s \hat{f}\left(\eta + \frac{\lambda_p}{b} k\right) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p} + \frac{k}{b}\right)} \right|^2 d\eta$$
$$= \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \int_0^{\lambda_p/b} F_p(\eta) \cdot \left(\sum_{k\in\mathbb{Z}} \overline{\left(1 + \left(\eta + \frac{\lambda_p}{b} k\right)^2\right)^s \hat{f}\left(\eta + \frac{\lambda_p}{b} k\right)} \hat{\psi}\left(\frac{\eta}{\lambda_p} + \frac{k}{b}\right) \right) d\eta, \tag{15}$$

where $F_p(\eta) = \sum_{k\in\mathbb{Z}} (1 + (\eta + \lambda_p/bk)^2)^s \hat{f}(\eta + \lambda_p/bk) \overline{\hat{\psi}(\eta/\lambda_p + k/b)}$. So, we can obtain

$$I = \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \int_{\mathbb{R}} F_p(\eta) \cdot \left(1 + \eta^2\right)^s \overline{\hat{f}(\eta)} \hat{\psi}\left(\frac{\eta}{\lambda_p}\right) d\eta$$
$$= \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \int_{\mathbb{R}} \left(1 + \eta^2\right)^{2s} \left|\hat{f}(\eta)\right|^2 \left|\hat{\psi}\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta$$
$$+ \frac{1}{b} \sum_{p=1}^{\infty} \lambda_p^{-2s} \int_{\mathbb{R}} \sum_{0\neq k\in\mathbb{Z}} \left(1 + \eta^2\right)^s \left(1 + \left(\eta + \frac{\lambda_p}{b} k\right)^2\right)^s \overline{\hat{f}(\eta)} \hat{f}\left(\eta + \frac{\lambda_p}{b} k\right) \hat{\psi}\left(\frac{\eta}{\lambda_p}\right) \overline{\hat{\psi}\left(\frac{\eta}{\lambda_p} + \frac{k}{b}\right)} d\eta. \tag{16}$$

This finishes the proof.

The proof of the following lemma is similar to Lemma 1 and we omit it. □

Lemma 3. *Let* $s \in \mathbb{R}$, $a > 0$, $\{\lambda_p: p = 1, 2, \cdots\} \subset \mathbb{R}^+$, *and* $g \in H^s(\mathbb{R})$. *Then we have*

$$\begin{aligned} I &= \sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), e^{2\pi i\lambda_p \cdot} g(\cdot - ak)\right\rangle_s\right|^2 \\ &= \frac{1}{a}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2 \sum_{p=1}^{\infty}\left|\widehat{g}\left(\eta-\lambda_p\right)\right|^2 d\eta \\ &+\frac{1}{a}\int_{\mathbb{R}}\sum_{0\neq k\in\mathbb{Z}}\left(1+\eta^2\right)^s\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s \widehat{f}(\eta)\overline{\widehat{f}\left(\eta+\frac{k}{a}\right)}\sum_{p=1}^{\infty}\overline{\widehat{g}\left(\eta-\lambda_p\right)}\widehat{g}\left(\eta+\frac{k}{a}-\lambda_p\right)d\eta \end{aligned}, \quad (17)$$

for $f \in \mathcal{D}$.

3. The Sufficient Conditions of IWF and IGF

This section is devoted to the sufficient conditions of IWS and IGS to be frames for $H^s(\mathbb{R})$ and some examples are given. We begin with the sufficient condition of IWS to be a frame.

Theorem 4. *Let* $s \in \mathbb{R}$, $b > 0$, $\{\lambda_p: p = 1, 2, \cdots\} \subset \mathbb{R}^+$, *and* $\Psi \subset H^s(\mathbb{R})$. *If*

$$B = \frac{1}{b}\sup_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s \sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right| < \infty, \quad (18)$$

then $X(\Psi)$ *forms a Bessel sequence in* $H^s(\mathbb{R})$ *with bound B. If furthermore,*

$$A = \frac{1}{b}\inf_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s\left(\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 - \sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{0\neq k\in\mathbb{Z}}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|\right) > 0, \quad (19)$$

then $X(\Psi)$ *forms a IWF for* $H^s(\mathbb{R})$ *with bounds A and B.*

Proof. By Lemma 2, we have

$$\begin{aligned} I &= \sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), \lambda_p^{(1/2-s)}\psi_l\left(\lambda_p \cdot - kb\right)\right\rangle_s\right|^2 \\ &= \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta \\ &+\frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\int_{\mathbb{R}}\sum_{0\neq k\in\mathbb{Z}}\left(1+\eta^2\right)^s\left(1+\left(\eta+\frac{\lambda_p}{b}k\right)^2\right)^s\overline{\widehat{f}(\eta)}\widehat{f}\left(\eta+\frac{\lambda_p}{b}k\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\overline{\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)}d\eta = I_1 + I_2. \end{aligned} \quad (20)$$

We first estimate I_2 as

$$\left|I_2\right| \leq \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^s\left(1+\left(\eta+\frac{\lambda_p}{b}k\right)^2\right)^s\left|\widehat{f}(\eta)\widehat{f}\left(\eta+\frac{\lambda_p}{b}k\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta. \tag{21}$$

By Cauchy–Schwartz inequality, we can get

$$\begin{aligned}
\left|I_2\right| \leq{}& \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{0\neq k\in\mathbb{Z}}\left(\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\
&\times\left(\int_{\mathbb{R}}\left(1+\left(\eta+\frac{\lambda_p}{b}k\right)^2\right)^{2s}\left|\widehat{f}\left(\eta+\frac{\lambda_p}{b}k\right)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\
\leq{}& \frac{1}{b}\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\lambda_p^{-2s}\left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\
&\times\left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\left(\eta+\frac{\lambda_p}{b}k\right)^2\right)^{2s}\left|\widehat{f}\left(\eta+\frac{\lambda_p}{b}k\right)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\
={}& \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}(\clubsuit)(\spadesuit).
\end{aligned} \tag{22}$$

Actually, $(\clubsuit)=(\spadesuit)$, due to

$$\begin{aligned}
(\spadesuit) &= \left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}-\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\
&= \left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} = (\clubsuit).
\end{aligned} \tag{23}$$

Then, we have

$$\left|I_2\right| \leq \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta. \tag{24}$$

