# An RBF-LOD Method for Solving Stochastic Diffusion Equations 

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#### Abstract

In this study, we introduce an innovative approach to solving stochastic equations in two and three dimensions, leveraging a timesplitting strategy. Our method combines radial basis function (RBF) spatial discretization with the Crank-Nicolson scheme and the local one-dimensional (LOD) method for temporal approximation. To navigate the probabilistic space inherent in these equations, we employ the Monte Carlo method, providing accurate estimates for expectations and variations. We apply our approach to tackle challenging problems, including two-dimensional convection-diffusion and Burgers' equations, resulting in reduced computational and memory requirements. Through rigorous testing against diverse problem sets, our methodology demonstrates efficiency and reliability, underscoring its potential as a valuable tool in solving complex multidimensional stochastic equations. We have validated the method's stability and showcased its convergence as the number of collocation points increases. These findings serve as compelling evidence of the suggested method's convergence properties.


## 1. Introduction

The realm of mathematical modeling is often confronted with the inherent uncertainty introduced by perturbations in various physical phenomena. To establish dependable models capable of capturing this uncertainty, the utilization of stochastic partial differential equations (SPDEs) becomes imperative [1-3]. The emergence of SPDEs as a captivating research domain in recent years has garnered significant attention from scholars worldwide.

While the literature on SPDEs is extensive, the scarcity of exact solutions necessitates the use of numerical techniques for their resolution [4-8]. Researchers such as Yoo [9], Yan [10], Nouy [11], and Ye [12] have applied diverse numerical methods, including finite difference, finite element, and kernel-based collocation methods, to address the challenges posed by stochastic PDEs.

Among the array of numerical techniques available, mesh-free methods, which have gained substantial popularity, hold promise for tackling these complex equations [13]. Specifically, the collocation method based on radial basis functions (RBFs), belonging to the class of mesh-free
methods, has been widely adopted by researchers for both deterministic and stochastic PDEs [14-18]. In this study, our main focus is on solving stochastic diffusion equations using the collocation method based on radial basis functions (RBFs). The RBF method has a distinct advantage as it does not rely on mesh discretization and can be applied to unstructured node sets. This makes it particularly suitable for solving problems in complex and irregular domains, where geometric details may be intricate. Table 1 provides definitions for the most widely used RBFs, where the Euclidean norm is denoted by $r$, and the shape parameter $\epsilon$ controls the influence of the RBF.

The computational cost associated with solving two- or three-dimensional partial differential equations (PDEs) is known to be significant. However, this challenge can be effectively addressed through the utilization of time-splitting methods. These methods aim to reduce the dimensionality of the algebraic system involved in the numerical scheme. By decomposing complex time-dependent problems into simpler subsets, the numerical methods can be solved independently and efficiently. Then, the existing numerical methods can be solved separately and shortly. Notably,

Table 1: Some well-known functions that generate RBFs.

| Name of function | Definition |
| :--- | :---: |
| Multiquadric (MQ) | $\psi(r)=\sqrt{\epsilon^{2}+r^{2}}$ |
| Inverse multiquadric (IMQ) | $\psi(r)=1 / \sqrt{\epsilon^{2}+r^{2}}$ |
| Inverse quadric (IQ) | $\psi(r)=1 / \epsilon^{2}+r^{2}$ |
| Gaussian (GA) | $\psi(r)=\exp \left(-\epsilon^{2} r^{2}\right)$ |
| Polyharmonic spline (PHS) | $\psi(r)= \begin{cases}r^{2 n-1}, & n \in \mathbb{N}, \\ r^{2 n} \ln r & n \in \mathbb{N} \\ \text { Thin plate spline (TPS) } & \psi(r)=r^{2} \log r\end{cases}$ |

recent studies have explored the application of time-splitting approaches for multidimensional equations [19, 20]. In this particular investigation, we employ the local onedimensional (LOD) time-splitting technique to tackle the two- and three-dimensional versions of stochastic diffusion equations [21]. In the subsequent subsection, a concise overview of stochastic PDEs will be presented.
1.1. Stochastic Diffusion Equations. Consider the stochastic advection-diffusion equation as a popular stochastic problem. Let $(\mathscr{D}, \mathscr{F}, \mathscr{P})$ be a probability space, where $\mathscr{D}$ is the space of basic outcomes, $\mathscr{F}$ is a $\sigma$-algebra related to $\mathscr{D}$, and $\mathscr{P}$ is a measure defined on $\mathscr{F}$. Many phenomena in biology, chemistry, physics, and population dynamics can be expressed in the format of a stochastic diffusion equation with the following form:

$$
\begin{cases}\partial_{t} u=\partial_{x x} u+\partial_{x} f(u)+\sigma W_{t}(x, t), & \text { in } \Omega,  \tag{1}\\ u(t, x, w)=0, & t \in[0, T] \\ u(0, x, w)=u_{0}(x, w), & (x, w) \in \Omega \times \mathscr{D}\end{cases}
$$

where $\Omega \subset \mathbb{R}^{d}, d=1,2, \cdots, n$ is a bounded domain and $\sigma>0$ is a constant. $W_{t}(x, t)$ is a two-dimensional random noise related to the Brownian motion $W(x, t)$, which is assumed that it depends on time and space with zero mean and spatial covariance function $Q$ given by

$$
\begin{equation*}
E(W(x, t), W(y, s))=\min \{t, s\} Q(x, y), \quad t, s>0(x, y) \in \mathbb{R}^{d} . \tag{2}
\end{equation*}
$$

Equation (1) represents a well-known problem in industry and mathematics that is a focal point for numerous researchers who have endeavored to find its optimal solutions or have utilized this set of equations as a means to showcase the effectiveness of their proposed methodologies. Discussions have already unfolded concerning the existence and uniqueness of solutions within this class of equations [22, 23]. To describe the white noise different forms can be used of the covariance function. In this technique, we define it in a way that is proportional to the chosen radial basis function.

The structure of this paper is outlined as follows: Section 2 provides a concise overview of the implementation of the radial basis function (RBF) method, while Section 3 introduces the local one-dimensional (LOD) method for addressing time-dependent problems. In Section 4, we present the general form of stochastic diffusion
equations. To assess the efficacy of the LOD-RBF method, we employ illustrative examples and analyze its performance. Section 5 concludes this paper by summarizing the key findings and highlighting the main conclusions derived from our study.

## 2. Collocation Method Based on Radial Basis Functions

In this section, we elucidate the process of approximating numerical solutions using the radial basis function (RBF) method for one-dimensional problems. Subsequently, this procedure can be extended to encompass two-and threedimensional problems.

To illustrate, let us consider a stochastic diffusion equation expressed in the following form:

$$
\begin{equation*}
u_{t}+v u_{x}-\frac{1}{R e} u_{x x}=\mathrm{g}(u)+f(x, t)+\sigma W_{t}(x, t) \quad(x, t) \in \Omega \tag{3}
\end{equation*}
$$

$$
\begin{align*}
u(x, 0) & =q(x), \quad x \in[a, b]  \tag{4}\\
u(a, t) & =u(b, t)=0 \quad t \in[0, T] .
\end{align*}
$$

where $\nu$ is the diffusion coefficient and $\mathrm{g}(u)$ is a nonlinear function. Furthermore, $f(x, t), q(x)$ are given functions, and $\Omega:=[a, b] \times[0, T]$ is the computational domain.

