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Research Article

Unbounded Order Convergence in Ordered Vector Spaces

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We consider an ordered vector space X. We define the net $\{x_\alpha\} \subseteq X$ to be unbounded order convergent to x (denoted as $x_\alpha \xrightarrow{\text{uo}} x$). This means that for every $0 \leq y \in X$, there exists a net $\{y_\beta\}$ (potentially over a different index set) such that $y_\beta \downarrow 0$, and for every β, there exists α_0 such that $\{\pm(x_\alpha - x)^u, y\}^l \subseteq \{y_\beta\}^l$ whenever $\alpha \geq \alpha_0$. The emergence of a broader convergence, stemming from the recognition of more ordered vector spaces compared to lattice vector spaces, has prompted an expansion and broadening of discussions surrounding lattices to encompass additional spaces. We delve into studying the properties of this convergence and explore its relationships with other established order convergence. In every ordered vector space, we demonstrate that under certain conditions, every *uo*-convergent net implies *uo*-Cauchy, and vice versa. Let X be an order dense subspace of the directed ordered vector space Y. If $J \subseteq Y$ is a *uo*-band in Y, then we establish that $J \cap X$ is a *uo*-band in X.

1. Introduction

In the domain of vector lattices, it is common to encounter partially ordered vector spaces that lack lattice structure, particularly in spaces of operators between vector lattices. To address this, one approach is to assume the Dedekind completeness of the codomain. Another strategy involves extending the necessary concepts from lattice theory to a broader class of partially ordered vector spaces, thereby yielding intrinsic definitions of various notions. In many instances, a lattice concept in a vector lattice can be expressed in multiple ways. When reformulated in partially ordered vector spaces, it may give rise to different concepts. The most valuable generalizations often align these two approaches: direct reformulation and the utilization of embeddings. In this paper, we adopt this perspective to further explore "vector lattice notions" in pre-Riesz spaces. In addition, this topic pertains to notations that involve unbounded order convergence (uo-convergence). Unbounded order convergence, as investigated in [1, 2], has found various applications in economics, particularly within the realms of economic theory, decision-making processes, and optimization problems. In addition, the authors delved into the introduction and exploration of positive operators on unbounded vector spaces in their paper [3].

The relevance of unbounded order convergence can be observed in various specific areas. We have extensively explored convergence in ordered vector spaces, which bears resemblance to the concept of unbounded order convergence examined in prior works, such as those mentioned in references [1, 4]. The main objective of this article is to examine the convergence properties in ordered vector spaces and extend the concepts and problems that have been previously explored in vector lattice spaces. While there exist numerous vector spaces that possess an ordered structure, not all of them exhibit lattice properties (see [5]). Therefore, the study of unbounded order in these spaces holds significant importance. Our research is motivated by the exploration of these convergences and the establishment of their generalizations within the framework of ordered vector spaces. The authors in [6] demonstrated that in vector lattices, *uo*-convergence does not arise from a topology. Consequently, it can be concluded that *uo*-convergence is not necessarily topological in ordered vector spaces in general. Thus, by [6], we see that there is no inherent relationship between the partial-order topology and their newly introduced concept of convergence in general. On the other hand, by Theorem 7.5 of [7], *uo*-convergence in a vector lattice X that agrees with the convergence of a locally convex-solid topology on X if and only if X is atomic.

For information on bounded and unbounded order (or norm) convergence in vector lattices, we recommend referring to [1, 6, 8]. In the following sections, we presented several fundamental definitions pertaining to ordered vector space.

We consider a real vector space X and a cone K in X. Here, K is defined as a wedge, meaning that if x and y belong to K, and λ and μ are nonnegative scalars, then $\lambda x + \mu y$ also belongs to K. In addition, $K \cap (-K) = 0$, indicating that the only element common to both K and its negation $-K$ is the zero vector. In X, we define a partial order denoted by $\leq$, where $x \leq y$ if and only if $y - x$ lies in K. Consequently, the space (X, K), or simply X, is referred to as an ordered vector space (or partially ordered vector space).

A subspace $M \subseteq X$ is said to be majorizing in X if, for every $x \in X$, there exists an element $m \in M$ such that $x \leq m$ (or, equivalently, if for each $x \in X$, there exists an element $m \in M$ with $m \leq x$).

