Research Article

Analytical Solutions for Bending of Nanoscaled Bars Based on Eringen's Nonlocal Differential Law

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The one-dimensional nanoscaled bar is commonly seen in current nanodevices, and it plays a significant role in nanoelectromechanical system (NEMS) and related nanotechnology. Motivated by this, the paper is concerned with the bending and stability of a nanoscaled bar especially a flexible bar subjected to vertical concentrated, vertical linearly distributed, and horizontal concentrated forces simultaneously. The theoretical model is developed by employing Eringen's nonlocal differential law. Hence, the nonlocal differential constitutions in terms of stress and bending moment at a nanoscale are applied to the classical equilibrium formulation in order to construct the nonlocal constitutive model and then determine the analytical solutions for bending of such a nanoscale bar. Subsequently, the effect of a nonlocal scale on the midpoint deflections and internal forces is shown numerically, and the internal and external characteristic length scales are also examined. It is revealed that the capacity of resisting compression decreases with increasing the nonlocal effect since the internal characteristic scale cannot be neglected by comparing with the external characteristic scale. There exists a threshold value for bending stiffness. Both the midpoint deflection and bending moment vary monotonously when the material bending stiffness exceeds the threshold value, while the mechanical quantities fluctuate when the bending stiffness is less than its threshold value, namely, for a flexible nanoscaled bar. Accordingly, two different kinds of nonlocal predictions and the related disputes are resolved in the context of Eringen's nonlocal differential law. The work is expected to be useful for the design, application, and optimization of nanoscaled bars sustaining bending with linear vertical and horizontal forces.

1. Introduction

Structural bending deformation exists extensively in mechanical engineering, civil engineering, and aerospace engineering, and it has received a great deal of research attention. In addition to the common bending, there is a special kind of bending deformation for bars subjected to vertical and horizontal forces simultaneously. For such a bending bar with vertical and horizontal forces, the effect of the horizontal force (tensile or compressive force) on the bending deformation cannot be ignored if the deformation resistance of the bar is sufficiently small. So the influence of horizontal force for a flexible bending bar has to be specifically taken into consideration. In particular, if the horizontal force is a

compression, the structure is called a compressive bar, and the obvious bending deformation occurs because of the combined actions of horizontal compressive force and vertical linear force. Therefore, the horizontal compressive force plays a more remarkable role and cannot be omitted in predicting the mechanical properties and stabilities of a compressive bar. Compared with the tensile forces, the compressive forces of a bar get more attention. This is because the combined actions of horizontal compressive force and vertical force may cause the bar to lose its stability. At present, the research work on static bending with vertical and horizontal compressive forces of a nanoscaled bar is seldom seen. The nanoscaled bar is one of the most important components in nanoactuators, nanosensors, nanoresonators,

and nanooscillators, etc. So it requires systematic studies to characterize and control the mechanical behaviors at a nanoscale including the present point nonlocal scale effect.

As we know, the classical theory is valid to describe the static bending of bars at a macroscale. However, when the size of bars decreases to a nanoscaled level, the classical continuum theory is incapable of predicting new properties of nanoscaled bars since the intrinsic internal length scale should be considered in the constitutive relation. This is because the internal and external characteristic scales remain at almost the same level for a nanoscaled bar, but the classical theory does not contain the effect of a material internal length scale. Thus, it requires a new approach to study the nanoscaled bars to meet the nanoengineering requirements, such as NEMS, nanorobots, and nanodevices for drug screening. In general, there are different approaches to study the nanostructures including the modified continuum theories, the molecular dynamics simulation, and experiments at a nanoscale. Since the molecular dynamics simulation is incapable of calculating a structural system including a large number of atoms and the nanoscaled experiment costs too much and is hard to control and measure, the modified continuum theoretical approach becomes a popular option. The modified continuum theory, as its name implies, is based on but different from the classical continuum theory. The constitutive relations of classical continuum theory are amended by introducing one or more internal characteristic length scales to suit for the nanomaterials and nanostructures, and that is the modified continuum theory. One of the most popular modified continuum theories is the nonlocal continuum theory, which was proposed by Eringen and Edelen [1, 2]. The nonlocal theory is based on the concept of long-range force between atoms and molecules. It is assumed that the stress at a point of a continuum is related not only to the strain of that point but also to all other points in the continuum. Such theory contains the differential model and integral model, and it shows unique advantages when dealing with the issues that cannot be explained by classical continuum mechanics. It should be pointed out that at first the nonlocal theory was applied to the classical mechanics including the fracture mechanics, damage mechanics, dislocation mechanics, fluid mechanics, and mechanics of composites. From the beginning of the twentieth century, the nonlocal theory especially the differential model has been extensively used in nanomechanics. It is suited for the nanomechanics since an internal length scale is involved in nonlocal constitution. On the other hand, as a hot research field, the nanomechanics makes great progress with the development of nonlocal theory and other generalized continuum mechanics theories; hence, an increasing number of research papers have been published in recent years [3–24]. Adali [6] proposed a variational principle for vibrating multiwalled carbon nanotubes based on the nonlocal Euler-Bernoulli beam model. It was shown that the boundary conditions are coupled due to nonlocal effects. Li and Wang [12] studied the morphological evolution and migration of void in bipiezoelectric material interface using a nonlocal phase field model and finite element method, and the work is useful for the safety evaluation of piezoelectric materials. Zhu and Li [16] employed the non-