Together with (3.1), (3.3), and (3.4), we have

$$
\begin{aligned}
I \leq I_1 + |I_2| &\leq \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta \\
&\quad + \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right| d\eta \\
&= \int_{\mathbb{R}}\left(1+\eta^2\right)^{s}\left|\widehat{f}(\eta)\right|^2\cdot\frac{1}{b}\left(1+\eta^2\right)^{s}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right| d\eta \\
&\leq B\|f\|_s^2.
\end{aligned}
\tag{25}
$$

In addition, by (19) and (20), we get

$$
\begin{aligned}
I \geq I_1 - |I_2| &\geq \frac{1}{b}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta \\
&\quad - \frac{1}{b}\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\lambda_p^{-2s}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right| d\eta \\
&= \int_{\mathbb{R}}\left(1+\eta^2\right)^{s}\left|\widehat{f}(\eta)\right|^2\cdot\frac{1}{b}\left(1+\eta^2\right)^{s}\left(\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 - \sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{0\neq k\in\mathbb{Z}}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|\right) d\eta \\
&\geq A\|f\|_s^2.
\end{aligned}
\tag{26}
$$

So, $X(\Psi)$ forms a IWF for $H^s(\mathbb{R})$ with bounds A and B by (25) and (26). The proof is thus finished. □

Corollary 5. *Suppose $\Psi \subset H^s(\mathbb{R})$ and supp $\widehat{\psi}_l \subset [c_l, d_l]$ with $d_l - c_l < 1/b$ for each $1 \leq l \leq L$, then we get the following.*

(i) If

$$
B = \frac{1}{b}\sup_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^{s}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 < +\infty, A = \frac{1}{b}\inf_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^{s}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 > 0, a.e., \tag{27}
$$

then, $X(\Psi)$ forms a IWF in $H^s(\mathbb{R})$ with bounds A and B.

(ii) Besides, if $X(\Psi)$ forms a IWF in $H^s(\mathbb{R})$ with bounds A and B, then we get

$$
A \leq \frac{1}{b}\left(1+\eta^2\right)^{s}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 \leq B \text{ a.e.on } \mathbb{R}. \tag{28}
$$

Proof. By Theorem 4, the statement (i) is correct. For (ii), we assume that $X(\Psi)$ forms a IWF. By (20), we can get

$$\begin{aligned}A\|f\|_s^2 &\leq \sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot),\lambda_p^{(1/2-s)}\psi_l\left(\lambda_p\cdot -kb\right)\right\rangle_s\right|^2\\ &= \int_{\mathbb{R}}\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta \leq B\|f\|_s^2,\end{aligned} \tag{29}$$

for all $f\in H^s(\mathbb{R})$. It follows that

$$\int_{\mathbb{R}}\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2\left(\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 - A\right)d\eta \geq 0. \tag{30}$$

If there exists $E\subset\mathbb{R}$ with $|E|>0$ such that

$$\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 < A, \tag{31}$$

on E, then by taking $\widehat{f}(\cdot) = (1+\eta^2)^{-s/2}\chi_E(\cdot)$ in (30) and the reduction to absurdity, we get

$$A\leq\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2, \tag{32}$$

a.e. on E. By a standard argument, we get

$$A\leq\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2, \tag{33}$$

which holds a.e. on $\mathbb{R}$. The other inequality in (28) is similarly provable. We thus finish the proof. □

Example 1. Suppose $s>0, L\in\mathbb{N}$. For each $1\leq l\leq L$, we assume that $N_l\leq\left|\widehat{\psi}_l(\xi)\right|\leq M_l$ almost everywhere with $0<N_l\leq M_l$ and supp $(\widehat{\psi}_l) = [c_l, d_l]$ with $0<c_l<d_l$. We take $\lambda_p = ((\alpha^p/p))^{2s}, p=1,2,\cdots$. Then, if $0<\alpha<1$ and $d_l - c_l < (1/b)$ for every $1\leq l\leq L$, we can get $X(\Psi)$ that forms an IWF for $\mathrm{H}^s(\mathbb{R})$.

Proof. We suppose that

$$N=\min\{N_1,N_2,\ldots,N_L\}, M=\max\{M_1,M_2,\ldots,M_L\}, \tag{34}$$

and

$$c=\min\{c_1,c_2,\ldots,c_L\}, d=\max\{d_1,d_2,\ldots,d_L\}. \tag{35}$$

Also, supp $(\widehat{\psi}_l) = [c_l,d_l]\subset[c,d]$, so $c\leq(\eta/\lambda_p)\leq d$, i.e., $c\lambda_p\leq\eta\leq d\lambda_p$, and by $\lambda_p = ((\alpha^p/p))^{2s}, p=1,2,\cdots$, we can obtain $0<\lambda_p<1$. Then, we have only $0<c\lambda_p\leq\eta\leq d\lambda_p<d$, where $\widehat{\psi}_l((\eta/\lambda_p))$ is not equal to 0, and this implies that $1<(1+\eta^2)^s<(1+d^2)^s$. It follows that

$$\frac{L}{b}N^2\sum_{p=1}^{\infty}\frac{\alpha^p}{p}\leq\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2\leq\frac{L}{b}M^2\left(1+d^2\right)^s\sum_{p=1}^{\infty}\frac{\alpha^p}{p}. \tag{36}$$

By a standard argument, if $0<\alpha<1$, the series $\sum_{p=1}^{\infty}(\alpha^p/p)$ is convergent, and its sum is $-\ln(1-\alpha)$. Then, we can get

$$\frac{L}{b}N^2(-\ln(1-\alpha)) \le \frac{1}{b}\left(1+\eta^2\right)^s \sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 \le \frac{L}{b}M^2\left(1+d^2\right)^s(-\ln(1-\alpha)). \tag{37}$$

This implies that $X(\Psi)$ forms an IWF for $H^s(\mathbb{R})$ by Corollary 5. □

Remark 6. Let $1 \le l \le L$ and ψ_l satisfy the conditions in Example 1. Taking λ_p^{-2s} as the general term of a convergent series and $\lambda_p > 0$ being bounded above, we can obtain $X(\Psi)$ that forms an IWF for $H^s(\mathbb{R})$ by Corollary 5. From this, we can construct many useful examples.

Then, we give the sufficient condition of IGF.