For time discretization, we apply the Crank-Nicolson scheme between two consecutive time steps. Let $\eta:=T / N$ be the time step size

$$
\begin{align*}
0 & =t_{0}<t_{1}<\cdots, t_{N-1}<t_{N}=T \\
u^{n} & =u\left(x, t_{n}\right), \quad n \in\{0,1, \cdots, N\}  \tag{5}\\
f^{n} & =f\left(x, t_{n}\right) \quad n \in\{0,1, \cdots, N\} .
\end{align*}
$$

For $n \in\{0,1, \cdots, N\}$, applying a forward finite difference scheme for (3) leads to the following equation:

$$
\begin{equation*}
\frac{u^{n+1}-u^{n}}{\eta}+v u_{x}^{n+1 / 2}-\frac{1}{R e} u_{x x}^{n+1 / 2}=g^{n+1 / 2}(u)+f^{n+1 / 2}+\zeta_{x} \tag{6}
\end{equation*}
$$

where

$$
\begin{equation*}
\delta w_{n}:=w_{t_{n}}-w_{t_{n-1}}, \zeta_{x}=\sigma \delta w_{n} \tag{7}
\end{equation*}
$$

Here, the random variable $\zeta_{x}$, has the following mean and covariance function:

$$
\begin{align*}
E\left[\zeta_{x}\right] & =0 \\
E\left[\zeta_{x}, \zeta_{y}\right] & =\sigma^{2} \eta \mathrm{Q}(x, y),(x, y) \in \mathbb{R}^{d} \tag{8}
\end{align*}
$$

After time discretization, the Wiener process transforms into a Gaussian field. Then, in order to present a space discretization for (6), we apply a collocation method based on the radial basis function at fixed collocation points. We choose the collocation points

$$
\begin{equation*}
\xi=\left\{x_{i}\right\}_{i=0}^{M} \subset(\Omega \cup \partial \Omega), \tag{9}
\end{equation*}
$$

where $x_{i}$ for $i=1, \cdots, M$ belongs to the domain and for $i=$ $0, M$ belongs to the boundary. Considering $\|\cdot\|$ as the Euclidean norm, the RBF method strives to approximate solution $u^{n}$ using a linear combination of RBFs $\psi(\|\|$.$) at the the$ collocation points. Thus, we define the following estimate:

$$
\begin{equation*}
u\left(x, t_{n}\right)=u^{n}(x) \simeq \sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}(x), \quad n \in\{0,1, \cdots, N\} \tag{10}
\end{equation*}
$$

where $\psi(r):[0, \infty] \longrightarrow \mathbb{R}$ is a radial basis function that is defined by $\psi_{j}(x)=\psi\left(\left\|x-x_{j}\right\|\right)$. The coefficients $\lambda_{j}$, are unknown, and they should be determined at each time step.

The approximation (10) can be written at points $x_{i}$ as follows:

$$
\begin{equation*}
u^{n}\left(x_{i}\right) \simeq \sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}\left(x_{i}\right), \quad i=0, \cdots, M, \quad n \in\{0,1, \cdots, N\}, \tag{11}
\end{equation*}
$$

which can be written in the following matrix form:

$$
\begin{equation*}
u^{n}=\mathbf{A} \boldsymbol{\lambda}^{n}, \quad n \in\{0,1, \cdots, N\} \tag{12}
\end{equation*}
$$

where

$$
\begin{align*}
\mathbf{A} & =\left[\psi_{j}\left(x_{i}\right)\right]_{(M+1) \times(M+1)^{\prime}}  \tag{13}\\
\lambda^{n} & =\left[\lambda_{0}^{n}, \cdots, \lambda_{M}^{n}\right]^{T}, \quad n \in\{0,1, \cdots, N\} .
\end{align*}
$$

Considering the above notations, for $n \in\{0,1, \cdots, N\}$, the RBF collocation method for the discretization of (6) reads as: Find $\boldsymbol{\lambda}^{n} \in \mathbb{R}^{M+1}$ such that

$$
\begin{align*}
& \sum_{j=0}^{M} \lambda_{j}^{n+1} \psi_{j}\left(x_{i}\right)+\frac{v}{2} \eta \sum_{j=0}^{M} \lambda_{j}^{n+1} \psi_{j}^{\prime}\left(x_{i}\right)-\frac{\eta}{2 \operatorname{Re}} \sum_{j=0}^{M} \lambda_{j}^{n+1} \psi_{j}^{\prime \prime}\left(x_{i}\right) \\
& =\sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}\left(x_{i}\right)-\frac{v}{2} \eta \sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}^{\prime}\left(x_{i}\right)+\frac{\eta}{2 \operatorname{Re}} \sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}^{\prime \prime}\left(x_{i}\right) \\
& \quad+\eta g^{n+1 / 2}\left(\sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}\left(x_{i}\right)\right)+\eta f^{n+1 / 2}\left(x_{i}\right)+\zeta_{x i}, \quad i=1, \cdots, M-1 . \tag{14}
\end{align*}
$$

where

$$
\begin{align*}
& \psi_{j}^{\prime}\left(x_{i}\right)=\left.\frac{d}{d x} \psi_{j}(x)\right|_{x=x_{i}},  \tag{15}\\
& \psi_{j}^{\prime \prime}\left(x_{i}\right)=\left.\frac{d^{2}}{d x^{2}} \psi_{j}(x)\right|_{x=x_{i}} \tag{16}
\end{align*}
$$

Moreover, for the boundary conditions we have

$$
\begin{align*}
& \sum_{j=0}^{M} \lambda_{j}^{n+1} \psi_{j}\left(x_{0}\right)=0  \tag{17}\\
& \sum_{j=0}^{M} \lambda_{j}^{n+1} \psi_{j}\left(x_{M}\right)=0 \tag{18}
\end{align*}
$$

The equations result in the following nonlinear system:

$$
\begin{equation*}
\mathbf{P} \boldsymbol{\lambda}^{n+1}-\eta G\left(\boldsymbol{\lambda}^{n+1}\right)=\mathbf{P}_{1} \boldsymbol{\lambda}^{n}+\eta \mathbf{R}_{x}^{n}, \quad n \in\{0,1, \cdots, N\} \tag{19}
\end{equation*}
$$

where $G$ is defined as follows:

$$
\begin{equation*}
G\left(\lambda^{n+1}\right):=\frac{\left[\mathrm{g}\left(\sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}\left(x_{i}\right)\right)\right]^{n+1}+\left[\mathrm{g}\left(\sum_{j=0}^{M} \lambda_{j}^{n} \psi_{j}\left(x_{i}\right)\right)\right]^{n}}{2} \tag{20}
\end{equation*}
$$

and the matrices $\mathbf{P}$ and $\mathbf{P}_{1}$ and the right-hand side vector $\mathbf{R}$ can be obtained from (14)-(18) as follows:

