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Research Article

Similarity of C_1: Operators and the Hyperinvariant Subspace Problem

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In the present paper, we first show that the existence of the solutions of the operator equation $S^{*}\mathrm{XT} = X$ is related to the similarity of operators of class $C_{1.}$, and then we give a sufficient condition for the existence of nontrivial hyperinvariant subspaces. These subspaces are the closure of ran $\varphi(T)$ for some singular inner functions φ. As an application, we prove that every C_{10}-quasinormal operator and C_{10}-centered operator, under suitable conditions, have nontrivial hyperinvariant subspaces.

1. Introduction

Let H be a separable infinite dimensional complex Hilbert space and let $B(H)$ be the algebra of all bounded linear operators acting on H (see [1] for basics and fundamentals). The commutant of T, denoted by $\{T\}$, is the algebra of all operators $A \in B(H)$ such that $\mathrm{AT} = \mathrm{TA}$. A closed subspace $M \subseteq H$ is called a nontrivial hyperinvariant subspace for T if $0 \neq M \neq H$ and $\mathrm{AM} \subseteq M$ for every $A \in \{T\}$. In particular, if $\mathrm{TM} \subseteq M$, then the subspace M is called a nontrivial invariant subspace for T. The invariant subspace problem asks whether every operator $T \in B(H)$ has a nontrivial invariant subspace with $T(M) \subset M$. In a similar fashion, the hyperinvariant subspace problem asks whether every bounded linear operator such that $T \neq \alpha I$ has a nontrivial hyperinvariant subspace. These problems are still unresolved, especially for operators $T \in B(H)$ such that $\|T^n x\| \nrightarrow 0$ for every nonzero x in H.

A power-bounded operator T of class C_{*0} which commutes with a nonzero quasinilpotent operator has a nontrivial invariant subspace. In the hyponormality case, it is well known that in [2], Kubrusly and Levan have shown that if the strong limit A, defined below, is a projection for every biquasitriangular contraction T, then every contraction not in C_{00} has a nontrivial invariant subspace. We recall the following standard definitions: for $T \in B(H)$, T is a normal operator if $T^{*}T = \mathrm{TT}^{*}$, T is a quasinormal operator if $\mathrm{TT}^{*}T = T^{*}\mathrm{TT}$, T is subnormal if there exist a complex Hilbert space $K \supset H$ and a normal operator $N \in B(K)$ such that H is an invariant subspace for N and T is the restriction of N to H (i.e., $N_{|H} = T$). T is hyponormal if $T^{*}T - \mathrm{TT}^{*} \geq 0$. The proper inclusions are well known (see, e.g., [3]).

$$\text{Normal} \subset \text{Quasinormal} \subset \text{Subnormal} \subset \text{Hyponormal}. \tag{1}$$

Recall that a unitary operator is singular (resp. absolutely continuous) if its spectral measure is singular (resp. absolutely continuous) with respect to the Lebesgue measure on the unit circle. Any contraction T can be decomposed uniquely as the direct sum $T = U_s \oplus U_a \oplus T_0$, where U_s and U_a are singular and absolutely continuous unitary operators, respectively, and T_0 is a completely nonunitary contraction. T is said to be absolutely continuous if in this decomposition U_s is absent. For this type of decomposition for polynomially bounded operators, see [2, 4].

It is well known that the equation $S^{*}\mathrm{XS} = X$, where S is the unilateral shift of multiplicity one, characterizes the class of Toeplitz operators, and that this type of equations for contraction operators was studied by many authors as in [5] and the references therein. For the invariant subspace problem, it is known that it was solved for the class of subnormal operators but the hyperinvariant subspace problem is still open for this class (for certain partial results, readers may wish to look at [6]). In [7], it was shown that if T is a hyponormal operator with a thin spectrum, then it has a nontrivial invariant subspace, and in the C_{10} case, some partial results were obtained by Kubrusly and Levan in [8], who have proved that if a hyponormal operator T has no nontrivial invariant subspace, then T is either a proper contraction of class C_{00} or a nonstrict proper contraction of class C_{10} for which the strong limit A of $(T^{*n}T^n)_{n\geq 1}$ is a completely nonprojective nonstrict proper contraction. For more details, see [4, 9–16].

In the present paper, we show that the existence of nontrivial solutions of the equation $S^{*}\mathrm{XT} = X$, where S is the unilateral shift of multiplicity one on the hardy space $\mathbb{H}^2(\mathbb{T})$ and T is a polynomially bounded operator on a Hilbert space H and is related to the similarity of operators of class $C_{1.}$. In other words, $S^{*}\mathrm{XT} = X$ has nontrivial solutions X if and only if the operator matrix

$$\begin{bmatrix} S & 1\otimes X^{*}z \\ 0 & T \end{bmatrix}, \tag{2}$$

is similar to $S\oplus T$.

Then, we study the hyperinvariant subspaces problem for polynomially bounded operators of class C_{10}, and we give sufficient conditions for the existence of nontrivial hyperinvariant subspaces for hyponormal operators of class C_{10}.

Now, we summarize our main results.