Theorem 7. *Let $s \in \mathbb{R}$, $a > 0$, $\{\lambda_p\colon p = 1, 2, \cdots\} \subset \mathbb{R}^+$, and $G \subset H^s(\mathbb{R})$. If*

$$B = \frac{1}{a}\sup_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s\left(\sum_{k\in\mathbb{Z}}\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\overline{\widehat{g}_l(\eta-\lambda_p)}\,\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|\right) < \infty, \tag{38}$$

then, $\mathcal{G}(G)$ is a Bessel sequence with bound B. Besides, if

$$A = \frac{1}{a}\inf_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s\left(\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l(\eta-\lambda_p)\right|^2 - \sum_{0\ne k\in\mathbb{Z}}\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\overline{\widehat{g}_l(\eta-\lambda_p)}\,\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|\right) > 0, \tag{39}$$

then, $\mathcal{G}(G)$ forms IGF with bounds A and B.

Proof. By Lemma 3, $\forall f \in \mathcal{D}$, we have

$$\begin{aligned} I &= \sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), e^{2\pi i\lambda_p\cdot} g_l(\cdot - ak)\right\rangle_s\right|^2 \\ &= \frac{1}{a}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|\left|\widehat{f}(\eta)\right|\left|\widehat{f}(\eta)\right|^2\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l(\eta-\lambda_p)\right|^2 d\eta \\ &\quad + \frac{1}{a}\int_{\mathbb{R}}\sum_{0\ne k\in\mathbb{Z}}\left(1+\eta^2\right)^s\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s\widehat{f}(\eta)\overline{\widehat{f}\left(\eta+\frac{k}{a}\right)}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\overline{\widehat{g}_l(\eta-\lambda_p)}\,\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)d\eta \\ &= I_1 + I_2. \end{aligned} \tag{40}$$

We first compute I_2 as

$$\begin{aligned}
|I_2| &\le \frac{1}{a}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^s\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{f}(\eta)\widehat{f}\left(\eta+\frac{k}{a}\right)\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta \\
&\le \frac{1}{a}\sum_{0\neq k\in\mathbb{Z}}\left(\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{R}}\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^{2s}\left|\widehat{f}\left(\eta+\frac{k}{a}\right)\right|^2\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta\right)^{1/2}
\end{aligned} \tag{41}$$

$$\begin{aligned}
&\le \frac{1}{a}\left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta\right)^{1/2} \\
&\quad\times\left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^{2s}\left|\widehat{f}\left(\eta+\frac{k}{a}\right)\right|^2\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta\right)^{1/2}
\end{aligned} \tag{42}$$

$$= \frac{1}{a}(*)(**). \tag{43}$$

Then, we prove that $(*) = (**)$. Actually,

$$\begin{aligned}
(**) &= \left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\frac{k}{a}-\lambda_p\right)\widehat{g}_l\left(\eta-\lambda_p\right)\right|d\eta\right)^{1/2} \\
&= \left(\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta\right)^{1/2} = (*).
\end{aligned} \tag{44}$$

So, we have

$$|I_2| \le \frac{1}{a}\sum_{0\neq k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta. \tag{45}$$

By (40), we have

$$I_1 - |I_2| \le I = \sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), e^{2\pi i\lambda_p \cdot} g_l(\cdot - ak)\right\rangle_s\right|^2 \le I_1 + |I_2|. \tag{46}$$

Thus, together with (45), we can obtain

$$\int_{\mathbb{R}}\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2\cdot\frac{1}{a}\left(1+\eta^2\right)^s\left(\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2-\sum_{0\neq k\in\mathbb{Z}}\left|\sum_{l=1}^{L}\sum_{j\in\mathbb{Z}}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|\right)d\eta$$
$$\leq I\leq\int_{\mathbb{R}}\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2\frac{1}{a}\left(1+\eta^2\right)^s\left(\sum_{k\in\mathbb{Z}}\left|\sum_{l=1}^{L}\sum_{p=1}^{\infty}\widehat{g}_l\left(\eta-\lambda_p\right)\widehat{g}_l\left(\eta+\frac{k}{a}-\lambda_p\right)\right|\right)d\eta. \quad (47)$$

Hence, we finish the proof by (3.10), (3.11), and (3.17). □

Corollary 8. *Suppose $G\subset H^s(\mathbb{R})$ and supp $\widehat{g}_l\subset[c_l,d_l]$ with $d_l-c_l<(1/a)$ for each $1\leq l\leq L$, then we can get the following.*

(i) If

$$B=\frac{1}{a}\sup_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2<+\infty, A=\frac{1}{a}\inf_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2>0\text{ a.e.}, \quad (48)$$

then $\mathcal{G}(G)$ is a IGF for $H^s(\mathbb{R})$ with bounds A and B.

(ii) If $\mathcal{G}(G)$ forms a IGF for $H^s(\mathbb{R})$ with bounds A and B, then we can obtain

$$A\leq\frac{1}{a}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2\leq B\text{ a.e.on }\mathbb{R}. \quad (49)$$

Proof. So, (i) is obviously right by Theorem 7. For (ii), by (40), we get

$$A\|f\|_s^2\leq\frac{1}{a}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|\left|\widehat{f}(\eta)\right|\left|\widehat{f}(\eta)\right|^2\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2d\eta\leq B\|f\|_s^2, \quad (50)$$

for all $f\in\mathcal{D}$. Then, we have

$$\int_{\mathbb{R}}\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2\left(\frac{1}{a}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2-A\right)d\eta\geq 0. \quad (51)$$

So,

$$A\leq\frac{1}{a}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2, \quad (52)$$

holds a.e. on $\mathbb{R}$ by a standard argument. We can similarly prove another inequality in (49). Thus, the proof is completed. □

Example 2. Suppose $\widehat{g}_l\in L^\infty(\mathbb{R})$ and supp $\widehat{g}_l\subset[c_l,d_l]$ with $d_l-c_l<1/a$ for each $1\leq l\leq L$, and $0<m\leq\lambda_p\leq M$ for arbitrary $p=1,2,\cdots$. Then we can get $\mathcal{G}(G)$ that forms a IGF for $H^s(\mathbb{R})$ after a simple calculation.

4. The Necessary Condition

We present the necessary conditions of IWF and IGF in $H^s(\mathbb{R})$. First, we give a necessary condition for $X(\Psi)$ to be a IWF for $H^s(\mathbb{R})$. By Proposition 19.1.3 in [4], if the IWS $\left\{\lambda_p^{1/2}\psi(\lambda_p x-kb)\text{: }p=1,2,\ldots,k\in\mathbb{Z}\right\}$ forms a IWF for $L^2(\mathbb{R})$, then $\left\{\lambda_p\text{: }p=1,2,\cdots\right\}$ has to be a finite union of logarithmically separated sets. Then, we get the following theorem.