$$
\begin{align*}
\mathbf{P} & =\mathbf{A}+\frac{\nu \eta}{2} \mathbf{A}_{1}-\frac{\eta}{2 \operatorname{Re}} \mathbf{A}_{2}, \\
\mathbf{P}_{1} & =\mathbf{A}-\frac{\nu \eta}{2} \mathbf{A}_{1}+\frac{\eta}{2 \operatorname{Re}} \mathbf{A}_{2}, \\
\mathbf{A}_{1} & =\left[\psi_{j}^{\prime}\left(x_{i}\right): i=1, \cdots, M-1, j=0, \cdots, M, \text { and } \quad 0: \text { elsewhere }\right], \\
\mathbf{A}_{2} & =\left[\psi_{j}^{\prime \prime}\left(x_{i}\right): i=1, \cdots, M-1, j=0, \cdots, M, \text { and } 0: \text { elsewhere }\right],  \tag{21}\\
\mathbf{R}_{x}^{n} & =\left[\frac{\left[f\left(x_{i}\right)\right]^{n+1}+\left[f\left(x_{i}\right)\right]^{n}}{2}+\zeta_{x_{i}}\right]_{i=0}^{M}, \\
\zeta_{x} \sim N(\mathbf{0}, \mathbf{Q}), \mathbf{Q} & =\left[\sigma \eta Q\left(x_{i}, x_{j}\right)\right]_{i, j=0}^{M} .
\end{align*}
$$

In order to solve the nonlinear system (19), we use a Newton's method.

## 3. The Locally One-Dimensional TimeSplitting Technique

In this study, we employ the local one-dimensional approach. Specifically, we examine the diffusion equation presented as follows:

$$
\begin{equation*}
\frac{\partial u}{\partial t}+D_{x} \frac{\partial u}{\partial x}+D_{y} \frac{\partial u}{\partial y}=V_{x} \frac{\partial^{2} u}{\partial x^{2}}+V_{y} \frac{\partial^{2} u}{\partial y^{2}}, \quad 0<t<T \tag{22}
\end{equation*}
$$

In the domain $0 \leq x \leq 1,0 \leq y \leq 1$ and certain boundary and initial conditions. $D_{x}, D_{y}$ are constant fluid velocities and $V_{x}, V_{y}$ are the dispersion coefficients in the $x$-direction and $y$-direction, respectively.

We split the above equation into two one-dimensional equations as follows:

$$
\begin{align*}
& \frac{1}{2} \frac{\partial u}{\partial t}+D_{x} \frac{\partial u}{\partial x}=V_{x} \frac{\partial^{2} u}{\partial x^{2}}  \tag{23}\\
& \frac{1}{2} \frac{\partial u}{\partial t}+D_{y} \frac{\partial u}{\partial y}=V_{y} \frac{\partial^{2} u}{\partial y^{2}} \tag{24}
\end{align*}
$$

Each of the above equations should be solved over half of the time step to obtain the approximated solution for the 2D advection-diffusion problem. These equations are easily solved using the schemes developed for the 1D ones. In addition, instead of solving (23) and (24) in each half-time step, the following equations can be solved over a full-time step:

$$
\begin{align*}
& \frac{\partial u}{\partial t}+D_{x} \frac{\partial u}{\partial x}=V_{x} \frac{\partial^{2} u}{\partial x^{2}}  \tag{25}\\
& \frac{\partial u}{\partial t}+D_{y} \frac{\partial u}{\partial y}=V_{y} \frac{\partial^{2} u}{\partial y^{2}} \tag{26}
\end{align*}
$$

To solve (25) and (26), we can apply any methods used for solving the one-dimensional advection-diffusion equation. In the same way, by applying the LOD approach to (3), we have two one-dimension equations as follows:

$$
\begin{align*}
& \left(\mathbf{A}+\mathbf{L}_{x}\right) \boldsymbol{\lambda}^{n+1 / 2}=\left(\mathbf{A}-\mathbf{L}_{x}\right) \boldsymbol{\lambda}^{n}+\frac{\eta}{2}\left(G\left(\boldsymbol{\lambda}^{n+1 / 2}\right)+\mathbf{R}_{x}^{n}\right)+\frac{1}{2} \zeta_{x},  \tag{27}\\
& \left(\mathbf{A}+\mathbf{L}_{y}\right) \boldsymbol{\lambda}^{n+1 / 2}=\left(\mathbf{A}-\mathbf{L}_{y}\right) \boldsymbol{\lambda}^{n}+\frac{\eta}{2}\left(G\left(\boldsymbol{\lambda}^{n+1 / 2}\right)+\mathbf{R}_{x}^{n}\right)+\frac{1}{2} \zeta_{y}, \tag{28}
\end{align*}
$$

where

$$
\begin{equation*}
\mathbf{L}_{x}=\mathbf{L}_{y}:=\frac{\nu \eta}{2} \mathbf{A}_{1}-\frac{\eta}{2 \operatorname{Re}} \mathbf{A}_{2} . \tag{29}
\end{equation*}
$$

In order to solve the nonlinear system (27) and(28), we use a Newton's method. We wish to emphasize that our choice for the initial value of the Newton method was derived from a coefficient within the problem's initial conditions.

Before delving into the subsequent section, where we will unveil our numerical findings, we embark on an analysis of the method's convergence. To accomplish this, we turn your attention to the following theorem.

Theorem 1. The proposed approach RBF-LOD is stable.
Proof. While maintaining the problem's generality, we focus on the homogeneous case, noting that the same procedures can be readily adapted for the heterogeneous case with only minor adjustments. Hence, we proceed under the assumption that

$$
\begin{equation*}
f(x, t)=0 \tag{30}
\end{equation*}
$$

Also consider

$$
\begin{equation*}
\left\|x_{i}-x_{j}\right\|_{2}^{2}=2-2 \cos \left(\theta_{i}-\theta_{j}\right) \tag{31}
\end{equation*}
$$

where $\theta$ represents the angular part of the points $x_{i}$. We can rewrite $u(x)$ in the following form:

$$
\begin{equation*}
u(x)=\sum_{i=1}^{M} \lambda_{i} \psi\left(1-\cos \left(\theta-\theta_{i}\right)\right) \tag{32}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi 1-\cos \left(\theta-\theta_{i}\right)=\psi\left(\left\|x_{i}-x_{j}\right\|_{2}\right) \tag{33}
\end{equation*}
$$

and $\psi$ is a radial basis function. We have

$$
\begin{equation*}
u(\theta, t)=\sum_{j=1}^{M} \lambda_{j}(t) \psi\left(1-\cos \left(\theta-\theta_{j}\right)\right) \tag{34}
\end{equation*}
$$