Let T be a polynomially bounded absolutely continuous operator. If

$$\sum_{n=0}^{\infty}\left|\widehat{\frac{1}{\phi}}(n)\right|\|T^{*n}x\| < \infty, \tag{3}$$

for some nonzero x such that the operator matrix

$$\begin{bmatrix} S & 1\otimes x \\ 0 & T \end{bmatrix}, \tag{4}$$

is similar to $S\oplus T$. Then, T has nontrivial hyperinvariant subspaces.

The nontrivial hyperinvariant subspaces obtained are the closure of $\operatorname{ran}\phi(T)$ where ϕ is a singular inner function. The operator $\phi(T)$ is the function of T obtained using the $\mathbb{H}^{\infty}$-functional calculus defined for absolutely continuous polynomially bounded operators [2, 4]. As an application, we prove that if T is a C_{10}-quasinormal operator, then T has a nontrivial hyperinvariant subspace. In particular if T is a completely nonnormal quasinormal operator, then the nontrivial hyperinvariant subspaces of T are the closure of $\operatorname{ran}\varphi(T)$ where φ is a singular inner function. In the case where T is a centered operator, we give a refinement of the result by showing that we may take $x\in D_T\operatorname{ran}A$.

Next, we provide other conditions sufficient for the existence of nontrivial hyperinvariant subspaces for T.

2. Preliminaries

2.1. Notations and Definitions. Throughout this paper, H denotes an infinite dimensional complex separable Hilbert space with inner product $\langle\cdot,\cdot\rangle$ and $B(H,K)$ denotes the space of all bounded linear operators acting from H to K. The kernel and the range of an operator T will be denoted by $\ker T$ and $\operatorname{ran}T$, respectively, and the rank one operator $x\otimes y$; $x,y\in H$ is defined by $(x\otimes y)h = \langle h,y\rangle x$, for all $h\in H$. The closure of a subspace M of H will be denoted by $\overline{M}$. For a contraction T, $\|T\|\leq 1$, the operators $D_T = (I-T^{*}T)^{1/2}$ and $D_{T^{*}} = (I-TT^{*})^{1/2}$ are the defect operators and $[T^{*},T] = T^{*}T-TT^{*}$ is the commutator of T.

Let A,B be bounded linear operators on the Hilbert spaces H and K, respectively. Consider the set

$$I(A,B) = \{X\in B(H,K)\colon \mathrm{XA} = \mathrm{BX}\}. \tag{5}$$

If there is an operator $X\in I(A,B)$ with a dense range, we set $A\prec^{d}B$. An operator $X\in B(H,K)$ will be said to be a quasiaffinity if it is injective and has a dense range and the operator A is a quasiaffine transform of the operator B and if there exists a quasiaffinity $X\in I(A,B)$, we set $A\prec B$.

2.2. Strong Limit for Contraction Operator. If T is a contraction, then $(T^{*n}T^n)_{n\geq 1}$ is a nonincreasing sequence of nonnegative contractions so that it converges strongly to an operator A which satisfies the following properties: $0\leq A\leq I$, $\|T^nx\|\longrightarrow\|A^{1/2}x\|$ as $n\longrightarrow\infty$ for all $x\in H$, $T^{*n}AT^n = A$ for all $n\geq 1$ and there exists an isometry V on $\overline{\operatorname{ran}A}$ such that $A^{1/2}\in I(T,V)$.

Furthermore, the subspace $\ker A = H_0 = \{x\colon \|T^nx\|\longrightarrow 0; n\longrightarrow\infty\}$ is a hyperinvariant subspace. We say that T is of class $C_{0.}$, that is strongly stable, if $H_0(T) = H$ and T is of class $C_{1.}$ if $H_0(T) = \{0\}$. T is of class $C_{.j}$: $j = 0,1$ if T^{*} is of class $C_{j.}$; $j = 0,1$ and T is of class C_{ij}: $i,j = 0,1$ if $T\in C_{i.}\cap C_{.j}$. For more details, see [4, 14].

We denote by $\mathbb{D}$ the open unit disc and by $\mathbb{T}$ the unit circle. Let m denote the normalized Lebesgue measure on the unit circle $\mathbb{T}$ (i.e., $m = d\theta/2\pi$) and let $L^2 = L^2(\mathbb{T})$ denote the space of all complex-valued Lebesgue measurable functions on $\mathbb{T}$ such that $\|f\|^2 = \int_{\mathbb{T}}|f(t)|^2dm(t)$ is finite. As such, L^2 is a Hilbert space, a simple calculation using the fact that $m(\mathbb{T}) = 1$ shows that this space has a canonical orthonormal basis $\{z^n\colon n\in\mathbb{Z}\}$ given by $z^n(\xi) = \xi^n$, for all $n\in\mathbb{Z}$; $\mathbb{Z}$ being the set of integers and z denotes the identity function, i.e., $z(\xi) = \xi$; $\xi\in\mathbb{T}$ and in the sequel, we set $\mathbf{1}\equiv z^0$.