Theorem 9. *Let $s\in\mathbb{R}$, $b>0$, $\left\{\lambda_p\text{: }p=1,2,\cdots\right\}\subset\mathbb{R}^+$, and $\Psi\subset H^s(\mathbb{R})$. We assume that $X(\Psi)$ forms an IWF in $H^s(\mathbb{R})$ and B is the upper frame bound. Then, we have*

$$\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=1}^{\infty}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 \leq B \text{ for } a.e. \quad \eta\in\mathbb{R}. \quad (53)$$

Proof. We just need to prove that for arbitrary $M, N \in \mathbb{N}_+$ and $M < N$,

$$\frac{1}{b}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=M}^{N}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 \leq B \text{ for a.e.} \quad \eta\in\mathbb{R}. \quad (54)$$

So, for arbitrary fix $f \in \mathcal{D}$, we get

$$A\|f\|_s^2 \leq \sum_{l=1}^{L}\sum_{p=M}^{N}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), \lambda_p^{(1/2-s)}\psi_l\left(\lambda_p\cdot -kb\right)\right\rangle_s\right|^2 \leq B\|f\|_s^2. \quad (55)$$

Hence,

$$\begin{aligned}\sum_{l=1}^{L}\sum_{p=M}^{N}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), \lambda_p^{(1/2-s)}\psi_l\left(\lambda_p\cdot -kb\right)\right\rangle_s\right|^2 &= \sum_{l=1}^{L}\sum_{p=M}^{N}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\left|\int_R\left(1+\eta^2\right)^s\widehat{f}(\eta)\overline{\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)}\frac{1}{\sqrt{\lambda_p}}e^{2\pi i\left(kb/\lambda_p\right)\eta}d\eta\right|^2\\ &= \frac{1}{b}\sum_{l=1}^{L}\sum_{p=M}^{N}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\left|\int_R\left(1+\eta^2\right)^s\widehat{f}(\eta)\overline{\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)}\sqrt{\frac{b}{\lambda_p}}e^{2\pi i\left(kb/\lambda_p\right)\eta}d\eta\right|^2.\end{aligned} \quad (56)$$

For $\forall p\in\mathbb{Z}$, $\left\{\sqrt{(b/\lambda_p)}\,e^{2\pi i(kb/\lambda_p)\eta}\colon k\in\mathbb{Z}\right\}$ is an o.n.b for $L^2(J)$, where $J\subset\mathbb{R}$ is a closed interval with length (λ_p/b). Suppose I is a closed interval with $I\subset(0,\infty)\cap J$ and supp $\widehat{f}\subset I$. Then, $(1+\eta^2)^s\widehat{f}(\eta)\overline{\widehat{\psi}_l((\eta/\lambda_j))}\in L^2(I)$. By Plancherel theorem, we have

$$\begin{aligned}\sum_{l=1}^{L}\sum_{p=M}^{N}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), \lambda_p^{(1/2-s)}\psi_l\left(\lambda_p\cdot -kb\right)\right\rangle_s\right|^2 &= \frac{1}{b}\sum_{l=1}^{L}\sum_{p=M}^{N}\lambda_p^{-2s}\int_I\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta\\ &= \frac{1}{b}\int_I\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=M}^{N}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta.\end{aligned} \quad (57)$$

We assume that $\eta_0>0$, and by taking $q>0$ such that $I_q=[\eta_0,\eta_0+q]$ and $I_q\subset I$. We choose $f_q\in\mathcal{D}$ in (57) such that

$$\widehat{f}_q(\eta)=\frac{1}{\left(1+\eta^2\right)^{s/2}}q^{-1/2}\chi_{I_q}, \|f\|_s=1. \quad (58)$$

Then, we have

$$\sum_{l=1}^{L}\sum_{p=M}^{N}\sum_{k\in\mathbb{Z}}\left|\left\langle f(\cdot), \lambda_p^{((1/2)-s)}\psi_l\left(\lambda_p\cdot -kb\right)\right\rangle_s\right|^2 = \frac{1}{qb}\int_{\eta_0}^{\eta_0+qh}\left(1+\eta^2\right)^s\sum_{l=1}^{L}\sum_{p=M}^{N}\lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 d\eta. \quad (59)$$

If $q \longrightarrow 0$, we can get

$$\frac{1}{b}\left(1+\eta_0^2\right)^s \sum_{l=1}^{L}\sum_{p=M}^{N} \lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta_0}{\lambda_p}\right)\right|^2 \leq B, \tag{60}$$

by (55). Then, we have

$$\frac{1}{b}\left(1+\eta^2\right)^s \sum_{l=1}^{L}\sum_{p=1}^{\infty} \lambda_p^{-2s}\left|\widehat{\psi}_l\left(\frac{\eta}{\lambda_p}\right)\right|^2 \leq B, \quad a.e.\eta > 0, \tag{61}$$

by $N \longrightarrow +\infty$ and the arbitrariness of $\eta_0 > 0$. The case $\eta < 0$ can be proved similarly. The proof is thus completed.

The next theorem is devoted to the necessary condition of $\mathscr{G}(G)$ that forms IGF in $H^s(\mathbb{R})$. □

Theorem 10. *Let $s \in \mathbb{R}$, $a > 0$, $\{\lambda_p\colon p = 1, 2, \cdots\} \subset \mathbb{R}^+$, and $G \subset H^s(\mathbb{R})$. We assume that $\mathscr{G}(G)$ is an IGF for $H^s(\mathbb{R})$ with bounds A and B. Then, we can get*

$$A \leq \frac{1}{a}\left(1+\eta^2\right)^s \sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2 \leq B. \tag{62}$$

Proof. Since $\mathscr{G}(G)$ is an IGF for $H^s(\mathbb{R})$ with bounds A and B. Then, we can obtain

$$A\|f\|_s^2 \leq \sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x} g_l(x-ak)\right\rangle_s\right|^2 \leq B\|f\|_s^2, \quad \text{for } \forall f \in H^s(\mathbb{R}). \tag{63}$$

Suppose $f \in \mathscr{D}$, then we get

$$\begin{aligned}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x} g_l(x-ak)\right\rangle_s\right|^2 &= \sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}}\left(1+\eta^2\right)^s\widehat{f}(\eta)\overline{\widehat{g}_l\left(\eta-\lambda_p\right)}e^{2\pi iak\eta}\right|^2\\ &= \frac{1}{a}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}}\left(1+\eta^2\right)^s\widehat{f}(\eta)\overline{\widehat{g}_l\left(\eta-\lambda_p\right)}a^{\frac{1}{2}}e^{2\pi iak\eta}\right|^2.\end{aligned} \tag{64}$$