local integral model to investigate the twisting static behaviors of through-radius functionally graded nanotubes, and they determined the closed-form solutions of a rotational angle. Xu et al. [18] studied the scale effects on the dynamics of nanorods based on the nonlocal strain gradient theory and clarified the variationally consistent boundary conditions. Hosseini [20] presented the Laplace transform-based analytical solutions for coupled thermoelasticity analysis with energy dissipation in a micro/nanobeam resonator using the Green-Naghdi theory of generalized coupled thermoelasticity and nonlocal Rayleigh beam theory, and the author revealed the small-scale effect in the transient behaviors of fields' variables using the Talbot method. Kiani [22] investigated the traveling transverse waves in vertically aligned jungles of single-walled carbon nanotubes in the presence of a longitudinal magnetic field using a nonlocal higher-order beam theory. Barati and Shahverdi [24] presented the large-amplitude vibration of a porous nanoplate resting on nonlinear hardening elastic foundations and obtained the closed-form solutions of nonlinear frequencies using the homotopy perturbation and Hamiltonian methods. Subsequently, Lim et al. [25] developed a new nonlocal strain gradient theory based on Eringen's nonlocal theory and Mindlin's strain gradient theory and examined the wave propagation in carbon nanotubes. Barati [26] continued to propose a general nonlocal stress-strain gradient theory and investigated the forced vibration of heterogeneous porous nanoplates. The refined theories were verified via a large number of theoretical applications to nanostructures [27–29]. For example, Ebrahimi and Barati [27] studied the buckling properties of a size-dependent curved functionally graded nanobeam based on the nonlocal strain gradient theory and a higher-order shear beam model without any shear correction factor. Barati and Shahverdi [28] constructed a new model for mass detection of a nanomechanical mass sensor based on vibrating nanoporous nanoplates using the nonlocal strain gradient theory. The frequency shift due to mass sensing was obtained via Galerkin's method, and it was found to change with the attached mass and number of nanoparticles. On the basis of the nonlocal stress-strain gradient theory, a recent work by Mirjavadi et al. [29] presented the transient behaviors of a porous functionally graded nanoplate subjected to various impulse forces including rectangular, triangular, and sinusoidal types via Galerkin's method and an inverse Laplace transform approach.

During the development of nonlocal theory, there were two kinds of nonlocal models in nonlocal differential constitution and both were studied extensively. When increasing the nonlocal scale parameter or when the nonlocal effect gets stronger, the increasing and decreasing trends of deformation result in different kinds of nonlocal differential models, namely, the nonlocal stiffness-weakening model (e.g., [3–5]) and the stiffness-strengthening model (e.g., [9, 11, 13]). The strengthening model declares that nanostructural stiffness is enhanced with stronger nonlocal effects, while the weakening model asserts an opposite conclusion. People were puzzled with such different nonlocal results. In recent years, Shen and Li [30] revealed the physical meanings of both nonlocal differential models and proved

that both models are reasonable. The stiffness-strengthening and stiffness-weakening models are found to be related to different types of surface effects, i.e., the repulsive or attractive interactions, respectively. Moreover, Xu et al. [18] indicated that both the weakening and strengthening effects can be captured by adjusting two material length parameters in the nonlocal strain gradient theory. It should be noted that the literature reviews aforementioned belong to the field of the nonlocal differential model. In particular, the nonlocal integral constitution has been investigated widely in very recent years [31–33]. The roles of the stress field and strain field are swapped in the nonlocal integral model, which is different from the nonlocal differential model. Consequently, the nonlocal differential and integral models can be considered strain-driven and stress-driven nonlocal models, respectively. Actually, the integral model is the original form of Eringen's nonlocal theory. Apuzzo et al. [31] investigated the free vibration of Bernoulli-Euler nanobeams using the nonlocal integral model, and the results obtained were compared with those by the nonlocal differential model and the strain gradient model, thereby proving the validity of the integral model. Barretta et al. [32] presented the size-dependent behaviors of inflected Timoshenko nanobeams based on the nonlocal integral model, where the error function of probability theory and the special biexponential function proposed by Helmholtz were selected as averaging kernels of nonlocal theory. Oskouie et al. [33] proposed a novel numerical approach for bending of Euler-Bernoulli nanobeams in the framework of nonlocal differential and integral models, respectively, the finite difference technique and trapezoidal integration rule were utilized, and the effectiveness of the developed numerical methodology was verified.