The Hardy space $\mathbb{H}^2 = \mathbb{H}^2(\mathbb{T})$ is the closed linear span of $\{z^n\colon n = 0, 1, \ldots\}$. The operators of multiplication by the identity function z on the spaces $\mathbb{H}^2$ and $\mathbb{H}^2_{-} = L^2\ominus\mathbb{H}^2$ are the unilateral forward shift S in $\mathbb{H}^2$ defined by $(Sf)(\xi) := \xi.f(\xi)$ and the unilateral forward shift S_{-} in $\mathbb{H}^2_{-}$ defined by $(S_{-}f)(\xi) := \overline{\xi}.f(\xi)$. It is clear that the bilateral forward shift U on L^2 has the following form with respect to the decomposition $L^2 = \mathbb{H}^2\oplus\mathbb{H}^2_{-}$:

$$U = \begin{bmatrix} S & \mathbf{1} \otimes z^{-1} \\ 0 & S_- \end{bmatrix}. \tag{6}$$

For a Borel set $\alpha \subset \mathbb{T}$, we write $L^2(\alpha) = L^2(\alpha, m), L^\infty(\alpha) = L^\infty(\alpha, m)$ and the operator of multiplication by the identity function z on the space $L^2(\alpha)$ will be denoted by U_α.

Definition 1 (See [11]). A dissymmetric weight is a non-increasing, unbounded function ω: $\mathbb{Z} \longrightarrow (1, \infty)$ satisfying the following conditions:

(1) $\omega(n) = 1, n \geq 0$

(2) $\limsup \omega(n-1)/\omega(n) < \infty$

(3) $\omega(-n)^{1/n} \longrightarrow 1$ when $n \longrightarrow \infty$

Definition 2.

(1) An inner function is a bounded analytic function f on $\mathbb{D}$ such that $|f(z)| = 1$ for almost every z in $\mathbb{T}$, where $f(z)$ is the radial limit of f (i.e., $f(z) = \lim_{r \longrightarrow 1^-} f(rz)$).

(2) Let μ be a positive, finite singular (with respect to the Lebesgue measure m) Borel measure on $\mathbb{T}$. A singular inner function is an analytic function defined by

$$\phi_\mu(z) = \exp\left(-\int \frac{\zeta + z}{\zeta - z} d_\mu \zeta\right), z \in \mathbb{D}, \tag{7}$$

If $\mu = \delta_1$ denotes the point mass at $\zeta = 1$, then

$$\phi_{\delta_1}(z) = \exp\left(\frac{z+1}{z-1}\right), z \in \mathbb{D}, \tag{8}$$

This type of inner function is called an (singular) atomic inner function.

(3) An outer function is an analytic function F on $\mathbb{D}$ of the form

$$F(z) = \exp^{i\gamma}\left(\int \frac{\zeta + z}{\zeta - z} \phi(\zeta) d_\mu \zeta\right), \tag{9}$$

where γ is a real constant and ϕ is a real-valued function in L^1.

Remark 3. It is well known that the only nonconstant invertible inner functions in the Hardy spaces are the outer functions. For more details, see [17, 18].

Theorem 4 (See [11]). *Let ω be a dissymmetric weight. Then, there is a singular inner function ϕ such that $m(supp\mu_\phi) = 0$ and*

$$\sum_{n=0}^{\infty} \frac{1}{\omega^2(-n-1)} \left|\widehat{\frac{1}{\phi}}(n)\right|^2 < \infty. \tag{10}$$

Lemma 5 (See [19]). *Let $(\beta_n)_{n \geq 0}$ be a sequence of positive numbers such that $\beta_n \longrightarrow \infty$. Then, there exists a dissymmetric weight ω such that $\omega(-n-1) \leq \beta_n$ for sufficiently large n.*

Definition 6. An operator $T \in B(H)$ is said to be polynomially bounded if there exists $C > 0$ such that $\|P(T)\| \leq C\|P\|_\infty$ for every polynomial P, where $\|P\|_\infty = \sup_{|z|<1} |P(z)|$.

We denote by $PB(H)$ for the set of polynomially bounded operator in $B(H)$. It is well known, by von Neumann's inequality, that every contraction operator is polynomially bounded.

Proposition 7 (See [19]). *Let $T \in PB(H)$ be an absolutely continuous operator and let ϕ be a singular inner function. If*

$$\sum_{n=0}^{\infty} \left|\widehat{\frac{1}{\phi}}(n)\right| \| T^{*n} x \| < \infty, \tag{11}$$

for some $x \in H$, then

$$\phi(T^*)\left(\sum_{n=0}^{\infty} \widehat{\frac{1}{\phi}}(n) T^{*n} x\right) = x. \tag{12}$$

Lemma 8 (See [7]). *Let $T \in B(H)$. If T is a polynomially bounded operator, then there is a contraction operator A such that $A \prec T$. Conversely, if T is a contraction operator, then there is a polynomially bounded operator A such that $A \prec T$.*

3. Similarity of Operators

Let R be an operator on $K = \mathbb{H}^2 \oplus H$ defined for every $x \in H$ by

$$R = R_x = \begin{bmatrix} S & 1 \otimes x \\ 0 & T \end{bmatrix}. \tag{13}$$

Set

$$\begin{aligned} Z_T &= \{x \in H: R_x \in PB(H)\}, \\ B_T &= \{x \in H: \exists L \in B(H, \mathbb{H}^2), \mathbf{1} \otimes x = SL - LT\}. \end{aligned} \tag{14}$$

Following [20], Z_T and B_T are called the subspaces of cocycles and coboundaries, respectively.