A subspace $M \subseteq X$ is called directed if for every $x, y \in M$, there is an element $z \in M$ such that $x \leq z$ and $y \leq z$. An ordered vector space X is directed if and only if X_+ is generating in X, that is, $X = X_+ - X_+$. An ordered vector space X is called Archimedean if for every $x, y \in X$ with $nx \leq y$ for every $n \in \mathbb{N}$, one has $x \leq 0$. The ordered vector space X has the Riesz decomposition property (RDP) if for every $x_1, x_2, z \in K$ with $z \leq x_1 + x_2$, there exist $z_1, z_2 \in K$ such that $z = z_1 + z_2$ with $z_1 \leq x_1$ and $z_2 \leq x_2$. We call a linear subspace M of an ordered vector space X order dense in X if for every $x \in X$ we have

$$x = \inf\{z \in M : x \leq z\}, \tag{1}$$

that is, the greatest lower bound of set $\{z \in M\colon x \leq z\}$ exists in X and equals to x (see page 360 of [?]). Clearly, if M is order dense in X, then M is majorizing in X. We denote for a subset M of X, the set of all upper bounds (resp, down bounds) by $M^u = \{x \in X\colon x \geq m \text{ for all } m \in M\}$ (resp, $M^l = \{x \in X\colon x \leq m \text{ for all } m \in M\}$). It is clear that for every subset A, B of X, $(A^l + B^l) \subseteq (A + B)^l$. Moreover, if X has a Riesz decomposition property and $A, B \subseteq K$, then $(A + B)^l \subseteq A^l + B^l$.

The elements $x, y \in X$ are called disjoint, in symbols $x \perp y$, if $\{\pm(x + y)\}^u = \{\pm(x - y)\}^u$. The disjoint complement of a subset $M \subseteq X$ is $M^d = \{x \in X \mid \forall y \in M\colon x \perp y\}$. A sequence $\{x_n\} \subseteq X$ is said to be disjoint, if for every $n \neq m$, $x_n \perp x_m$. A linear subspace B of an ordered vector space X is called a band in X if $B = B^{dd}$. A subset M of an ordered vector space X is called solid if for every $x \in X$ and $y \in M$ the relation $\{\pm y\}^u \subseteq \{\pm x\}^u$ implies that $x \in M$. A solid subspace M of X is called an ideal.

We recall that a linear map $i\colon X \longrightarrow Y$ is said to be bipositive if for every $x \in X$, one has $i(x) \geq 0$ if and only if $x \geq 0$. A partially ordered vector space (X, K) is called a pre-Riesz space if for every $x, y, z \in X$, the inclusion $\{x + y, x + z\}^u \subseteq \{y, z\}^u$ implies that $x \in K$. Clearly, each vector lattice is a pre-Riesz space, since the inclusion in the definition of pre-Riesz space reduces to inequality $(x + y) \vee (x + z) \geq y \vee z$, so $x + (y \vee z) \geq y \vee z$, which implies that $x \geq 0$. By Theorem 4.3 of [9], partially ordered vector space X is a pre-Riesz space if and only if there exists a vector lattice Y and a bipositive linear map $i\colon X \longrightarrow Y$ such that $i(X)$ is order dense in Y. The pair (Y, i) (or, loosely Y) is then called a vector lattice cover of X. The theory of pre-Riesz spaces and their vector lattice covers is due to van Haandel (see [10]). Let X be an ordered vector space. An ideal I in X is supremum closed (short s-closed), if for every $z \in X^+$, the relation $z = \sup(I \cap [0, z])$ implies that $z \in I$ (see [11]).

A net $\{x_\alpha\} \subseteq X$ is said to be decreasing (in symbols, $x_\alpha \downarrow$), whenever $\alpha \geq \beta$ implies that $x_\alpha \leq x_\beta$. For $x \in X$, the notation $x_\alpha \downarrow x$ which means that $x_\alpha \downarrow$ and $\inf_\alpha \{x_\alpha\} = x$ hold. The meanings of $x_\alpha \uparrow x$ are analogous. We say that a net $\{x_\alpha\} \subseteq X$ (o)-converges (respectively, $\tilde{o}$-converges) to $x \in X$ (in symbols, $x_\alpha \xrightarrow{o} x$, respectively) if there is a net $\{y_\alpha\} \subseteq X$ (respectively, $\{y_\beta\}$ possibly over a different index set) such that $y_\alpha \downarrow 0$ (respectively, $y_\beta \downarrow 0$) and for all α (for every β, there exists α_0 such that for all, $\alpha \geq \alpha_0$), one has $\pm(x_\alpha - x) \leq y_\alpha$, (respectively, $\pm(x_\alpha - x) \leq y_\beta$). For two elements, $y, z \in X$ with $y \leq z$ denotes the according order interval by $[y, z] = \{x \in X\colon y \leq x \leq z\}$. A set $M \subset X$ is called order-bounded if there are $y, z \in X$ such that $M \subseteq [y, z]$. Let X and Z be ordered vector spaces.