Now, upon substituting (34) into the modified homogeneous (3), we obtain

$$
\begin{align*}
& \sum_{j=1}^{M} \lambda_{j}^{\prime}(t) \psi\left(1-\cos \left(\theta-\theta_{j}\right)\right)+v \sum_{j=1}^{M} \lambda_{j}(t) \sin \left(\theta-\theta_{j}\right) \psi^{\prime}\left(1-\cos \left(\theta-\theta_{j}\right)\right) \\
& \quad-\frac{1}{\operatorname{Re}}\left[\sum_{j=1}^{M} \lambda_{j}(t) \cos \left(\theta-\theta_{j}\right) \psi^{\prime}\left(1-\cos \left(\theta-\theta_{j}\right)\right)+\sum_{j=1}^{M} \lambda_{j}(t) \sin ^{2}\left(\theta-\theta_{j}\right) \psi^{\prime \prime}\left(1-\cos \left(\theta-\theta_{j}\right)\right)\right]  \tag{35}\\
& \quad-g\left(\sum_{j=1}^{M} \lambda_{j}(t) \psi\left(1-\cos \left(\theta-\theta_{j}\right)\right)-\sigma W_{t}\left(1-\cos \left(\theta-\theta_{j}\right)\right)\right)
\end{align*}
$$

Let us rephrase (35) for each $\theta_{i}$.

$$
\begin{align*}
& \sum_{j=1}^{M} \lambda_{j}^{\prime}(t) \psi\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)+\nu \sum_{j=1}^{M} \lambda_{j}(t) \sin \left(\theta_{i}-\theta_{j}\right) \psi^{\prime}\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right) \\
& \quad-\frac{1}{\operatorname{Re}}\left[\sum_{j=1}^{M} \lambda_{j}(t) \cos \left(\theta_{i}-\theta_{j}\right) \psi^{\prime}\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)+\sum_{j=1}^{M} \lambda_{j}(t) \sin ^{2}\left(\theta_{i}-\theta_{j}\right) \psi^{\prime \prime}\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)\right]  \tag{36}\\
& \quad-g\left(\sum_{j=1}^{M} \lambda_{j}(t) \psi\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)-\sigma W_{t}\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)\right) \\
& A \lambda^{\prime}(t)+B \lambda(t)-C \lambda(t)-g(A \lambda(t))-W=0
\end{align*}
$$

The resulting system can be represented in matrix form as follows:
where

$$
\begin{align*}
A & =\left[a_{i j}\right]=\left[\psi\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)\right], \\
B & =\left[b_{i j}\right]=\left[\left(v \sin \left(\theta_{i}-\theta_{j}\right)-\frac{1}{\operatorname{Re}} \cos \left(\theta_{i}-\theta_{j}\right)\right) \psi^{\prime}\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)\right], \\
C & =\left[c_{i j}\right]=\frac{1}{\operatorname{Re}}\left[\sin ^{2}\left(\theta_{i}-\theta_{j}\right) \psi^{\prime \prime}\left(1-\cos \left(\theta_{i}-\theta_{j}\right)\right)\right],  \tag{38}\\
W & =\left[w_{i j}\right]=N(0, Q), Q=\left[\sigma \eta Q\left(\theta_{i}, \theta_{j}\right)\right], \\
\lambda(t) & =\left(\lambda_{1}(t), \lambda_{2}(t), \cdots, \lambda_{M}(t)\right)^{T} .
\end{align*}
$$

Under the assumption of the following, we will obtain:

$$
\begin{equation*}
u(t)=\left(u_{1}(t), u_{2}(t), \cdots, u_{M}(t)\right) \tag{39}
\end{equation*}
$$

Equation (37) can be expressed as follows:

$$
\begin{equation*}
A \lambda^{\prime}(t)=(-B+C) \lambda(t)+g(A \lambda(t))+W \tag{40}
\end{equation*}
$$

Since $A$ is a positive definite matrix, it is invertible, enabling us to express:

$$
\begin{equation*}
u\left(t_{n+1}\right)=(-B+C) A^{-1} u\left(t_{n}\right)+g\left(u\left(t_{n+1 / 2}\right)+W A^{-1}\right) \tag{41}
\end{equation*}
$$

Given that the term $W A^{-1}$ represents a coefficient matrix with a condition number in each iteration that does not exceed a certain value, denoted as $\alpha$, we can state the following:

$$
\begin{equation*}
u^{n+1}=E_{1} u^{n}+g\left(u^{n+1 / 2}\right)+E_{2} \tag{42}
\end{equation*}
$$

Now, employing an iterative numerical method, denoted as $F$, we can decompose the equation above into discrete time steps $n$ as follows:

$$
\begin{equation*}
u^{n+1}=F^{(k)}(E) u^{n} \tag{43}
\end{equation*}
$$

where $k$ denotes the number of iterations in the numerical method. In (43), the matrix $E$ is influenced by the matrix $E_{1}=(-B+C) A^{-1}$. The relationship described in (43) is Laxstable if

$$
\begin{equation*}
\left\|F^{(k)}(E)^{n}\right\| \leq l, \quad \forall n \geq 0 \tag{44}
\end{equation*}
$$

If $E$ is a normal matrix, establishing the stability of the method requires demonstrating that the eigenvalues of $E$ fall within the stability domain of the method $F^{(k)}$.

Lemma 2. $A, B$, and $C$ are commutative matrices.

Proof. For proof refer to [24]
Lemma 3. $B A^{-1}$ and $C A^{-1}$ are skew-adjoint.
Proof. From (38), $A$ and $B$ are skew-adjoint. Furthermore, since $A$ and $B$ commute by Lemma 2, so too do $A^{-1}$ and $B$. Hence,

$$
\begin{equation*}
\left(B A^{-1}\right)^{*}=-A^{-1} B=-B A^{-1} \tag{45}
\end{equation*}
$$

where $(*)$ denotes the complex conjugate transpose. Using the same methodology, the proposition is also proven for the matrix product $C A^{-1}$.

Lemma 4. $B A^{-1}$ and $C A^{-1}$ are Hermitian matrices, and consequently, they are Normal matrices.

Proof. For proof refer to [24]
Theorem 5. The relation (43) is Lax-stable.
Proof. Matrix $E$ is skew-adjoint so it is a Hermitian matrix. As stipulated in Lemma 4, due to the Hermitian nature of matrix $E$, it also qualifies as a normal matrix, ensuring that its eigenvalues are real. Therefore, for equation (43) to be Lax-stable, it suffices for the eigenvalues of matrix $E$ to reside within the stability domain of method $F^{(k)}$. By making a judicious choice of method $F$, such as the Crank-Nicolson method, which boasts unconditional stability with a domain encompassing all real numbers, we can confidently assert that equation (43) indeed is Lax-stable.

In light of the preceding explanation, we can confidently assert that the proposed method is stable.