Proposition 9 (See [20])

(1) $x \in Z_T$ if and only if $\sum_{n=0}^{\infty} |\langle h, T^{*n} x\rangle|^2 < \infty$, for every $h \in H$.

(2) $x \in B_T$ if and only if the operator matrix R_x is similar to $S \oplus T$.

It is clear that $B_T \subseteq Z_T$ for every $T \in PB(H)$ and B_T, Z_T are hyperinvariant subspaces (not necessarily closed) for T^*.

Remark 10. We note here that if $T = U$ is a unitary operator on $L^2(\alpha)$, then $B_U = Z_U = L^\infty(\alpha)$. If $T = S$ is a unilateral shift on $\mathbb{H}^2$, then $B_S = Z_S = \mathbb{H}^\infty$ (see [20, 21] for further details).

Let Y be an operator (not necessarily bounded) from H to $\mathbb{H}^2$ defined by

$$Yh = \sum_{n=0}^{\infty} \langle h, T^{*n}x\rangle z^n; x \in H. \tag{15}$$

It is easy to check that $S^*Y = YT$ for every bounded operator T on H.

Lemma 11. *Let $T \in PB(H)$. Then, Y is a bounded operator from H to $\mathbb{H}^2$ for all $x \in ranX^* D_R$, where X is a quasiaffinity in $I(T,R)$ and R is a contraction operator.*

Proof. According to Lemma 8, there exists a contraction operator R and a quasiaffinity X in $I(T,R)$. Let $x = D_R y$. Then, for all $h \in H$, we have

$$\| Yh\|^2 = \sum_{n=0}^{\infty} |\langle h, R^{*n} D_R y\rangle|^2 = \sum_{n=0}^{\infty} |\langle D_R R^n h, y\rangle|^2 \le \|y\|^2 \sum_{n=0}^{\infty} \|D_R R^n h\|^2. \tag{16}$$

A simple calculation shows, for all $h \in H$, that

$$\|D_R R^n h\|^2 = \left\langle R^{*n} D_R^2 R^n h, h\right\rangle = \langle R^{*n} R^n h, h\rangle - \left\langle R^{*n+1} R^{n+1} h, h\right\rangle = \|R^n h\|^2 - \|R^{n+1} h\|^2. \tag{17}$$

Hence,

$$\|Yh\|^2 \le \|y\|^2 \left(\|h\|^2 - \|A^{\frac{1}{2}} h\|^2 \right); \forall h \in H, \forall y \in H, \tag{18}$$

where A is the strong limit defined in Section 2 for the contraction R. Hence, an easy computation shows that the following operator

$$Yh = \sum_{n=0}^{\infty} \langle h, T^{*n} X^* D_R x\rangle z^n, \tag{19}$$

is bounded for every $x \in H$. □

Proposition 12. *Let $T \in B(H)$. Then, the following conditions are equivalent:*

(1) $B_T \neq \{0\}$

(2) The equation $S^*XT = X$ has nontrivial solutions in $B(H, \mathbb{H}^2)$

(3) There exist nonzero operators $X \in I(T,U)$ such that $X^* z^{-1} \in B_T$

Proof. (1) ⇔ (2): If $B_T \neq \{0\}$, then there is $L \neq \{0\} \in B(H, \mathbb{H}^2)$ such that $\mathbf{1} \otimes k = SL - LT$; $k \in B_T$. Hence, $S^*(\mathbf{1} \otimes k) = L - S^*LT = 0$, and then $S^*LT = L$.

Conversely, let L be a nonzero solution of the equation $S^*XT = X$, then $S^*(LT - SL) = 0$. So, we either have (i) $SL = LT$ or (ii) $\overline{\text{ran}}(SL - LT) \subseteq \ker S^*$.

(i). If $\overline{\text{ran}}L = \mathbb{H}^2$ that is $T \prec^d S$, then by Remark 10, there exist $\phi \in \mathbb{H}^\infty$ and $L' \in B(\mathbb{H}^2), L \in B(H, \mathbb{H}^2)$ such that $\mathbf{1} \otimes L^*\phi = SL'L - L'LT$. Thus, $L^*\phi \in B_T$. If $\overline{\text{Ran}L} \neq \mathbb{H}^2$, then $\overline{\text{ran}}L$ is a nontrivial invariant subspace for S. Hence, by Beurling's theorem, there exists an inner function θ such that $\overline{\text{ran}}L = \theta\mathbb{H}^2$ and the restriction $S|_{\theta\mathbb{H}^2}$ is a unilateral shift on $\theta\mathbb{H}^2$. Thus, $T \prec^d S|_{\theta\mathbb{H}^2}$. So, using the same argument as in (i), we get $B_T \neq \{0\}$. (ii). If $\overline{\text{ran}}(SL - LT) \subseteq \ker S^*$, then for every $h \in H$, there is a scalar α_h such that $(LT - SL)h = \alpha_h \mathbf{1}$. The function $H \longrightarrow \mathbb{C}; h \longmapsto \langle (LT - SL)h, \mathbf{1}\rangle = \alpha_h$ is a bounded functional. Hence, by Riesz representation's theorem, there exists $k \in H$ such that $\langle h, k\rangle = \alpha_h$ for every $h \in H$. Therefore, $\langle h, k\rangle \mathbf{1} = LTh - SLh$ for every $h \in H$. This means that $\mathbf{1} \otimes k = LT - SL$.