We recall that the net $\{x_\alpha\}$ in vector lattice E is said to be order convergent to x (or, o-convergent for short) if there is a net $\{y_\beta\}$, possibly over a different index set, such that $y_\beta \downarrow 0$ and for every β, there exists α_0 such that $|x_\alpha - x| \leq y_\beta$, whenever $\alpha \geq \alpha_0$. In this case, we write $x_\alpha \xrightarrow{o} x$. In a vector lattice E, a net $\{x_\alpha\}$ is unbounded order convergent (or, *uo*-convergent for short) to x if $|x_\alpha - x| \wedge y \xrightarrow{o} 0$ for all $0 \leq y \in E$. In this case, we write $x_\alpha \xrightarrow{uo} x$.

2. Unbounded Order Convergence on Ordered Vector Spaces

Definition 1. Let X be an ordered vector space. The net $\{x_\alpha\} \subseteq X$ is unbounded order convergent (or, *uo*-convergent for short) to x if for each $0 \leq y \in X$, there is a net $\{y_\beta\}$, possibly over a different index set, such that $y_\beta \downarrow 0$ and for every β, there exists α_0 such that $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \geq \alpha_0$. In this case, we write $x_\alpha \xrightarrow{uo} x$.

Remark 2. The assertion that $\tilde{o}$-convergence implies *uo*-convergence is valid, but the reverse is not generally true.

Let $\{x_\alpha\} \subseteq X$ and $x_\alpha \xrightarrow{\tilde{o}} x$. Therefore, there exists a net $\{y_\beta\} \subseteq X$ and $y_\beta \downarrow 0$, and for each β, there is an α_0 so that for each $\alpha \geq \alpha_0$, $\pm(x_\alpha - x) \leq y_\beta$. Let $z \in \{\{\pm(x_\alpha - x)\}^u, y\}^l$ for all $0 \leq y \in X$, where $\alpha \geq \alpha_0$. Since for $\alpha \geq \alpha_0$, $\pm(x_\alpha - x) \leq y_\beta$, hence $z \leq y_\beta$. So, $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$. Therefore, $x_\alpha \xrightarrow{uo} x$.

On the other hand, it is worth noting that the standard basis of c_0, denoted by $\{e_n\}_{n=1}^{\infty}$, exhibits *uo*-convergence but not $\tilde{o}$-convergence. It should be emphasized, however, that for each order-bounded net, *uo*-convergence is equivalent to $\tilde{o}$-convergence.

Proposition 3. *If E is a vector lattice, then the net $\{x_\alpha\} \subseteq E$ is unbounded order convergent to x (in the sense of Definition 1) if and only if it is an unbounded order convergent to x.*

Proof. Let $\{x_\alpha\} \subseteq E$ and $x_\alpha \xrightarrow{uo} x$. For each $y \in E^+$, there exists a net $\{y_\beta\} \subseteq E$ such that $y_\beta \downarrow 0$ and for each β, there exists an α_0 such that for each $\alpha \geq \alpha_0$ we have $|x_\alpha - x| \wedge y \leq y_\beta$. Let $z \in \{\{\pm(x_\alpha - x)\}^u, y\}^l$. It is clear that $z \leq |x_\alpha - x| \wedge y$ and therefore for each $\alpha \geq \alpha_0$, $z \leq y_\beta$. Therefore, $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$. It means that $x_\alpha \xrightarrow{uo} x$.

Conversely, let $x_\alpha \xrightarrow{uo} x$ (in the sense of Definition 1). It is clear that $(|x_\alpha - x| \wedge y) \in \{\{\pm(x_\alpha - x)\}^u, y\}^l$ for each $y \in E^+$. Therefore, for each β, there exists α_0 such that for each $\alpha \geq \alpha_0$, $(|x_\alpha - x| \wedge y) \in \{y_\beta\}^l$. Hence, $|x_\alpha - x| \wedge y \leq y_\beta$ for each $y \in E^+$ whenever $\alpha \geq \alpha_0$. It means that $x_\alpha \xrightarrow{uo} x$. □

Lemma 4. *Let X be an ordered vector space and $\{x_\alpha\} \subseteq X$, then*

(1) $x_\alpha \xrightarrow{uo} x$ iff $(x_\alpha - x) \xrightarrow{uo} 0$.

(2) if for each α, $x_\alpha \leq y$ and $x_\alpha \xrightarrow{uo} x$, then $x \leq y$.

(3) if $0 \leq x_\alpha \xrightarrow{uo} x$, then $0 \leq x$.

Moreover, if X has the RDP property, then

(4) if $x_\alpha \xrightarrow{uo} x$ and $y_\alpha \xrightarrow{uo} y$, then $\lambda x_\alpha + \mu y_\alpha \xrightarrow{uo} \lambda x + \mu y$ for each scalar $\lambda, \mu \in \mathbb{R}$.

(5) if $x_\alpha \xrightarrow{uo} x$, $z_\alpha \xrightarrow{uo} z$, and $x_\alpha \leq z_\alpha$ for all α, then $x \leq z$.