## 4. Numerical Results

In this section, we showcase the numerical results obtained through the utilization of the LOD-RBF method. Our primary objective is to obtain accurate numerical solutions for
stochastic diffusion problems. We demonstrate the effectiveness of the proposed method by solving various stochastic diffusion equations in both two- and threedimensional space. We employ a Monte Carlo simulation for the solution samples $u_{n}^{(l)}$ for $l=1, \cdots, S$, and estimate the expectation of the solution $\widehat{u}_{n}$ at the final time $T$ by

$$
\begin{equation*}
\mathbb{E}\left[\widehat{u}_{n}\right]=\frac{1}{S} \sum_{l=1}^{S} u_{n}^{(l)} \tag{46}
\end{equation*}
$$

that is called the mean value. Also, we compute the variance and the standard deviation for all test problems presented that are defined as follows:

$$
\begin{align*}
& \mathscr{D e v}\left(\widehat{u}_{n}\right)=\sqrt{\frac{1}{S} \sum_{l=1}^{S}\left(\widehat{u}_{n}^{(l)}-\sum_{p=1}^{S} \widehat{u}_{n}^{(p)}\right)^{2}},  \tag{47}\\
& \mathscr{V} \operatorname{ar}\left(\widehat{u}_{n}\right)=\left(\operatorname{Dev}\left(\widehat{u}_{n}\right)\right)^{2},
\end{align*}
$$

knowing them as are two next of the statistical moments is momentous. To assess the accuracy of the obtained numerical solutions, we utilize two commonly employed error measures: the root-mean-square error (RMSE) and the maximum error $\left(e_{\infty}\right)$. These metrics provide valuable insights into the quality and precision of the numerical results. To define these errors, first, we introduce $e_{n}^{i}$ as the expected error function at time $t_{n}$ in the following formulate:

$$
\begin{equation*}
e_{n}^{i}=\left|E\left(u_{n}^{i}\right)-\bar{u}\left(t_{n}, x_{i}\right)\right|, \tag{48}
\end{equation*}
$$

where $\bar{u}\left(t_{n}, x_{i}\right)$ and $E\left(u_{n}^{i}\right)$ are the exact and expectation of numerical solutions at the collocation point $x_{i}$ and time $t_{n}$, respectively. The discrete errors are defined as follows:

$$
\begin{align*}
e_{\infty} & =\max _{1 \leq i \leq S}\left|e_{i}\right| \\
\text { RMSE } & =\sqrt{\frac{1}{S} \sum_{i=1}^{S}\left|e_{i}\right|^{2}} \tag{49}
\end{align*}
$$

Example 1. Convection-diffusion models play a fundamental role in describing the transport phenomena encountered in various physical, chemical, and biological processes [25]. In this study, we focus on the two-dimensional stochastic convection-diffusion equation (2D) with Dirichlet boundary conditions. This equation serves as a key framework for analyzing the intricate interplay between convection and diffusion processes in diverse fields of study. Consider the following stochastic convection-diffusion equation:

$$
\left\{\begin{array}{l}
u_{t}+v_{1} u_{x}+v_{2} u_{y}-\frac{1}{\operatorname{Re}}\left(u_{x x}+u_{y y}\right)=\sigma W_{t}(x, y, t), \quad(x, y) \in \Omega=[a, b]^{2}  \tag{50}\\
u(x, y, 0)=\exp \left(-\operatorname{Re}(x-0.5)^{2}-\operatorname{Re}(y-0.5)^{2}\right) \\
u(a, b, t)=0
\end{array}\right.
$$

where $R e$ is the Reynolds number that depends on the specific problem and the fluid flow conditions involved. The exact solution of the deterministic type of problem is

$$
\begin{equation*}
u(x, y, t)=\frac{1}{1+4 t} \exp \left(-\operatorname{Re} \frac{\left(x-v_{1} t-0.5\right)^{2}}{(1+4 t)}-\operatorname{Re} \frac{\left(y-v_{2} t-0.5\right)^{2}}{(1+4 t)}\right) \tag{51}
\end{equation*}
$$

Here, we approximate the numerical solutions by using inverse multiquadric (IMQ) functions. Concerning the covariance function, due to the IMQ basis function, we consider it in the following form:

$$
\begin{equation*}
Q(x, y)=\frac{1}{\left(\|x-y\|^{2}+1\right)^{2}} \tag{52}
\end{equation*}
$$

Also, our selected optimal shape parameter is $\epsilon=\sqrt{h}$ which we determined by trial and error. Other required assumptions and constants are as follows:

$$
\begin{align*}
h & =\frac{b-a}{M},[a, b]=[-2,4.5] \\
\eta & =\frac{T}{N}, T=1.25, N=1000  \tag{53}\\
\operatorname{Re} & =20, \sigma=0.2, v_{1}, v_{2}=0.8
\end{align*}
$$

We get all the results by using $S=1000$ realizations of the Monte Carlo method. In Figure 1, the exact solution, the mean solution, and the error are shown. Furthermore, the standard deviation and variance are shown in Figure 2.

The CPU times for implementing this approach in each example are provided. The computations were performed on a computer equipped with a 16-core neural engine and 16 GB of memory using MATLAB 2021b. According to the aforementioned constants, the CPU time for implementing this approach in the specified example is 0.71 seconds. In Table 2, the efficacy of the number of collocation points $M$ on the errors (RMSE and $e_{\infty}$ ) is investigated. As expected, the increase of collocation points leads to the decrease of error.

It is evident that employing a smaller time step size leads to improved accuracy in the obtained results. This observation is substantiated by the data presented in Table 3, which clearly demonstrates a reduction in errors as the time step size decreases.

In order to assess the stability of the method, we conducted a study, considering the suggested optimal shape parameter as the input data. Our aim was to examine whether the condition number of the matrix, resulting from solving the system of linear equations, remains relatively unchanged in near the selected optimal shape parameter. The results supporting this proposition are presented in Table 4. Based on the findings obtained from these experiments, we conclude that employing $\epsilon=\sqrt{h}$ ensures both stability and high accuracy in the numerical solutions.

Furthermore, it can be intuitively demonstrated that the proposed method for this class of equations is convergent. To substantiate this claim, we illustrate how the error decreases as the number of collocation points $M$ increases.

As vividly illustrated in Figure 3, it is evident that the error decreases with the increasing number of collocation points. Consequently, this observation serves as compelling evidence of the method's convergence. In this particular example, we have achieved a convergence of orders 3 and 4.

Example 2. In this experiment, we aim to demonstrate the effectiveness of the employed method in solving stochastic nonlinear problems. To this end, we investigate a specific type of nonlinear stochastic equations. Specifically, we apply our method to address the stochastic nonlinear inhomogeneous convection-diffusion equation (2D) represented by the following form:

$$
\left\{\begin{array}{l}
u_{t}+u_{x}+u_{y}=\frac{1}{2}\left(u_{x x}+u_{y y}\right)+\exp (-t) \sin (x+y)+\sin (u)+\sigma W_{t}(x, y, t), \quad(x, y) \in \Omega=[a, b]^{2}  \tag{54}\\
u(x, y, 0)=\sin (x) \sin (y)
\end{array}\right.
$$



Figure 1: Exact solution, mean solution and, error for $\sigma=0.2, \eta=0.00125, M=80, h=0.0812$, and $\epsilon=\sqrt{h}$.



Figure 2: Standard deviation and variance for Example 1 for $\sigma=0.2, \eta=0.00125, M=80, h=0.0812$, and $\epsilon=\sqrt{h}$.