(1) ⇔ (3): If $B_T \neq \{0\}$, then there is $L \neq \{0\} \in B(H, \mathbb{H}^2)$ such that $\mathbf{1} \otimes x = SL - LT$, $k \in B_T$. Set

$$X = \begin{bmatrix} L \\ X_- \end{bmatrix}, \tag{20}$$

where $X\_h = \sum_{n=1}^{\infty} \langle h, T^{*n-1}x\rangle z^{-n}$ is an operator from H to $\mathbb{H}^2_-$ and $X^* z^{-1} = X_-^* z^{-1} = x \in B_T$.

By the same argument as in Lemma 11, it is seen that X_- is a bounded operator if $x \in B_T$, and therefore, X is a bounded operator.

An easy computation then shows that $XT = UX$, and as the converse is clear, the proof is complete.

In what follows we show that if T is a polynomially bounded operator of class $C_{1.}$, then $B_T \neq \{0\}$. □

Corollary 13. *If T is a polynomially bounded operator of class $C_{1.}$, then $B_T \neq \{0\}$, and if T is a contraction operator, then $T^* A(RanD_T) \subseteq B_T$.*

Proof. According to Lemma 8, we can suppose that T is a contraction operator. Then, by Lemma 11, we can find a certain y (in the range of D_T) such that the operator Y: $Yh = \sum_{n=0}^{\infty} \langle h, T^n y\rangle z^n$ is a bounded operator

Since $S^*Y = YT^*$, by Subsection 2.2, we get

$$S^*YAT = YT^*AT = YA. \tag{21}$$

Since the strong limit A is an injective positive operator ($T \in C_{1.}$), $YA \neq 0$. Hence, the equation $S^*LT = L$ in $B(H, \mathscr{H}^2)$ has a nontrivial solution.

$$L = YA. \tag{22}$$

The result now follows from Proposition 12. It follows from the proof of the previous proposition that the solutions of the equation $S^*LT = L$ in $B(H, \mathscr{H}^2)$ have the form $L = YA$, where $Y \in I(T^*, S^*)$. If $x \in B_T$, then there is $L = YA$ such that $\mathbf{1} \otimes x = SYA - YAT$. An easy computation shows that $x = -T^*Ay$ for some $y \in \mathrm{ran}_{D_T}$.

Now, we recall some well-known facts: an operator T is said to be binormal if T^*T and TT^* commute, see [5, 20]. An operator T is said to be centered if the following sequence

$$\ldots T^3T^{*3}, T^2T^{*2}, TT^*, T^*T, T^{*2}T^2, T^{*3}T^3 \ldots \tag{23}$$

is commutative. In [4], Morrel and Muhly showed some properties and obtained a nice structure of centered operators. We also recall that binormal operators are called weakly centered operators in [18]. The following result is due to V. Paulsen, C. Pearcy, and S. Petrovic [18]. □

Theorem 14. *Every power bounded centered operator is similar to a contraction.*

It is easy to see that the following results hold true.

Lemma 15. *The class of binormal is self-adjoint and closed under multiplication by complex numbers, taking inverses and formation of direct sums.*

Proposition 16. *If T is a centered contraction operator, then the asymptotic limit A commutes with D_T^2.*

Proof. Since A is the strong limit of the sequence $T^{*n}T^n$, $n \geq 1$,

$$\begin{aligned} \|\left(AD_T^2 - D_T^2A\right)x\| &= \|(AT^*T - T^*TA)x\| \\ &= \|AT^*Tx - T^{*n}T^nT^*Tx - T^*T\left(Ax - T^{*n}T^nx\right) + \\ &\quad + T^{*n}T^nT^*Tx - T^*TT^{*n}T^nx\|. \end{aligned} \tag{24}$$

Since T is a centered operator, T^*T commutes with $T^{*n}T^n$ for all $n \geq 1$, and therefore,

$$\|\left(AD_T^2 - D_T^2A\right)x\| = \|AT^*Tx - T^{*n}T^nT^*Tx - T^*T\left(Ax - T^{*n}T^nx\right)\|. \tag{25}$$

Hence,

$$\|\left(AD_T^2 - D_T^2A\right)x\| \leq \|AT^*Tx - T^{*n}T^nT^*Tx\| + \|T^*T\| \, \|Ax - T^{*n}T^nx\| \longrightarrow_n 0. \tag{26}$$

This means that $AD_T^2x = D_T^2Ax$, for all $x \in H$. Accordingly, $AD_T^2 = D_T^2A$, as needed.