(6) if $x_\alpha \xrightarrow{uo} x$ and $x_\alpha \xrightarrow{uo} y$, then $x = y$.

(7) if $x_\alpha \xrightarrow{uo} 0$, then for each subnet $\{z_\gamma\}$ of $\{x_\alpha\}$, $z_\gamma \xrightarrow{uo} 0$.

Proof

(1) By Definition 1, it is established.

(2) Based on the assumption made, it is evident that $\{x_\alpha\}$ is order-bounded. Consequently, we can conclude that $x_\alpha \xrightarrow{\tilde{o}} x$. As a result, the proof becomes clear by applying Lemma 2.1 from [12].

(3) Let $0 \leq x_\alpha \xrightarrow{uo} x$, therefore for each $0 \leq y \in X$, there exists net $\{y_\beta\} \subseteq X$ such that $y_\beta \downarrow 0$ and for every β, there exists α_0 such that $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$ whenever $\alpha \geq \alpha_0$. We know that $\{x_\alpha - x\} \subseteq \{\pm(x_\alpha - x)\}$ for all α. Therefore, it is clear that for all α, $\{\pm(x_\alpha - x)\}^u \subseteq \{x_\alpha - x\}^u$. So, we have $\{\{(x_\alpha - x)\}^u, y\}^l \subseteq \{\{\pm(x_\alpha - x)\}^u, y\}^l$ for all α. Therefore, for $\alpha \geq \alpha_0$, $\{\{\{x_\alpha - x\}\}^u, y\}^l \subseteq \{y_\beta\}^l$ for each $y \in \{-x, 0\}^u$. It is clear that $-x \in \{\{(x_\alpha - x)\}^u, y\}^l$ for each $y \in \{-x, 0\}^u$. Hence, $-x \leq y_\beta$. So, $x \geq 0$.

(4) Let $\lambda > 0$ and $x_\alpha \xrightarrow{uo} x$. Then, we have $\{\{\pm(x_\alpha - x)\}^u, y\}^l = \lambda(\{\{\pm(x_\alpha - x)\}^u, y\}^l) \subseteq \{\lambda y_\beta\}^l$. Note that, if $\lambda < 0$, then $\lambda(\pm x_\alpha) = -\lambda(\pm x_\alpha)$.

We have $\{\{\pm(x_\alpha - x)\}^u + \{\pm(y_\alpha - y)^u\} \subseteq \{\pm(x_\alpha - x + y_\alpha - y)\}^u$. Therefore, $\{\{\{\pm(x_\alpha - x + y_\alpha - y)\}\}^u, y\}^l \subseteq \{\{\pm(x_\alpha - x)\}^u + \{\pm(y_\alpha - y)\}^u\}, y^l$ for each $0 \leq y \in X$. Since X has the *RDP* property, $\{\{\pm(x_\alpha - x + y_\alpha - y)\}^u, y\}^l \subseteq \{\{\{\pm(x_\alpha - x)\}^u, y\}^l + \{\{\pm(y_\alpha - y)\}^u\}, y\}^l$. Hence, $x_\alpha + y_\alpha \xrightarrow{uo} x + y$.

(5) For each $0 \leq y \in X$, we have $\{\{x - z\}^u, y\} = \{\{x - x_\alpha + x_\alpha - z\}^u, y\}$. It is clear that $\{\{x - x_\alpha + z_\alpha - z\}^u, y\} \subseteq \{\{x - x_\alpha + z_\alpha - z\}^u, y\}$ and therefore $\{\{x - x_\alpha + z_\alpha - z\}^u, y\}^l \subseteq \{\{x - x_\alpha + z_\alpha - z\}^u, y\}^l$. Since X has the *RDP* property, the rest of the proof is clear.

(6) Similar to the proof of Proposition 17, since for each α, we have $x_\alpha \leq x_\alpha$, therefore $x \leq y$ and $y \leq x$. Therefore, $x = y$.

(7) We have $\{z_\gamma\} \subseteq \{x_\alpha\}$. Therefore, $\{x_\alpha\}^u \subseteq \{z_\gamma\}^u$ and so $\{\{z_\gamma\}^u, y\}^l \subseteq \{\{x_\alpha\}^u, y\}^l$ for each $0 \leq y \in X$. Hence, the proof is complete. □

Proposition 5. *Let B be a projection band of an ordered Banach space X, and let P_B denote the corresponding band projection. If $\{x_\alpha\} \subseteq X$ is uo-null in X, then $\{P_B(x_\alpha)\}$ is uo-null in B.*