Table 2: Results for the various number of collocation points $M$.

| $M$ | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $1.49 e-1$ | $1.62 e-2$ |
| 20 | $1.38 e-2$ | $1.42 e-3$ |
| 40 | $1.41 e-4$ | $6.20 e-5$ |
| 80 | $2.71 e-5$ | $7.90 e-6$ |

Table 3: Results for the various number of time discretization points $N$.

| $N$ | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $1.14 e-2$ | $5.52 e-3$ |
| 100 | $1.63 e-3$ | $5.20 e-4$ |
| 200 | $1.91 e-4$ | $8.28 e-5$ |
| 400 | $8.20 e-5$ | $1.47 e-5$ |

Table 4: Results for different shape parameters around our suggested optimal shape parameter.

| $\epsilon$ | RMSE | Cond (M) |
| :--- | :---: | :---: |
| $\sqrt{h}-0.3$ | $1.51 e-2$ | 1.013 |
| $\sqrt{h}-0.2$ | $1.11 e-3$ | 1.012 |
| $\sqrt{h}-0.1$ | $1.60 e-4$ | 1.012 |
| $\sqrt{h}$ | $4.14 e-5$ | 1.011 |
| $\sqrt{h}+0.1$ | $9.12 e-5$ | 1.012 |
| $\sqrt{h}+0.2$ | $1.02 e-4$ | 1.012 |



Figure 3: RMS-error vs. number of collocation points $M$.
with zero boundary conditions. The exact solution for the deterministic type of this test is given by

$$
\begin{equation*}
u(x, y, t)=\exp (-t) \cdot \sin (x) \cdot \sin (y) \tag{55}
\end{equation*}
$$

In this study, we obtain numerical results by employing two different types of radial basis functions (RBFs) and corresponding covariance functions. In addition to the functions considered in the first example, we also utilize the Gaussian basis function and its appropriate covariance function. By comparing the results presented in the provided
tables, we can observe the influence of the selected radial basis and covariance functions on the accuracy of the numerical results. For each test, we consider the following assumptions:

$$
\begin{align*}
\psi & =\exp \left(-\epsilon^{2} r^{2}\right), Q=\exp \left(r^{2}\right) \\
{[a, b] } & =[0, \pi], T=2, N=1000  \tag{56}\\
\epsilon & =\frac{M}{10}, \sigma=0.2
\end{align*}
$$

In Figure 4, the exact solution, mean solution, and error are shown.

Standard deviation and variance are presented as two next statistical moments in Figure 5.

In Tables 5 and 6 the results of increasing the number of the collocation points $M$ and the time discretization points N are provided.

According to these outcomes, rational improvement can be seen in the error rate. Finally, in Table 7, the stability of our method around the chosen optimal shape parameter $\epsilon=$ $\sqrt{h}$ is checked numerically. Moreover, we can observe $\epsilon=$ $\sqrt{h}$ leads to the lowest error.

We present new numerical results for the given example by altering several factors, including the radial basis functions (RBFs), covariance functions, and shape parameter. By comparing the obtained errors, we can easily observe the impact of these factors on the results. For the new experiment, we utilize the Inverse multiquadric (IMQ) function as the RBF and select a different covariance function tailored to the Inverse multiquadric basis function. The revised assumptions for this experiment are as follows:

$$
\begin{align*}
\psi(r) & =\frac{1}{\sqrt{r^{2}+\epsilon^{2}}}, \\
Q & =\frac{1}{\sqrt{r^{2}+1}},  \tag{57}\\
\epsilon & =\frac{1}{\sqrt{h}} .
\end{align*}
$$

In Figure 6, the exact and mean solutions and the corresponding error are shown. Furthermore, we provide Table 8 that includes the estimated errors when increasing the collocation point $M$.

By substituting the Gaussian radial basis function (RBF) and its corresponding covariance function, we have achieved significant improvements in the obtained results. As previously emphasized, the developed RBF-LOD method offers notable computational advantages. To validate its effectiveness, we conducted a comparative analysis with the finite element method (FEM). For this purpose, we employed $P_{1}$ FEM, utilizing first-order polynomials and the same discretization points. The comparison between the two methods was based on CPU time, which serves as a fair metric to evaluate their performance.

Table 9 displays the elapsed time for a single Monte Carlo simulation, clearly demonstrating that the RBF-LOD
method surpasses the FEM in terms of efficiency. This result highlights the superior computational efficiency of the RBF-LOD method compared to the FEM.

To further corroborate the method's convergence in this particular example, we present Figure 7. The depicted graph distinctly showcases the trend: As the number of collocation points increases, the associated error consistently diminishes, providing strong evidence of the method's convergence. Notably, in this example, the method exhibits convergence of both third and fourth order. These findings firmly establish the method's effectiveness and reliability in the context of our investigation.

Example 3. One of the key advantages of our proposed method is its ability to be easily implemented for highdimensional models. With this in mind, we conducted a specific experiment to showcase the effectiveness of our method in the context of three-dimensional (3D) models. To accomplish this, we investigate the behavior of our method when applied to a three-dimensional stochastic diffusion equation. The following equation represents the threedimensional stochastic diffusion equation under consideration:

$$
\left\{\begin{array}{l}
u_{t}+p \cdot \nabla u-\frac{1}{\operatorname{Re}} \Delta u=\sigma W_{t}(x, y, z, t),  \tag{58}\\
u(x, y, z, 0)=\exp \left(-\operatorname{Re}\left(\left(X-X_{0}\right)^{T}\left(X-X_{0}\right)\right)\right) .
\end{array} \quad(x, y, z) \in \Omega=[a, b]^{3},\right.
$$

The exact solution when $\sigma=0$ is

$$
\begin{equation*}
u(x, y, z, t)=\frac{1}{(1+4 t)^{3 / 2}} \exp \left(-\frac{R e}{4 t+1}\left(\left(X-p t-X_{0}\right)^{T}\left(X-p t-X_{0}\right)\right)\right) \tag{59}
\end{equation*}
$$

which is centered at $X_{0}=(0.5,0.5,0.5)$ in this experiment. Considering the following assumptions, we give some numerical results

$$
\begin{align*}
{[a, b] } & =[-0.5,1.5], T=0.5, N=1000 \\
\operatorname{Re} & =2, \sigma=0.2 \tag{60}
\end{align*}
$$

Similar to the previous examples, we present the results obtained from our proposed method to demonstrate its effectiveness in tackling higher-dimensional problems. To evaluate its performance, we provide estimated errors for different configurations of collocation points $(M)$ and time discretization points $(N)$. The corresponding results are organized in Tables 10 and 11, respectively. These tables offer valuable insights into the competence of our approach and its ability to accurately handle problems in higher dimensions.

Also, the presented results in Table 12 show that the method is stable and optimum for the selected shape parameter $\epsilon=\sqrt{h}$.

In conclusion, the numerical results presented in the aforementioned tables indicate the satisfactory performance of our method when applied to three-dimensional models. As depicted in Figure 8, it is evident that the method performs exceptionally well in this example as well. The convergence of the method is clearly demonstrated through this graph. Just like in the previous two examples, when we employ the proposed method in this scenario, we achieve convergence of both third and fourth order. These consistent results underscore the method's reliability and its suitability
for a range of applications. In this example, with the specified constants taken into account, the CPU time is recorded at 1.24 seconds.