As a consequence of Corollary 13 as well as the preceding proposition, we obtain □

Corollary 17. *If T is a centered contraction operator of class $C_{1.}$, then*

$$D_T \mathrm{ran} A \subseteq B_T. \tag{27}$$

4. Hyperinvariant Subspaces

First, we recall some well-known facts in complex analysis: for every analytic function f in $\mathbb{D}$ the function $\widetilde{f}$ defined on $\mathbb{D}$ by $\widetilde{f}(z) = \overline{f(\overline{z})}$ is analytic in $\mathbb{D}$ and $\widehat{\widetilde{f}}(n) = \overline{\widehat{f}}(n), n \geq 0$.

If T is an absolutely continuous (PB)-operator and $f \in \mathbb{H}^\infty$, then $f(T^*)^* = \widetilde{f}(T)$, see [2]. For $f \in \mathbb{H}^\infty$ and for $t \in \mathbb{T}$, we set

$f_t(z) = f(tz)$, for every $z \in \mathbb{D}$.

Then,

$$\widetilde{f}_t = \widetilde{f}\,\overline{t}. \tag{28}$$

If f is a singular inner function, then it has no zeros in $\mathbb{D}$, and so the function $1/f$ is analytic in $\mathbb{D}$, i.e., $1/f(z) = \sum_{n=0}^{\infty} \widehat{1/f}(n) z^n, z \in \mathbb{D}$.

From the equality $f(z).1/f(z) = 1$, we get

$$\begin{aligned} &\widehat{f}(0).\widehat{\frac{1}{f}}(0) = 1, \sum_{k=0}^{k=n} \widehat{f}(n-k)\widehat{\frac{1}{f}}(k) = 0, n \geq 1,\\ &\widehat{\frac{1}{f_t}}(n) = \widehat{\frac{1}{f}}(n)t^n, n \geq 0. \end{aligned} \tag{29}$$

Theorem 18. *Let $T \in PB(H)$ be an absolutely continuous operator. If*

$$\sum_{n=0}^{\infty} \left|\widehat{\frac{1}{\theta}}(n)\right| \| T^{*n} x \| < \infty, \tag{30}$$

for some nonzero $x \in B_T$, then T has nontrivial hyperinvariant subspaces.

Proof. Let $0 \neq x \in B_T$. Then, by Proposition 12, there exists

$$X = \begin{bmatrix} X_+ \\ X_- \end{bmatrix}, \tag{31}$$

in $I(T, U)$, where $X\_h = \sum_{n=0}^{\infty} \langle h, T^{*n-1} x\rangle z^{-n}$ is an operator from H to $\mathbb{H}^2_-$ and $X^* z^{-1} = X_-^* z^{-1} = x$. Since $X_+ \in B(H, \mathbb{H}^2)$, X_+ may be written as $X_+ h = \sum_{n\geq 0} \langle X_+ h, z^n\rangle z^n$.

Set $X_+^* z^n = x_n, n \geq 0$. If $f = \sum_{n\geq 0} \widehat{f}(n) z^n \in \mathbb{H}^2$, then

$$X_+^* f = \sum_{n\geq 0} \widehat{f}(n) x_n. \tag{32}$$

It then follows from $SX_+ - X_+ T = -1 \otimes x$, by an easy computation, that $x = T^* x_0$.

Similarly, we have

$$X^* \widetilde{\phi}_t z^{-1} = [X_+^*, X_-^*]\widetilde{\phi}_t z^{-1} = X_+^* \psi_t + X_-^* \overline{\widehat{\phi}}(0) z^{-1} = X_+^* \psi_t + \overline{\widehat{\phi}}(0) x, \tag{33}$$

where $\widehat{\psi}_t(n) = \overline{\widehat{\phi}}(n+1), n \geq 0$.

On the other hand, we have

$$\phi_t(T^*)\left(X^* \widetilde{\phi}_t z^{-1}\right) = X^* \phi_t(U^*)\left(\widetilde{\phi}_t z^{-1}\right) = X^*\left(z^{-1}\right) = x. \tag{34}$$

By Proposition 7,

$$\phi_t(T^*)\left(\sum_{n=0}^{\infty} \widehat{\frac{1}{\phi_t}}(n) T^{*n} x\right) = x, \tag{35}$$

so that

$$\phi_t(T^*)\left(\sum_{n=0}^{\infty} \widehat{\frac{1}{\phi_t}}(n) T^{*n} x - X^* \widetilde{\phi}_t z^{-1}\right) = 0. \tag{36}$$

Suppose to the contrary that, for every $t \in \mathbb{T}$, $\phi_t(T^*)$ has no nonzero eigenvectors, that is,

$$\forall t \in \mathbb{T}, X^*\left(\widetilde{\phi}_t z^{-1}\right) = \sum_{n=0}^{\infty} \widehat{\frac{1}{\phi_t}}(n) T^{*n} x. \tag{37}$$

Then,

$$\forall h \in H, \forall t \in \mathbb{T}, \left\langle X^*\left(\widetilde{\phi}_t z^{-1}\right), h\right\rangle = \sum_{n=0}^{\infty} \widehat{\frac{1}{\phi_t}}(n) \langle T^{*n} x, h\rangle. \tag{38}$$