Proof. Let $x_\alpha \xrightarrow{uo} 0$ in X, therefore for each $0 \leq y \in X$, there is a net $\{y_\beta\}$, possibly over a different index set, such that $y_\beta \downarrow 0$ and for every β, there exists α_0 such that $\{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \geq \alpha_0$. We know that $\pm P_B(x_\alpha) \leq \pm(x_\alpha)$ for all α. Therefore, $\{\pm(x_\alpha)\}^u \subseteq \{\pm(P_B(x_\alpha))\}^u$. So, for each fixed $0 \leq y \in B$, we have $\{\{\pm(P_B(x_\alpha))\}^u, P_B(y)\}^l \subseteq \{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{y_\beta\}^l$. Note that, $P_B(y_\beta) \leq y_\beta$ for all β and $P_B(y_\beta) \downarrow 0$ in B. We suppose that $z \in \{\{\pm(P_B(x_\alpha))\}^u, y\}^l$. Hence, $z = P_B(z) \leq P_B(y_\beta)$. It means that $z \in \{P_B(y_\beta)\}^l$. Therefore, $\{\{\pm(P_B(x_\alpha))\}^u, P_B(y)\}^l \subseteq \{P_B(y_\beta)\}^l$ and so $P_B(x_\alpha) \xrightarrow{uo} 0$. □

Definition 6. A net $\{x_\alpha\}$ in ordered vector space X is said to be *uo*-Cauchy, if $\{x_\alpha - x_\beta\}_{(\alpha,\beta)}$ *uo*-converges to 0 in X.

Proposition 7. *Let X be an ordered vector space with the property RDP. Then,*

(1) if an uo-Cauchy net $\{x_\alpha\}$ has an uo-convergent subnet whose uo-limit is x, then $x_\alpha \xrightarrow{uo} x$.

(2) each uo-convergent net is a uo-Cauchy net.

Proof

(1) Let $\{x_\alpha\}$ be a *uo*-Cauchy net in X and $\{z_\gamma\}$ be a subnet of $\{x_\alpha\}$ such that $z_\gamma \xrightarrow{uo} x$. We have $x_\alpha - x = x_\alpha - z_\gamma + z_\gamma - x$ and $-(x_\alpha - x) = -(x_\alpha - z_\gamma + z_\gamma - x)$, and $\{\{\pm(x_\alpha - z_\gamma)\}^u + \{\pm(z_\gamma - x)\}^u\} \subseteq \{\pm(x_\alpha - z_\gamma + z_\gamma - x)\}^u$. Therefore, $\{\pm(x_\alpha - x)\}^u, y^l = \{\{\pm(x_\alpha - z_\gamma + z_\gamma - x)\}^u, y\}^l \subseteq \{\{\pm(x_\alpha - z_\gamma)\}^u + \{\pm(z_\gamma - x)\}^u, y\}^l$ for each $0 \le y \in X$. Since X has the *RDP* property, $\{\{\pm(x_\alpha - x)\}^u, y^l\} \subseteq \{\{\{\pm(x_\alpha - z_\gamma)\}^u, y\}^l + \{\{\pm(z_\gamma - x)\}^u\}, y^l\}$. So, by assumption, $x_\alpha - z_\gamma \xrightarrow{uo} 0$ and $z_\gamma \xrightarrow{uo} x$. Hence, the proof is complete.

(2) Without loss of generality, by Lemma 4, we assume that $\{x_\alpha\}$ is an unbounded order convergent to 0 in X. Then, there is a net $\{y_\sigma\}$ such that $y_\sigma \downarrow 0$ and for every σ, there exists α_0 such that $\{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{y_\sigma\}^l$, whenever $\alpha \ge \alpha_0$ and for each $0 \le y \in X$, we have

$$\{\pm(x_\alpha)\}^u + \{\pm(x_\beta)\}^u \subseteq \frac{1}{2}\{\pm(x_\alpha - x_\beta)\}^u. \tag{2}$$

Let $y > 0$. Then, by property *RDP*, we have

$$\left\{\frac{1}{2}\{\pm(x_\alpha - x_\beta)\}^u, y\right\}^l \subseteq \{\{\pm(x_\alpha)\}^u, y\}^l + \{\{\pm(x_\beta)\}^u, y\}^l. \tag{3}$$

It follows that $\{1/2\{\pm(x_\alpha - x_\beta)\}^u, y\}^l \subseteq 2\{y_\sigma\}^l$, whenever $\alpha, \beta \ge \alpha_0$, and so the proof follows. □

Theorem 8. *Let X be an order dense subspace of an ordered vector space Y and $\{x_\alpha\} \subseteq X$. $x_\alpha \xrightarrow{uo} 0$ in X if $x_\alpha \xrightarrow{uo} 0$ in Y.*

Proof. Let $x_\alpha \xrightarrow{uo} 0$ in X and $z \in \{\{\pm(x_\alpha)\}^u, y\}^l$ for fixed $0 \le y \in Y$. Since X is order dense in Y, therefore there exists $y' \in X$ such that $y \le y'$ and $z \in \{\{\pm(x_\alpha)\}^u, y'\}^l$. By assumption, there is a net $\{y_\beta\} \subseteq X$ such that $y_\beta \downarrow 0$ in X and $z \in \{y_\beta\}^l$. By Proposition 5.1 of [9], $y_\beta \downarrow 0$ in Y. Therefore, for each $0 \le y \in Y$, there is a net $\{y_\beta\} \subseteq Y$ such that $y_\beta \downarrow 0$ and for each β, there is an α_0 such that for each $\alpha \ge \alpha_0$, $\{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{y_\beta\}^l$. So, $x_\alpha \xrightarrow{uo} 0$ in Y.