We suppose that $I \subset \mathbb{R}$ with length $(1/a)$ and $\operatorname{supp}\widehat{f} \subset I$. Since $f \in \mathscr{D}$, so for $\forall \eta \in I$, $\exists M > 0$ such that $|(1+\eta^2)^s\widehat{f}(\eta)| < M$. Then, we have

$$\begin{aligned}\int_I\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2 d\eta &\leq \int_I\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\\ \left(1+\left(\eta-\lambda_p\right)^2\right)^s\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2 d\eta &\leq M^2\|g\|_s^2, \quad \forall 1 \leq l \leq L.\end{aligned} \tag{65}$$

So, $(1+\eta^2)^s\widehat{f}(\eta)\,\overline{\widehat{g}_l(\eta-\lambda_p)} \in L^2(I)$ and

$$\sum_{l=1}^{L}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x} g_l(x-ak)\right\rangle_s\right|^2 = \frac{1}{a}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\int_I\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2 d\eta. \tag{66}$$

Then by (63), we get

$$A\|f\|_s^2 \leq \frac{1}{a}\sum_{l=1}^{L}\sum_{p=1}^{\infty}\int_I\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2 d\eta \leq B\|f\|_s^2. \tag{67}$$

So, we get

$$A \leq \frac{1}{a}\left(1+\eta^2\right)^s \sum_{l=1}^{L}\sum_{p=1}^{\infty}\left|\widehat{g}_l\left(\eta-\lambda_p\right)\right|^2 \leq B \text{a.e.} \quad \eta \in \mathbb{R}. \tag{68}$$

This finishes the proof. □

5. The Perturbation of IWF and IGF

In this section, we give the perturbation theorems of IWF and IGF.

Theorem 11. *Let* $\varphi, \psi \in H^s(\mathbb{R})$, $\{\lambda_p\colon p = 1, 2, \cdots\} \subset \mathbb{R}^+$, *and* $b > 0$ *be given. We assume that* $\{\lambda_p^{((1/2)-s)}\psi(\lambda_p x - kb)\colon p = 1, 2, \ldots, k \in \mathbb{Z}\}$ *is an IWF with bounds A, B. If* φ *and* ψ *satisfy that*

$$R = \frac{1}{b}\sup_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s \sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right| < A, \tag{69}$$

then $\{\lambda_p^{((1/2)-s)}\varphi(\lambda_p x - kb)\colon p = 1, 2, \ldots, k \in \mathbb{Z}\}$ *forms an IWF for* $\mathrm{H}^s(\mathbb{R})$ *with frame bounds*

$$A\left(1-\sqrt{\frac{R}{A}}\right)^2, B\left(1+\sqrt{\frac{R}{A}}\right)^2. \tag{70}$$

Proof. Fix $f \in \mathscr{D}$, Similar to Lemma 2 proof, we get

$$\begin{aligned} I &= \sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), \lambda_p^{(1/2-s)}(\psi-\varphi)\left(\lambda_p x - kb\right)\right\rangle_s\right|^2 \\ &= \frac{1}{b}\sum_{p=1}^{\infty}\lambda_p^{-2s}\int_{\mathbb{R}}\left(1+\eta^2\right)^s\overline{\widehat{f}(\eta)}(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)\sum_{k\in\mathbb{Z}}\left(1+\left(\eta+\frac{\lambda_p}{b}k\right)^2\right)^s\widehat{f}\left(\eta+\frac{\lambda_p}{b}k\right)\overline{(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)}d\eta. \end{aligned} \tag{71}$$

Then, we have

$$\begin{aligned} I &\leq \frac{1}{b}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\ &\quad\times\left(\int_{\mathbb{R}}\left(1+\left(\eta+\frac{\lambda_p}{b}k\right)^2\right)^{2s}\left|\widehat{f}\left(\eta+\frac{\lambda_p}{b}k\right)\right|^2\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\ &\leq \frac{1}{b}\left(\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\ &\quad\times\left(\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\left(\eta+\frac{\lambda_p}{b}k\right)^2\right)^{2s}\left|\widehat{f}\left(\eta+\frac{\lambda_p}{b}k\right)\right|^2\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2} \\ &\leq \frac{1}{b}(*)(**). \end{aligned} \tag{72}$$

Actually, $(*)=(**)$, due to

$$\begin{aligned}(**)&=\left(\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}-\frac{k}{b}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)\right|d\eta\right)^{1/2}\\&=\left(\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\right)^{1/2}=(*).\end{aligned}\tag{73}$$

So, by (69), we have

$$\begin{aligned}I&\leq\frac{1}{b}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\\&=\int_{\mathbb{R}}\left(1+\eta^2\right)^{s}\left|\widehat{f}(\eta)\right|^2\cdot\frac{1}{b}\left(1+\eta^2\right)^{s}\sum_{p=1}^{\infty}\lambda_p^{-2s}\sum_{k\in\mathbb{Z}}\left|(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}\right)(\widehat{\psi}-\widehat{\varphi})\left(\frac{\eta}{\lambda_p}+\frac{k}{b}\right)\right|d\eta\\&\leq R\|f\|_s^2.\end{aligned}\tag{74}$$

Thus, we finish the proof by Lemma 1. □

Example 3. Let $s>0$ and suppose $\psi\in H^s(\mathbb{R})$ satisfies $N_1\leq|\widehat{\psi}|\leq N_2$ with $N_1,N_2>0$ and supp $\widehat{\psi}\subset[c_1,d_1]$ *with* $d_1-c_1<(1/b)$. Then, we assume that $\varphi\in H^s(\mathbb{R})$ satisfies $M_1\leq|\widehat{\varphi}|\leq M_2$ with $M_1,M_2>0$ and supp $\widehat{\varphi}\subset[c_2,d_2]$ with $c_1<c_2<d_1$ and $d_2-d_1<(1/b)$. Taking $\lambda_p=((p/3^p))^{2s}$ for each $p\in\mathbb{N}_+$, we have $\{\lambda_p^{((1/2)-s)}\varphi(\lambda_p x-kb)\colon p=1,2,\ldots,k\in\mathbb{Z}\}$ which forms a frame for $H^s(\mathbb{R})$ after calculations similar to Example 1.