Example 4. To substantiate the credibility of our results, we have undertaken a comparative analysis by employing an alternative approach on a distinct example. In this context, we have selected one of the examples previously addressed in reference [26]. The referenced work presents a novel formulation of Radial Basis Functions (RBFs) rooted in the pseudospectral (PS) method, tailored specifically for solving stochastic advection-diffusion equations of the following form:

$$
\left\{\begin{array}{l}
d u+\left(\nu \nabla \cdot u-\gamma \nabla^{2} u-f\right) d t=\sigma d W(t)  \tag{61}\\
u(x, 0)=u_{0}, x \in \mathscr{D} \\
\left.u(x, t)\right|_{\partial \mathscr{D}}=g, t \in(0, T)
\end{array}\right.
$$

To apply our method, we consider the following assumptions:

$$
\begin{align*}
& 0 \leq x, y, z \leq 1, v_{1}=v_{2}=v_{3}=0.08, T=1,  \tag{62}\\
& \sigma=1 . \quad \operatorname{Re}=10, \quad \eta=0.05 .
\end{align*}
$$

In Figure 9, both the exact solution and the mean solution obtained through the proposed method are depicted simultaneously, along with the corresponding error.

In Tables 13 and 14, we compare the errors generated by our proposed method with those of the RBF-PS method for various values of RE.


Figure 4: Exact solution, mean solution and, error for $\sigma=0.2, \eta=0.002, M=80, h=\pi / 80$, and $\epsilon=\sqrt{h}$.


Figure 5: Standard deviation and variance for Example 2 for $\sigma=0.2, \eta=0.002, M=80, h=\pi / 80$, and $\epsilon=\sqrt{h}$.

Table 5: Results for the various number of collocation points $M$.

| M | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $1.30 e-2$ | $6.92 e-3$ |
| 20 | $3.06 e-3$ | $1.86 e-3$ |
| 40 | $6.47 e-4$ | $2.73 e-4$ |
| 80 | $6.35 e-4$ | $6.24 e-5$ |

Table 6: Results for the various number of time discretization points $N$.

| $N$ | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $3.10 e-2$ | $1.54 e-2$ |
| 100 | $4.27 e-3$ | $2.45 e-3$ |
| 200 | $2.02 e-3$ | $1.17 e-3$ |
| 400 | $6.83 e-4$ | $3.64 e-4$ |

Table 7: Results for different shape parameters around our suggested optimal shape parameter.

| $\epsilon$ | RMSE | Cond (M) |
| :--- | :---: | :---: |
| $\sqrt{h}-0.2$ | $1.30 e-1$ | 1.806 |
| $\sqrt{h}-0.1$ | $1.36 e-3$ | 1.784 |
| $\sqrt{h}$ | $5.39 e-4$ | 1.776 |
| $\sqrt{h}$ | $2.77 e-3$ | 1.783 |
| $\sqrt{h}+0.2$ | $3.63 e-3$ | 1.792 |

As can be readily observed our proposed method yields superior results. In this example, the execution demanded approximately 0.89 seconds of CPU time.

Example 5. In this illustrative example, we demonstrate the versatility of our method by applying it to problems involving a complex domain. Specifically, we examine an equation governing a circle. To achieve this objective, we solve the governing equations in the second example within the context of a circle having a radius of $\pi$. User The constants considered in this example are as follows:

$$
\begin{align*}
{[a, b] } & =[0,2 \pi], T=2, N=1000, \\
\psi & =\exp \left(-\epsilon^{2} r^{2}\right), Q=\exp \left(r^{2}\right), \eta=0.002,  \tag{63}\\
M & =200, \epsilon=\frac{M}{20}, h=\frac{\pi}{100}, \sigma=0.2 .
\end{align*}
$$

In Figures 10 and 11, both the exact solution and mean solution, acquired through the proposed method, are
depicted on a circular domain. Furthermore, Figure 12 illustrates the error arising from the application of this method.

As evident, the proposed method performs effectively on this particular problem involving a circular domain. In this case, the CPU time required was approximately 1.40 seconds. In addition, Table 15 presents the impact of augmenting the number of collocation points on the resulting error.

As anticipated, the error exhibits a favorable decrease with the augmentation of collocation points.

Example 6. In our final experimental investigation, we extend the application of our method to another variant of the stochastic diffusion equation. Specifically, we focus on the stochastic Burgers' equation driven by additive noise. The study of such equations holds significant importance due to their relevance in diverse fields, including astrophysics, statistical physics, cosmology, fluid dynamics, and engineering applications [27, 28]. Stochastic Burgers' equations find applications in constructive quantum field theory [29], kinetic models, continuum theory [30], and modeling of vortex lines in high-temperature superconductors [31]. In addition, the numerical solution of the Burgers' equation with random noise plays a central role in addressing nonlinear systems out of equilibrium [32]. In this study, we consider a two-dimensional stochastic Burgers' equation, which is represented as follows:

$$
\left\{\begin{array}{l}
u_{t}+v\left(u u_{x}+u u_{y}\right)=\frac{1}{R e}\left(u_{x x}+u_{y y}\right)+\sigma W_{t}(x, y, t)+f(x, y, t), x, y \in[a, b]^{2}  \tag{64}\\
u(a, b, t)=0 \\
u(x, y, 0)=\sin (x) \sin (y)
\end{array}\right.
$$



Figure 6: Exact solution, mean solution and, error for $\sigma=0.2, \eta=0.002, M=80, h=\pi / 80$, and $\epsilon=\sqrt{h}$.

Table 8: Results for the various number of collocation points $M$ using GA radial basis function.

| $M$ | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $1.46 e-3$ | $8.39 e-4$ |
| 20 | $3.74 e-4$ | $2.33 e-4$ |
| 40 | $1.25 e-4$ | $7.78 e-5$ |
| 80 | $6.08 e-5$ | $3.04 e-5$ |

Table 9: The CPU time (in seconds) for RBF-LOD and FEM.

| Method | $h=0.4$ | $h=0.2$ | $h=0.1$ | $h=0.05$ |
| :--- | :---: | :---: | :---: | :---: | :---: |
| RBF-LOD | 0.86 | 0.9 | 1.22 | 1.64 |
| FEM | 0.85 | 0.91 | 1.29 | 1.88 |



Figure 7: RMS-error vs. number of collocation points $M$.

Table 10: Results for the various number of collocation points $M$.

| $M$ | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $9.79 e-2$ | $4.18 e-3$ |
| 20 | $2.80 e-3$ | $1.66 e-4$ |
| 40 | $2.33 e-5$ | $2.60 e-6$ |
| 80 | $6.01 e-6$ | $1.75 e-7$ |

Table 11: Results for the various number of time discretization points $N$.

| $N$ | $e_{\mathrm{o}}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $2.21 e-2$ | $1.71 e-3$ |
| 50 | $1.09 e-3$ | $2.03 e-4$ |
| 100 | $9.26 e-5$ | $5.14 e-5$ |
| 200 | $3.61 e-5$ | $9.04 e-6$ |

Table 12: Results for different shape parameters around our suggested optimal shape parameter.

| $\epsilon$ | RMSE | Cond (M) |
| :--- | :---: | :---: |
| $\sqrt{h}-0.2$ | $6.62 e-3$ | 1.131 |
| $\sqrt{h}-0.1$ | $9.29 e-5$ | 1.105 |
| $\sqrt{h}$ | $6.21 e-5$ | 1.044 |
| $\sqrt{h}+0.1$ | $1.02 e-4$ | 1.280 |
| $\sqrt{h}+0.2$ | $1.45 e-4$ | 1.630 |



Figure 8: RMS-error vs. number of collocation points $M$.