Consequently, let $x_\alpha \xrightarrow{uo} x$ in Y. Therefore, for each $0 \le y \in Y$, there exists a net $\{y_\beta\} \subseteq Y$ such that $y_\beta \downarrow 0$ and for every β, there exists α_0 such that $\{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \ge \alpha_0$. Since X is order dense in Y, therefore X is majorizing in Y. Hence, for each β, there is a $z \in X$ such that $y_\beta \le z$. We define $z_\beta = \inf\{z \in X\colon y_\beta \le z\}$. Hence, $z_\beta \downarrow 0$ in X. It is obvious that $\{y_\beta\}^l \subseteq \{z_\beta\}^l$. Therefore, for each $0 \le y \in X$ and for each β, there is an α_0 such that for each $\alpha \ge \alpha_0$, $\{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{z_\beta\}^l$. □

Corollary 9. *Let X be a pre-Riesz space with order complete vector lattice cover (Y, i) and $\{x_\alpha\} \subseteq X$ is a uo-null in X. By Theorem 8, $x_\alpha \xrightarrow{uo} 0$ in Y. If Y has a weak unit u and $u \in X$, then by Lemma 3.2 of [1], $x_\alpha \xrightarrow{uo} 0$ in X if there is a net $\{y_\beta\}$, possibly over a different index set, such that $y_\beta \downarrow 0$ and for every β, there exists α_0 such that $\{\{\pm(x_\alpha - x)\}^u, u\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \ge \alpha_0$.*

Theorem 10. *If X is a pre-Riesz space with a vector lattice cover (Y, i), then the following assertions are true:*

(1) $\{x_\alpha\} \subseteq X$ is uo-null if $\{i(x_\alpha)\}$ is uo-null in Y.
(2) If the sequence $\{x_n\} \subseteq X$ is disjoint, then $x_n \xrightarrow{uo} 0$ in X. Moreover, if X is normed space and
(3) Y, Y' have an order continuous norm and $\{x_\alpha\} \subseteq X$ is norm bounded and uo-null, then $\{x_\alpha\}$ is w-null.
(4) $\{x_\alpha\} \subseteq X$ is order-bounded, $x_\alpha \xrightarrow{uo} 0$ in Y and Y is a Banach lattice with order continuous norm, then $\{x_\alpha\}$ is norm-null.

Proof

(1) Let $\{x_\alpha\} \subseteq X$ be *uo*-null in X. Then, for each $0 \le y \in X$, there is a net $\{y_\beta\}$, possibly over a different index set, such that $y_\beta \downarrow 0$ and for every β, there exists α_0 such that $\{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \ge \alpha_0$. By Lemma 4 of [13], $i(y_\beta) \downarrow 0$ in $i(X)$. It is obvious that $\{i(x_\alpha)\}$ is *uo*-null in $i(X)$. Since $i(X)$ is order dense in Y, then by Proposition 3, $\{i(x_\alpha)\}$ is *uo*-null in Y.
Conversely, let $\{x_\alpha\} \subseteq X$ and $i(x_\alpha) \xrightarrow{uo} 0$ in Y. Since $i(x_\alpha) \xrightarrow{uo} 0$ in Y, therefore it is *uo*-null in $i(X)$. Therefore, for each $0 \le i(y) \in i(X)$, there is a net $\{i(y_\beta)\}$, possibly over a different index set, such that $i(y_\beta) \downarrow 0$ and for every β, there exists α_0 such that

$$\{\{\pm i(x_\alpha)\}^u, i(y)\}^l \subseteq \{i(y_\beta)\}^l, \tag{4}$$

whenever $\alpha \ge \alpha_0$. It follows that

$$\begin{aligned} i\left(\{\{\pm(x_\alpha)\}^u, (y)\}^l\right) &= \{\{\pm i(x_\alpha)\}^u, i(y)\}^l \\ &\subseteq \{i(y_\beta)\}^l = i\left(\{y_\beta\}^l\right). \end{aligned} \tag{5}$$

Since i is a bipositive operator, therefore $\{\{\pm(x_\alpha)\}^u, y\}^l \subseteq \{y_\beta\}^l$ and so $x_\alpha \xrightarrow{uo} 0$ in X.