Hence, by relations (22), (41), (42), and (43), we get

$$\forall h \in H, \forall t \in \mathbb{T}, \sum_{n\geq 0} \overline{\widehat{\phi}}(n+1) \langle x_n, h\rangle \overline{t}^n + \overline{\widehat{\phi}}(0) x = \sum_{n\geq 0} \widehat{\frac{1}{\phi}}(n) \langle T^{*n} x, h\rangle t^n. \tag{39}$$

In particular,

$$\forall h \in H, \widehat{\phi}(1)\langle x_0, h\rangle = \left(\widehat{\frac{1}{\phi}}(0) - \overline{\widehat{\phi}}(0)\right)\langle x, h\rangle. \tag{40}$$

Since $x = T^* x_0$, it is seen that

$$\forall h \in H, \widehat{\phi}(1)\langle x_0, h\rangle = \left(\widehat{\frac{1}{\phi}}(0) - \overline{\widehat{\phi}}(0)\right)\langle T^* x_0, h\rangle. \tag{41}$$

Therefore,

$$\widehat{\phi}(1) x_0 = \left(\widehat{\frac{1}{\phi}}(0) - \overline{\widehat{\phi}}(0)\right) T^* x_0, \tag{42}$$

by (40), we have $\widehat{1/\phi}(0) \neq \widehat{\phi}(0)$, and therefore, $\lambda = \widehat{\phi}(1)/\widehat{1/\phi}(0) - \overline{\widehat{\phi}}(0)$ is an eigenvalue for T^*.

So, if the point spectrum of T^* is empty, then there exists $t \in \mathbb{T}$ such that

$$X^*\phi_t\left(z^{-1}\right) \neq \sum_{n=0}^{\infty} \widehat{\frac{1}{\phi_t}}(n) T^{*n} x. \tag{43}$$

Thus, by (44), $\overline{\operatorname{ran}\widetilde{\phi}_t(T)}$ are nontrivial hyperinvariant subspaces for T. □

Corollary 19. *Let $T \in PB(H)$. If there exists a solution X to the equation $S^*XT = X$ such that*

$$\sum_{n=0}^{\infty} \left|\widehat{\frac{1}{\phi}}(n)\right| \| T^{*n} X^* z^{-1} \| < \infty, \tag{44}$$

then either the point spectrum of T^* is not empty or T has nontrivial hyperinvariant subspaces of the form $\overline{\operatorname{ran}\phi(T^*)}$, where ϕ is a singular inner function.

Remark 20. Since the operator

$$R = \begin{bmatrix} S & 1\otimes x \\ 0 & T \end{bmatrix}, \tag{45}$$

is of class C_{10}, by Proposition 9 and Corollary 13, there exists a dense linear manifold B_T for T (otherwise it will be a nontrivial hyperinvariant subspace for T^*) such that the operator R is similar to $S \oplus T$. Hence, Theorem 18 means that if $T \in PB(H)$ is an absolutely continuous operator and there exists some $x \in H$ such that R is similar to $S \oplus T$, and

$$\sum_{n=0}^{\infty} \left|\widehat{\frac{1}{\phi}}(n)\right| \| T^{*n} x \| < \infty, \tag{46}$$

then either the point spectrum of T^* is not empty or T has nontrivial hyperinvariant subspaces of the form $\overline{\operatorname{ran}\phi(T^*)}$, where ϕ is a singular inner function.

In the sequel, we give some applications of the previous theorem.

Proposition 21. *If T is a $C_{.0}$−contraction quasinormal operator, then for every nonzero $h \in \operatorname{ran} D_T$, there is an increasing sequence of positive numbers $(\alpha_n)_{n\geq 0}$: $\alpha_0 = 0$ and $\alpha_n \longrightarrow \infty$ such that*

$$\sum_{n=0}^{\infty} \alpha_{n+1} \| T^{*n} h \|^2 < \infty. \tag{47}$$

Proof. Let $h = D_{T^*}x$; $0 \neq x \in H$. Recall that if T is a quasinormal operator, then $T^*D_T = D_T T^*$. Hence,

$$\| T^{*n} D_T x \|^2 = \| D_T T^{*n} x \|^2 = \left\langle x, T^n D_T^2 T^{*n} x \right\rangle = \| T^{*n} x \|^2 - \| T T^{*n} x \|^2. \tag{48}$$

By the hyponormality of T, we get that $\|T^{*n}D_{T*}x\|^2 \leq \|T^{*n}x\|^2 - \|T^{*n+1}x\|^2$ for every $n \geq 0$. Since T is of class $C_{.0}$, we claim that there is a singular inner function ϕ such that

$$\sum_{n=0}^{\infty} \left|\widehat{\frac{1}{\phi}}(n)\right|^2 \| T^{*n} h \|^2 < \infty. \tag{49}$$

Indeed, by Lemma 5, there exists a dissymmetric weight ω such that $\omega(-n-1) \leq 1/\|T^{*n}h\|$ for sufficiently large n. By Theorem 4, there exists a singular inner function ϕ *such that* $\sum_{n=0}^{\infty} 1/\omega^2(-n-1)|\widehat{1/\phi}(n)|^2 \leq \infty$. For sufficiently large m, we get

$$\sum_{n=m}^{\infty} \left|\widehat{\frac{1}{\phi}}(n)\right|^2 \| T^{*n} h \|^2 \leq \sum_{n=m}^{\infty} \frac{1}{\omega^2(-n-1)} \left|\widehat{\frac{1}{\phi}}(n)\right|^2. \tag{50}$$

That means that $\sum_{n=0}^{\infty} |\widehat{1/\phi}(n)|^2 \|T^{*n}h\|^2 < \infty$, for every $h \in H$.