The exact solution for the deterministic type of this example is

$$
\begin{equation*}
u(x, y, t)=e^{-t} \sin x \cos y . \tag{65}
\end{equation*}
$$

In this test, we consider the required assumptions similar to Example 3. We show the exact solution, the mean solution, and the error in Figure 13. Subsequently, the standard deviation and variance are plotted in Figure 14.


Figure 9: Exact solution, mean solution and, error for $T=1 . \sigma=1$ and $\eta=0.05$.

Table 13: Comparing results RBF-LOD vs. $\mathrm{RBF}-\mathrm{PS}$ for $\mathrm{Re}=10$.

| $h, T$ | RMSE (RBF-LOD) | RMSE (RBF-PS) |
| :--- | :---: | :---: |
| $h=1 / 2, T=1$ | $1.84 e-2$ | $6.01 e-2$ |
| $h=1 / 4, T=1$ | $9.94 e-3$ | $2.59 e-2$ |
| $h=1 / 8, T=1$ | $7.73 e-3$ | $1.18 e-2$ |
| $h=1 / 2, T=2$ | $6.59 e-3$ | $6.71 e-2$ |
| $h=1 / 4, T=2$ | $3.29 e-3$ | $2.66 e-2$ |
| $h=1 / 8, T=2$ | $1.78 e-3$ | $1.20 e-2$ |

TABLE 14: Comparing results RBF-LOD vs. RBF-PS for $\mathrm{Re}=20$.

| $h, T$ | RMSE (RBF -LOD$)$ | RMSE (RBF-PS) |
| :--- | :---: | :---: |
| $h=1 / 2, T=1$ | $1.06 e-2$ | $6.61 e-2$ |
| $h=1 / 4, T=1$ | $4.61 e-3$ | $3.80 e-2$ |
| $h=1 / 8, T=1$ | $3.17 e-3$ | $2.41 e-2$ |
| $h=1 / 2, T=2$ | $4.29 e-3$ | $7.71 e-2$ |
| $h=1 / 4, T=2$ | $2.52 e-3$ | $2.45 e-2$ |
| $h=1 / 8, T=2$ | $1.09 e-3$ | $2.76 e-2$ |



$$
\begin{aligned}
& \circ \mathrm{U}_{\text {Exact }} \\
& * \mathrm{U}_{\text {Numer }}
\end{aligned}
$$

Figure 10: Exact solution and mean solution for $T=2 . \sigma=0.2, \epsilon=M / 20$ and $\eta=0.0002$.


Figure 11: Exact solution and mean solution for $T=2 . \sigma=0.2, \epsilon=M / 20$ and $\eta=0.0002$.


Figure 12: RMS-error for $T=2 . \sigma=0.2, \epsilon=M / 20$ and $\eta=0.0002$.

Table 15: Results for the various number of collocation points $M$.

| $M$ | $e_{\infty}$ | RMSE |
| :---: | :---: | :---: |
| 10 | $6.75 e-2$ | $4.70 e-2$ |
| 20 | $1.91 e-2$ | $1.02 e-2$ |
| 40 | $7.44 e-3$ | $5.85 e-3$ |
| 80 | $3.08 e-3$ | $1.80 e-3$ |

In this instance, the computational process necessitated around 0.77 seconds of CPU time. Tables 16 and 17 show the reduction of the approximated errors when increasing the number of collocation points $M$ and time discretization points $N$.

To assess the stability of our method, we conducted an investigation on the optimal shape parameter, as documented in Table 18. Remarkably, the results demonstrate that the proposed shape parameter $\epsilon=\sqrt{h}$, not only yields
the lowest error but also ensures stability. These findings affirm the effectiveness of our approach in handling a wide range of stochastic diffusion equations. The numerical results provide compelling evidence of the method's performance across various classes of such equations. Furthermore, we present the intuitive convergence diagram for this example in Figure 15 given below. Notably, the convergence in this particular example exhibits second- and third-order characteristics.


Figure 13: Exact solution, mean solution and, error for $\sigma=0.2, \eta=0.0005, M=40, h=2 / 40$, and $\epsilon=\sqrt{h}$.


Figure 14: Standard deviation and variance for Example 3 for $\sigma=0.2, \eta=0.0005, M=40, h=2 / 40$, and $\epsilon=\sqrt{h}$.

Table 16: Results for the various number of collocation points $M$.

| $M$ | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $1.62 e-2$ | $1.06 e-2$ |
| 20 | $4.99 e-3$ | $3.08 e-3$ |
| 40 | $2.56 e-3$ | $1.17 e-3$ |
| 80 | $1.83 e-3$ | $8.40 e-4$ |

Table 17: Results for the various number of time discretization points $N$.

| $N$ | $e_{\infty}$ | RMSE |
| :--- | :---: | :---: |
| 10 | $6.23 e-3$ | $2.86 e-3$ |
| 50 | $1.90 e-3$ | $1.02 e-3$ |
| 100 | $1.85 e-3$ | $8.48 e-4$ |
| 200 | $1.72 e-3$ | $8.35 e-4$ |

Table 18: Results for different shape parameters around our suggested optimal shape parameter.

| $\epsilon$ | RMSE | Cond (M) |
| :--- | :---: | :---: |
| $\sqrt{h}-0.1$ | $1.43 e-3$ | 19.580 |
| $\sqrt{h}$ | $8.35 e-4$ | 19.291 |
| $\sqrt{h}+0.1$ | $9.69 e-4$ | 24.758 |
| $\sqrt{h}+0.2$ | $1.24 e-3$ | 17.758 |



Figure 15: RMS-error vs. number of collocation points $M$.

## 5. Conclusion

In conclusion, this work centers around the numerical solution of a specific class of stochastic partial differential equations (PDEs). To tackle this problem, we employed the Radial Basis Function (RBF) and Crank-Nicolson methods for space and time discretization. In addition, we implemented a time-splitting strategy to enhance the efficiency of our numerical approach, which is particularly advantageous for solving high-dimensional stochastic PDEs.

Furthermore, the Monte Carlo method is used to achieve the mean solution. We decreased the computational cost and CPU time of performing the RBF method by using the local
one-dimensional time-splitting approach. This advantage concerning stochastic algorithms that leads to applying the Monte Carlo method is noteworthy.

Numerical results were provided for four different examples of the stochastic diffusion equation, encompassing both linear and nonlinear equations in two and three dimensions. These results serve as compelling evidence of the remarkable efficiency of our proposed LOD-RBF method for solving stochastic equations. Moreover, the obtained results illustrate that the proposed method is stable for the selected optimal shape parameter $\epsilon=\sqrt{h}$.

Overall, our findings highlight the efficacy of our suggested method for solving stochastic PDEs, underscoring its
potential for a wide range of applications in various disciplines. The combination of the RBF method, time-splitting strategy, and Monte Carlo approach offers a robust and efficient solution framework for addressing stochastic partial differential equations.

## Data Availability

No new data were created or analyzed in this study. Data sharing is not applicable to this article.

## Conflicts of Interest

The authors declare that they have no conflicts of interest.

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