(2) Since X is an order dense subspace of Y, then by Proposition 5.9 of [9], $\{x_n\}$ is disjoint in Y. By

Corollary 3.6 of [4], $x_n \xrightarrow{uo} 0$ in Y. Therefore, $x_n \xrightarrow{uo} 0$ in $i(X)$. By 1, $x_n \xrightarrow{uo} 0$ in X.

(3) Let $\{x_\alpha\} \subseteq X$ be *uo*-null. Then by Proposition 3, $x_\alpha \xrightarrow{uo} 0$ in Y. Since Y and Y' have order continuous norm, then by Theorem 5 of [2], $x_\alpha \xrightarrow{w} 0$ in Y. By Theorem 3.6 of [14], $x_\alpha \xrightarrow{w} 0$ in X.

(4) We have $x_\alpha \xrightarrow{uo} 0$ in Y. Since Y is a Banach lattice with order continuous norm and the net $\{x_\alpha\}$ is almost order-bounded in Y, therefore by Proposition 3.7 of [1], $\{x_\alpha\}$ is norm-null in Y and so it is norm-null in X. □

Definition 11. A subset J of ordered vector space X is said to be *uo*-closed, if $\{x_\alpha\} \subseteq J$ with $x_\alpha \xrightarrow{uo} x$ implies that $x \in J$.

Let $A \subseteq X$ be a solid subset of a vector lattice X. By Lemma 8.1 from [7], A is (sequentially) o-closed if and only if it is (sequentially) *uo*-closed.

Remark 12. Let $J \subseteq X$ be a *uo*-closed set and $\{x_\alpha\} \subseteq J$ such that $x_\alpha \xrightarrow{\tilde{o}} x$. It is obvious that $x_\alpha \xrightarrow{uo} x$. Since J is *uo*-closed, therefore $x \in J$. So, J is $\tilde{o}$-closed.

Theorem 13. *The following assertions are true.*

(1) Let X be an order dense subspace of an ordered vector space Y. If $J \subseteq Y$ is uo-closed in Y, then $J \cap X$ is uo-closed in X.

(2) Let X be a pre-Riesz space with vector lattice cover (Y, i). $J \subseteq X$ is uo-closed if $i(J)$ is uo-closed in $i(X)$.

Proof

(1) Let $\{x_\alpha\} \subseteq J \cap X$ and $x_\alpha \xrightarrow{uo} x$ in X. Hence, for each $0 \le y \in X$, there exists a net $\{y_\beta\} \subseteq X$ such that $y_\beta \downarrow 0$ and for each β, there exists α_0 such that $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \ge \alpha_0$. Since X is order dense in Y, then for each $z \in Y$, $\{\{\pm(x_\alpha - x)\}^u, z\}^l \subseteq \{\{\pm(x_\alpha - x)\}^u, z\}^l \subseteq \{y_\beta\}^l$ and $y_\beta \downarrow 0$ in Y. Since $\{x_\alpha\} \subseteq J$ and J is *uo*-closed in Y, therefore $x \in J$.

(2) Let $\{x_\alpha\} \subseteq J$ and $x_\alpha \xrightarrow{uo} x$. It is obvious that $i(x_\alpha) \xrightarrow{uo} i(x)$. Since $i(J)$ is *uo*-closed, therefore $i(x) \in i(J)$. So, $x \in J$.

Conversely, let $\{i(x_\alpha)\} \subseteq i(J)$ and $i(x_\alpha) \xrightarrow{uo} i(x)$. So, for each $0 \le i(y) \in i(X)$, there exists a net $\{i(y_\beta)\} \subseteq i(X)$ such that $i(y_\beta) \downarrow 0$ and for each β, there exists α_0 such that $\{\{\pm(i(x_\alpha - x)\}^u, i(y)\}^l \subseteq \{i(y_\beta)\}^l$, whenever $\alpha \ge \alpha_0$. By Lemma 4 of [13], $y_\beta \downarrow 0$ in X. Therefore, $x_\alpha \xrightarrow{uo} x$. Since, $\{x_\alpha\} \subseteq J$ and J is *uo*-closed, hence $x \in J$ and therefore $i(x) \in i(J)$. □

Theorem 14. *Let J be a uo-closed subspace in Archimedean ordered vector space X and $\{x_\alpha\} \subseteq J$. Then $x_\alpha \xrightarrow{uo} x$ in J if $x_\alpha \xrightarrow{uo} x$ in X.*

Proof. Let $\{x_\alpha\} \subseteq J$ and $x_\alpha \xrightarrow{uo} x$ in X. Since J is *uo*-closed it is obvious that $x_\alpha \xrightarrow{uo} x$ in J.