Theorem 12. *Let* $g,h\in H^s(\mathbb{R})$, $\{\lambda_p\colon p=1,2,\cdots\}\subset\mathbb{R}^+$, *and* $a>0$ *be given. We assume that* $\{e^{2\pi i\lambda_p x}g(x-ak)\colon p=1,2,\ldots,k\in\mathbb{Z}\}$ *forms an IGF for* $H^s(\mathbb{R})$ *with bounds A and B. If*

$$R=\frac{1}{a}\sup_{\eta\in\mathbb{R}}\left(1+\eta^2\right)^s\sum_{k\in\mathbb{Z}}\left|\sum_{p=1}^{\infty}\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}(\widehat{g}-\widehat{h})\left(\eta-\lambda_p-\frac{k}{a}\right)\right|<A,\tag{75}$$

then $\{e^{2\pi i\lambda_p x}h(x-ak)\colon p=1,2,\ldots,k\in\mathbb{Z}\}$ *forms a IGF for* $H^s(\mathbb{R})$ *with bounds*

$$A\left(1-\sqrt{\frac{R}{A}}\right)^2,B\left(1+\sqrt{\frac{R}{A}}\right)^2.\tag{76}$$

Proof. If fix $f\in\mathcal{D}$, then we have

$$\begin{aligned}&\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x),e^{2\pi i\lambda_p x}(g-h)(x-ak)\right\rangle_s\right|^2\\&\quad=\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\int_{\mathbb{R}}\left(1+\eta^2\right)^s\widehat{f}(\eta)\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}e^{2\pi iak\eta}d\eta\right|^2.\end{aligned}\tag{77}$$

By the periodic process, we get

$$\begin{aligned}
&\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x}(g-h)(x-ak)\right\rangle_s\right|^2\\
&=\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}\int_{m/a}^{m+1/a}\left(1+\eta^2\right)^s\hat{f}(\eta)\overline{(\hat{g}-\hat{h})\left(\eta-\lambda_p\right)}e^{2\pi iak\eta}d\eta\right|^2\\
&=\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\sum_{m\in\mathbb{Z}}\int_{0}^{1/a}\left(1+\left(\eta+\frac{m}{a}\right)^2\right)^s\hat{f}\left(\eta+\frac{m}{a}\right)\overline{(\hat{g}-\hat{h})\left(\eta+\frac{m}{a}-\lambda_p\right)}e^{2\pi iak\eta}d\eta\right|^2\\
&=\frac{1}{a}\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\int_{0}^{1/a}\sum_{m\in\mathbb{Z}}\left(1+\left(\eta+\frac{m}{a}\right)^2\right)^s\hat{f}\left(\eta+\frac{m}{a}\right)\overline{(\hat{g}-\hat{h})\left(\eta+\frac{m}{a}-\lambda_p\right)}a^{1/2}e^{2\pi iak\eta}d\eta\right|^2.
\end{aligned}\tag{78}$$

A simple calculation implies that

$$\sum_{m\in\mathbb{Z}}\left(1+\left(\eta+\frac{m}{a}\right)^2\right)^s\hat{f}\left(\eta+\frac{m}{a}\right)\overline{(\hat{g}-\hat{h})\left(\eta+\frac{m}{a}-\lambda_p\right)}\in L^2[0,1/a],\tag{79}$$

and $\{a^{1/2}e^{2\pi iak\eta}\}_{k\in\mathbb{Z}}$ is an orthonormal basis for $L^2[0,1/a]$. It follows that

$$\begin{aligned}
&\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x}(g-h)(x-ak)\right\rangle_s\right|^2\\
&=\frac{1}{a}\sum_{p=1}^{\infty}\int_{0}^{1/a}\left|\sum_{k\in\mathbb{Z}}\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s\hat{f}\left(\eta+\frac{k}{a}\right)\overline{(\hat{g}-\hat{h})\left(\eta+\frac{k}{a}-\lambda_p\right)}\right|^2d\eta\\
&=\frac{1}{a}\sum_{j\in\mathbb{Z}}\int_{0}^{1/a}F_p(\eta)\cdot\sum_{k\in\mathbb{Z}}\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s\hat{f}\left(\eta+\frac{k}{a}\right)\overline{(\hat{g}-\hat{h})\left(\eta+\frac{k}{a}-\lambda_p\right)}d\eta,
\end{aligned}\tag{80}$$

where

$$F_p(\eta)=\sum_{k\in\mathbb{Z}}\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s\overline{\hat{f}\left(\eta+\frac{k}{a}\right)}(\hat{g}-\hat{h})\left(\eta+\frac{k}{a}-\lambda_p\right).\tag{81}$$

Then, we can obtain

$$\begin{aligned}
&\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x}(g-h)(x-ak)\right\rangle_s\right|^2 \\
&= \frac{1}{a}\sum_{p=1}^{\infty}\int_{\mathbb{R}} F_p(\eta)\cdot\left(1+\eta^2\right)^s \widehat{f}(\eta)\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}d\eta \\
&= \frac{1}{a}\sum_{p=1}^{\infty}\int_{\mathbb{R}}\sum_{k\in\mathbb{Z}}\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s\overline{\widehat{f}\left(\eta+\frac{k}{a}\right)}(\widehat{g}-\widehat{h})\left(\eta+\frac{k}{a}-\lambda_p\right)\cdot\left(1+\eta^2\right)^s\widehat{f}(\eta)\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}d\eta.
\end{aligned} \tag{82}$$

So, we get

$$\begin{aligned}
&\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x}(g-h)(x-ak)\right\rangle_s\right|^2 \\
&\le \frac{1}{a}\sum_{k\in\mathbb{Z}}\int_{\mathbb{R}}\left(1+\eta^2\right)^s\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^s\left|\widehat{f}(\eta)\right|\left|\widehat{f}\left(\eta+\frac{k}{a}\right)\right|\left|\sum_{p=1}^{\infty}\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}(\widehat{g}-\widehat{h})\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta \\
&\le \frac{1}{a}\sum_{k\in\mathbb{Z}}\left(\int_{\mathbb{R}}\left(1+\eta^2\right)^{2s}\left|\widehat{f}(\eta)\right|^2\left|\sum_{p=1}^{\infty}\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}(\widehat{g}-\widehat{h})\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta\right)^{1/2} \\
&\quad\times\left(\int_{\mathbb{R}}\left(1+\left(\eta+\frac{k}{a}\right)^2\right)^{2s}\left|\widehat{f}\left(\eta+\frac{k}{a}\right)\right|^2\left|\sum_{p=1}^{\infty}\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}(\widehat{g}-\widehat{h})\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta\right)^{1/2} \\
&= \frac{1}{a}(*)(**).
\end{aligned} \tag{83}$$