Next, let $(\alpha_n)_n$ be a sequence defined by $\alpha_0 = 0$; $\alpha_{n+1} = \sum_{n=0}^{\infty} |\widehat{1/\phi}(n)|^2, n \geq 0$. It is clear that $(\alpha_n)_n$ is a positive increasing sequence. By Remark 3, $1/\phi \notin \mathbb{H}^2$. That is, $(\alpha_n)_n$ is an unbounded sequence ($\alpha_n \longrightarrow \infty$). An easy computation shows that

$$\begin{aligned} \sum_{k=0}^{n} \alpha_{k+1} \| T^{*k} h \|^2 &\leq \sum_{k=0}^{n} \alpha_{k+1} \left(\| T^{*k} x \|^2 - \| T^{*k+1} x \|^2 \right) \\ &= \sum_{k=0}^{n} (\alpha_{k+1} - \alpha_k) \| T^{*k} x \|^2 - \alpha_{n+1} \| T^{*n} x \|^2 \\ &= \sum_{k=0}^{n} \left|\widehat{\frac{1}{\phi}}(k)\right|^2 \| T^{*k} x \|^2 - \alpha_{n+1} \| T^{*n} x \|^2, \end{aligned} \tag{51}$$

for every $n \geq 0$.

Therefore, $\sum_{n=0}^{\infty}\alpha_{k+1}\|T^{*k}h\|^2 \leq \sum_{k=0}^{n}|\widehat{1/\phi}(k)|^2\|T^{*k}x\|^2$, for every $n \geq 0$.

Thus, by (45), we get $\sum_{n=0}^{\infty}\alpha_{k+1}\|T^{*k}h\|^2 < \infty$. □

Lemma 22. *If T is a contraction quasinormal operator, then $D_T^2(B_T) \subseteq B_T$.*

Proof. If $x \in B_T$, then there exists $L \in B(H, \mathbb{H}^2)$ such that

$$1 \otimes x = SL - LT. \quad (52)$$

Multiplying (52) by D_T^2 and using the quasinormality of T give.

$$1 \otimes D_T^2 x = SLD_T^2 - LD_T^2 T. \quad (53)$$

In other words, $D_T^2 x \in B_T$, for each $x \in B_T$. □

Theorem 23. *Let T be a $C_{1.}$-quasinormal operator, then either the point spectrum of T^* is nonempty or T has nontrivial hyperinvariant subspaces of the form $\overline{\operatorname{ran}\phi(T^*)}$, where ϕ is a singular inner function.*

Proof. According to Theorem 18 and the previous lemma, it suffices to show that

$$\sum_{n=0}^{\infty}\left|\widehat{\frac{1}{\phi}}(n)\right|\|T^{*n}D_T^2 y\| < \infty, y \in B_T. \quad (54)$$

Set $x = D_T^2 y$. By Proposition 21, there exists a positive unbounded sequence $(\alpha_n)_n$ such that (47) holds.

By Lemma 5, there exists a dissymmetric weight ω such that

$$\omega(-n-1) \leq \sqrt{\alpha_n}, \quad (55)$$

for sufficiently large n. Also, by Theorem 4, there exists a singular inner function θ satisfying (10). Therefore,

$$\begin{aligned}\sum_{n=0}^{\infty}\left|\widehat{\frac{1}{\theta}}(n)\right|\|T^{*n}x\| &= \sum_{n=0}^{\infty}\left|\widehat{\frac{1}{\theta}}(n)\right|\frac{1}{\sqrt{\alpha_n}}\sqrt{\alpha_n}\|T^{*n}x\| \\ &\leq \sum_{n=0}^{\infty}\left|\widehat{\frac{1}{\theta}}(n)\right|^2\frac{1}{\alpha_n}\sum_{n=0}^{\infty}\alpha_n\|T^{*n}x\|^2 \\ &\leq \sum_{n=0}^{\infty}\left|\widehat{\frac{1}{\theta}}(n)\right|^2\frac{1}{\omega^2(-n-1)}\sum_{n=0}^{\infty}\alpha_n\|T^{*n}x\|^2 \\ &< \infty.\end{aligned} \quad (56)$$

Finally, the result follows from Theorem 18, and this completes the proof.

As a consequence of Theorem 18 and Corollary 17, we get the following result that gives a refinement of the condition cited in Theorem 18 for the centered $C_{1.}$-operators. □

Theorem 24. *Let T be a $C_{1.}$-centered operator. If there exists $h \in H$ such that*

$$\sum_{n=0}^{\infty}\left|\widehat{\frac{1}{\phi}}(n)\right|\|T^{*n}D_T A h\| < \infty, \quad (57)$$

for some singular inner function ϕ, then T has nontrivial hyperinvariant subspaces of the form $\overline{\operatorname{ran}\phi(T)}$.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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