Conversely, let $\{x_\alpha\} \subseteq J$ and $x_\alpha \xrightarrow{uo} x$ in J. Therefore, for each $0 \le y \in J$, there is a net $\{y_\beta\} \subseteq J$, possibly over a different index set, such that $y_\beta \downarrow 0$ in J and for every β, there exists α_0 such that $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \ge \alpha_0$. It is obvious that $y_\beta \downarrow$ in X. Since X is Archimedean, there exists a z such that $y_\beta \downarrow z$ in X. It is clear that $y_\beta \xrightarrow{\tilde{o}} z$ in X and so $y_\beta \xrightarrow{uo} z$ in X. Therefore, $y_\beta \xrightarrow{uo} z$ in J. Since by Lemma 4, *uo*-limits are unique, therefore $z = 0$. Hence, $y_\beta \downarrow 0$ in X. Let $k \in \{\{\pm(x_\alpha - x)\}^u, y\}^l$ for each $0 \le y \in X$. It is clear that $k \in \{\{\pm(x_\alpha - x)\}^u, y'\}^l$ for each $0 \le y' \in J$. So, $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$ for each $0 \le y \in X$. Therefore, $x_\alpha \xrightarrow{uo} x$ in X. □

Proposition 15. *A solid subset J of an ordered vector space X is uo-closed if $\{x_\alpha\} \subseteq J$ and $0 \le x_\alpha \uparrow x$ imply that $x \in J$.*

Proof. Let $\{x_\alpha\} \subseteq J$ and $0 \le x_\alpha \uparrow x$. It is clear that $x_\alpha \xrightarrow{\tilde{o}} x$ and therefore $x_\alpha \xrightarrow{uo} x$. Since J is *uo*-closed, it implies that $x \in J$.

Conversely, let a net be $\{x_\alpha\} \subseteq J$ and $x_\alpha \xrightarrow{uo} x$. Then, for each $0 \le y \in X$, there is a net $\{y_\beta\} \subseteq X$, possibly over a different index set, such that $y_\beta \downarrow 0$ in X and for every β, there exists α_0 such that $\{\{\pm(x_\alpha - x)\}^u, y\}^l \subseteq \{y_\beta\}^l$, whenever $\alpha \ge \alpha_0$. It is clear that $\{\pm x_\alpha\}^u \subseteq \{\pm(x - y_\beta)\}^u$. Since J is solid, we have $\{x - y_\beta\} \subseteq J$. It is obvious that $0 \le x - y_\beta \uparrow x$. It follows that $x \in J$. Hence, J is *uo*-closed.

Pre-Riesz spaces are precisely defined as the order dense linear subspaces of vector lattices. The research paper [9] explores the restriction and extension properties of ideals, solvex ideals, and bands within this context. The authors specifically investigate these properties in Archimedean-directed partially ordered vector spaces, as they are all considered pre-Riesz spaces. For ordered vector spaces, in the subsequent discussion, we will introduce these concepts for the unbounded case while highlighting their relevant properties.

Let B be an ideal in ordered vector space X. Then, B is said to be a *uo*-band in X if B is *uo*-closed in X. Based on this definition, we will obtain the following outcome. □

Corollary 16

(1) Let X be an order dense subspace of directed ordered vector space Y. If $J \subseteq Y$ is uo-band in Y, then by Theorem 13, $J \cap X$ is uo-closed in X and by Proposition 5.3 of [9], $J \cap X$ is an ideal in X. Therefore, $J \cap X$ is a uo-band in X.

(2) *Let X be a pre-Riesz space with vector lattice cover (Y, i). It is clear that $J \subseteq X$ is an ideal in X if $i(J)$ is an ideal in $i(X)$ and by Theorem 13, $J \subseteq X$ is uo-closed in X if $i(J)$ is uo-closed in $i(X)$. So, $J \subseteq X$ is a uo-band in X if $i(J)$ is uo-band in $i(X)$.*

Proposition 17. *Let X be a vector lattice. If an ideal $B \subseteq X$ is a uo-band in X, then it is a band in X.*

Proof. Let an ideal B be a *uo*-band in X and $\{x_\alpha\} \subseteq X$ be a net such that $x_\alpha \xrightarrow{o} x$. Therefore, $x_\alpha \xrightarrow{uo} x$ in X. Since X is a vector lattice, then by Proposition 3, $x_\alpha \xrightarrow{uo} x$ in X. Given the assumption that x belongs to B, we can conclude that B forms a band in X.

Malinowski [11] demonstrated that in an Archimedean pre-Riesz space, a directed band may not be s-closed. However, in the following, by Proposition 15, every *uo*-band is indeed s-closed. □

Corollary 18. *Let X be an Archimedean pre-Riesz space. Then, every uo-band is s-closed.*

Data Availability

No data were used to support this study.

Disclosure

This article has already been pre-printed in [15].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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