Hence,

$$\begin{aligned}
&\sum_{p=1}^{\infty}\sum_{k\in\mathbb{Z}}\left|\left\langle f(x), e^{2\pi i\lambda_p x}(g-h)(x-ak)\right\rangle_s\right|^2 \\
&\le \int_{\mathbb{R}}\left(1+\eta^2\right)^s\left|\widehat{f}(\eta)\right|^2\cdot\frac{1}{a}\left(1+\eta^2\right)^s\left|\sum_{p=1}^{\infty}\overline{(\widehat{g}-\widehat{h})\left(\eta-\lambda_p\right)}(\widehat{g}-\widehat{h})\left(\eta+\frac{k}{a}-\lambda_p\right)\right|d\eta \\
&\le R\|f\|_s^2.
\end{aligned} \tag{84}$$

Similar to the proof mentioned above and using condition (5.2), the proof is thus completed by Lemma 1. □

Example 4. Suppose $\widehat{g} \in L^{\infty}(\mathbb{R})$ and supp $\widehat{g} \subset [c_1, d_1]$ with $d_1 - c_1 < 1/a$. We assume $0 < m \leq \lambda_p \leq M$ for arbitrary $p \in \mathbb{N}_+$. Let $c_1 < c_2 < d_1$ and $d_2 - d_1 < 1/a$. Then, by taking the function $h \in H^s(\mathbb{R})$ such that $\widehat{h} \in L^{\infty}(\mathbb{R})$ and supp $\widehat{h} \subset [c_2, d_2]$, we have $\left\{e^{2\pi i \lambda_p x} h(x - ak)\colon p = 1, 2, \ldots, k \in \mathbb{Z}\right\}$ which forms a IGF for $\mathrm{H}^s(\mathbb{R})$.

6. Conclusion

In this paper, we introduced the concept of IWS and IGS in Sobolev space $H^s(\mathbb{R})$. Then, we provided the necessary and sufficient conditions for IWS and IGS to be IWF and IGF in $H^s(\mathbb{R})$. Using these conditions, we also constructed specific IWF and IGF. At last, we discussed the perturbation theorem of IWF and IGF. The obtained results can provide theoretical reference for the practical application of frame in image restoration and DCNNs [47–50]

Data Availability

No data were used to support the study.

Disclosure

A preprint has previously been published ((Yu Tian, 2022); see [51]).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

This work was supported by the Science and Technology Research Project of Henan Province (Grant no. 222102210335).

References

[1] J. Duffin and A. C. Schaeffer, "A class of nonharmonic Fourier series," *Transactions of the American Mathematical Society*, vol. 72, pp. 341–366, 1952.

[2] I. Daubechies, A. Grossmann, and Y. Meyer, "Painless nonorthogonal expansions," *Journal of Mathematical Physics*, vol. 27, no. 5, pp. 1271–1283, 1986.

[3] J. F. Cai, J. K. Choi, J. Li, and K. Wei, "Image restoration: structured low rank matrix framework for piecewise smooth functions and beyond," *Applied and Computational Harmonic Analysis*, vol. 56, pp. 26–60, 2022.

[4] O. Christensen, "An introduction to frames and Riesz bases," *Applied and Numerical Harmonic Analysis*, 2nd edition, 2016.

[5] O. Christensen and M. Hasannasab, "Gabor frames in $${\ell}ˆ 2({\mathbb {Z}})$$ ℓ 2 (Z) and linear dependence," *Journal of Fourier Analysis and Applications*, vol. 25, no. 1, pp. 101–107, 2019.

[6] I. Daubechies, "The wavelet transform, time-frequency localization and signal analysis," *IEEE Transactions on Information Theory*, vol. 36, no. 5, pp. 961–1005, 1990.

[7] C. Diao and B. Han, "Quasi-tight framelets with high vanishing moments derived from arbitrary refinable functions," *Applied and Computational Harmonic Analysis*, vol. 49, no. 1, pp. 123–151, 2020.

[8] C. J. Liu and H. Wechsler, "Gabor feature based classification using the enhanced Fisher linear discriminant model for face recognition," *IEEE Transactions on Image Processing*, vol. 11, no. 4, pp. 467–476, 2002.

[9] H. Ji, Z. Shen, and Y. Zhao, "Directional frames for image recovery: multi-scale discrete Gabor frames," *Journal of Fourier Analysis and Applications*, vol. 23, no. 4, pp. 729–757, 2017.

[10] Y. C. Xiao and X. S. Zhuang, "Adaptive directional haar tight framelets on bounded domains for digraph signal representations," *Journal of Fourier Analysis and Applications*, vol. 27, no. 2, p. 7, 2021.

[11] H. G. Feichtinger, P. Balazs, and D. Haider, "Double preconditioning for Gabor frame operators: algebraic, functional analytic and numerical aspects," *Applied and Computational Harmonic Analysis*, vol. 66, no. 9, pp. 101–137, 2023.

[12] Y.-Z. Li and M. Yang, "Rationally sampled Gabor frames on the half real line," *Journal of Mathematical Analysis and Applications*, vol. 532, no. 1, p. 127919, 2024.

[13] Y.-Z. Li and X.-L. Zhang, "Characterization of rationally sampled quaternionic dual Gabor frames," *Mathematical Methods in the Applied Sciences*, vol. 47, no. 2, pp. 1095–1112, 2024.

[14] K. Poumai, N. Khanna, and S. Kaushik, "Finite Gabor systems and uncertainty principle for block sliding discrete Fourier transform," *Filomat*, vol. 37, no. 8, pp. 2361–2376, 2023.

[15] J. F. Cai, B. Dong, S. Osher, and Z. Shen, "Image restoration: total variation, wavelet frames and beyond," *Journal of the American Mathematical Society*, vol. 25, no. 4, pp. 1033–1089, 2012.

[16] J. F. Li, H. Feng, and D. X. Zhou, "Approximation analysis of CNNs from a feature extraction view," *Analysis and Applications*, vol. 22, no. 03, pp. 635–654, 2024.

[17] T. Mao, Z. J. Shi, and D. X. Zhou, "Approximating functions with multi-features by deep convolutional neural networks," *Analysis and Applications*, vol. 21, no. 01, pp. 93–125, 2023.

[18] D. X. Zhou, "Universality of deep convolutional neural networks," *Applied and Computational Harmonic Analysis*, vol. 48, no. 2, pp. 787–794, 2020.

[19] L. Dayong and L. Dengfeng, "A characterization of orthonormal wavelet families in sobolev spaces," *Acta Mathematica Scientia*, vol. 31, no. 4, pp. 1475–1488, 2011.

[20] M. Bownik, "A characterization of affine dual frames in L2Rn," *Applied and Computational Harmonic Analysis*, vol. 8, no. 2, pp. 203–221, 2000.

[21] K. Grochenig, *Foundations of Time-Frequency Analysis*, Birkhauser, Boston, MA, USA, 2000.

[22] C. Heil, *A Basis Theory Primer*, Birkhauser, New York, NY, USA, 2011.

[23] E. Hernandez and G. Weiss, *A First Course on Wavelets*, CRC Press, Boca Raton, FL, USA, 1996.

[24] Y.-Z. Li and Y. Tian, "Partial affine system-based frames and dual frames," *Mathematical Methods in the Applied Sciences*, vol. 40, no. 18, pp. 6927–6943, 2017.

[25] A. Benavente, O. Christensen, M. Hasannasab, H. O. Kim, R. Y. Kim, and F. D. Kovac, "Approximately dual pairs of

wavelet frames," *Journal of Mathematical Analysis and Applications*, vol. 507, no. 2, 2022.

[26] M. Bownik, K. Dziedziul, and A. Kamont, "Parseval wavelet frames on riemannian manifold," *Journal of Geometric Analysis*, vol. 32, no. 1, p. 4, 2022.

[27] D. T. Stoeva, "On compactly supported dual windows of Gabor frames," *Journal of Mathematical Analysis and Applications*, vol. 505, no. 1, 2022.

[28] R. Lu, "A structural characterization of compactly supported OEP-based balanced dual multiframelets," *Analysis and Applications*, vol. 21, no. 04, pp. 1039–1066, 2023.

[29] M. Ehler, "The multiresolution structure of pairs of dual wavelet frames for a pair of Sobolev spaces," *Jaen Journal on Approximation*, vol. 2, pp. 193–214, 2010.

[30] B. Han and Z. Shen, "Dual wavelet frames and Riesz bases in Sobolev spaces," *Constructive Approximation*, vol. 29, no. 3, pp. 369–406, 2009.

[31] B. Han and Z. Shen, "Characterization of Sobolev spaces of arbitrary smoothness using nonstationary tight wavelet frames," *Israel Journal of Mathematics*, vol. 172, no. 1, pp. 371–398, 2009.

[32] Y.-Z. Li and J.-P. Zhang, "Nonhomogeneous dual wavelet frames and mixed oblique extension principles in Sobolev spaces," *Applicable Analysis*, vol. 97, no. 7, pp. 1049–1073, 2018.

[33] Y.-Z. Li and H.-F. Jia, "Weak nonhomogeneous wavelet biframes for reducing subspaces of sobolev spaces," *Numerical Functional Analysis and Optimization*, vol. 38, no. 2, pp. 181–204, 2017.

[34] B. Han and M. Michelle, "Derivative-orthogonal Riesz wavelets in Sobolev spaces with applications to differential equations," *Applied and Computational Harmonic Analysis*, vol. 47, no. 3, pp. 759–794, 2019.

[35] Y. Li and S. Yang, "Multiwavelet sampling theorem in Sobolev spaces," *Science China Mathematics*, vol. 53, no. 12, pp. 3197–3214, 2010.

[36] Y. Li, "Sampling approximation by framelets in Sobolev space and its application in modifying interpolating error," *Journal of Approximation Theory*, vol. 175, pp. 43–63, 2013.

[37] Y. Li, S. Yang, and D. Yuan, "Bessel multiwavelet sequences and dual multiframelets in Sobolev spaces," *Advances in Computational Mathematics*, vol. 38, no. 3, pp. 491–529, 2013.

[38] Z. Jing, "On the stability of wavelet and Gabor frames (Riesz bases)," *Journal of Fourier Analysis and Applications*, vol. 5, no. 1, pp. 105–125, 1999.

[39] O. Christensen, "Perturbation of frames and applications to Gabor frames," in *Gabor Analysis and Algorithms. Applied and Numerical Harmonic Analysis*, pp. 193–209, Birkhauser, Boston, MA, USA, 1998.

[40] W. Sun and X. Zhou, "On the stability of gabor frames," *Advances in Applied Mathematics*, vol. 26, no. 3, pp. 181–191, 2001.

[41] M. Bownik and O. Christensen, "Characterization and perturbation of Gabor frame sequences with rational parameters," *Journal of Approximation Theory*, vol. 147, no. 1, pp. 67–80, 2007.

[42] W. Sun and X. Zhou, "Irregular wavelet/Gabor frames," *Applied and Computational Harmonic Analysis*, vol. 13, no. 1, pp. 63–76, 2002.

[43] W. Sun and X. Zhou, "Density of irregular wavelet frames," *Proceedings of the American Mathematical Society*, vol. 132, no. 8, pp. 2377–2387, 2004.

[44] O. Christensen, "Irregular wavelet frames and Gabor frames," *Approximation Theory and its Applications*, vol. 17, no. 3, pp. 90–101, 2001.

[45] A. Aldroubi, C. Cabrelli, and U. M. Molter, "Wavelets on irregular grids with arbitrary dilation matrices and frame atoms for $L^2(\mathbb{R}^d)$," *Applied and Computational Harmonic Analysis*, vol. 17, no. 2, pp. 119–140, 2004.

[46] I. Iglewska-Nowak and M. Holschneider, "Irregular gabor frames," *Kyushu Journal of Mathematics*, vol. 67, no. 1, pp. 237–247, 2013.

[47] P. G. Casazza and O. Christensen, "Weyl-Heisenberg frames for subspaces of $L^2(\mathbb{R})$," *Proceedings of the American Mathematical Society*, vol. 129, no. 1, pp. 145–154, 2000.

[48] H. G. Feichtinger and T. Strohmer, *Gabor Analysis and Algorithms: Theory and Applications*, Birkhauser, Boston, MA, USA, 1998.

[49] R. Q. Jia, J. Z. Wang, and D. X. Zhou, "Compactly supported wavelet bases for Sobolev spaces," *Applied and Computational Harmonic Analysis*, vol. 15, no. 3, pp. 224–241, 2003.

[50] A. Ron and Z. Shen, "Affine systems in $L^2(\mathbb{R}^d)$II. Dual systems," *Journal of Fourier Analysis and Applications*, vol. 3, no. 5, pp. 617–637, 1997.

[51] Y. Tian, "Irregular wavelet/Gabor Frames in Sobolev Space," preprint, 2022.