In this paper, the nonlocal differential law is employed to study the bending and stability of a nanoscaled bar subjected to both vertical and horizontal compressive forces. Previous studies have considered either the bending deformation of nanobeams under vertical forces or the tension/compression deformation of nanorods under horizontal forces. But there is no relevant research on the bending due to both vertical and horizontal forces of nanoscaled bars. Based on such a motivation, we explore the special characteristics in bending and stability of the nanoscaled bar, and this is the main contribution of the present paper. The results obtained are compared with the classical solutions (without the nonlocal scale effect) to show the nonlocal properties in vertical and horizontal bending. The work may be useful for nanoengineering where the bar-like structure acts as a basic element.

2. Nonlocal Methodology and Solution

2.1. Nonlocal Differential and Integral Methodologies. With the rising of nanomechanics, the nonlocal theory has attracted more and more researchers' interest during the past several decades. Such theory is different from the classical continuum mechanics because in nonlocal constitutive relation, the stress at a point is related to the strains of all points in the continuum while the latter considers that the stress of a point is only a function of the strain at the same point. Hence,

the property of nonlocal interactions between the local components of the continuum is long-range effort. The basic equations of Eringen's nonlocal integral theory for an isotropic, homogeneous, and elastic continuum without body force is given by [1]

$$t_{kl}(\mathbf{x}) = \int_V \alpha(|\mathbf{x}' - \mathbf{x}|, \tau) \sigma_{kl}(\mathbf{x}') dv(\mathbf{x}'), \quad (1)$$

where t_{kl} is the nonlocal stress tensor at a given point $\mathbf{x}$, while σ_{kl} is the local (classical) stress tensor at $\mathbf{x}'$, and α is the nonlocal kernel function that is associated with distance between points $\mathbf{x}$ and $\mathbf{x}'$, as well as a dimensionless nonlocal scale parameter τ . According to the physical equation of classical elasticity, $\sigma_{kl} = \lambda \varepsilon_{rr} \delta_{kl} + 2\mu \varepsilon_{kl}$, where ε_{kl} or ε_{rr} is the corresponding local strain tensor, δ_{kl} is the Kronecker delta function, and λ and μ are Lamé constants. According to the geometric equation of classical elasticity, $\varepsilon_{kl} = (u_{k,l} + u_{l,k})/2$, where u_k or u_l is the displacement. Consequently, Equation (1) shows the only difference between the nonlocal theory and local theory. In addition to Equation (1), the other formulas or components of the nonlocal constitutive relation are the same as those of the classical local theory. Nonlocal theory has shown great advantages in dealing with some typical mechanical problems. For example, according to the classical continuum theory, the stress field near the crack tip is singular and it is hard to understand from the physical mechanism since the experimental results and atomistic simulations are inconsistent with the classical prediction. However, the application of the nonlocal theory shows that stress near the crack tip is nonsingular and this is a reasonable result. Another example is the phonon scattering experiment. It has proved that the high-frequency elastic wave is dispersive and its propagation speed is related to frequency. Using the nonlocal continuum theory, one can obtain the same conclusion, but the wave velocity is a constant and it does not change with the frequency according to the classical mechanics. For this reason, the nonlocal theory has been applied widely in micro/nanomechanics in recent years, of which the nonlocal constitutive relation acts a natural role in nanomechanics because the theory can assess the scale effect via the ratio of internal to external characteristic length scales.

For one-dimensional nanoscaled structures, such as the nanobeams, nanorods, nanobars, and nanotubes, the nonlocal integral relation, i.e., the second equation of Equation (1), provides the following consequent nonlocal differential equation as [2]

$$t_{kl} - (e_0 a)^2 \nabla^2 t_{kl} = \sigma_{kl}. \quad (2)$$

It is noticed that the basic nonlocal integral constitutive relation can be represented by a simplified differential tensor equation. The approximate transformation from a set of partial integro-differential equations to a differential constitutive equation requires some certain conditions using Green's function with a certain approximation error [2]. However, in some special cases, such a simplification is not valid. For

FIGURE 1: Sketch of a nanoscaled bar under bending with vertical and horizontal forces.

instance, when the nonlocal differential model is adopted, the nanobeam's deflections under most boundary constraints are softening for increasing values of the nonlocal scale parameter, but the deflection solution of a cantilever nanobeam with uniformly or linearly distributed forces is stiffening [11]. Although the paper is only concerned with the simply supported boundary constraint, it can be predicted that the nonlocal differential-based deflection for a cantilever nanoscaled bar is less than that of a classical continuum bar without a nonlocal effect; namely, the equivalent stiffness of cantilever nanoscaled bars is enhanced in the context of the nonlocal differential model. In this case, the nonlocal differential model is invalid; instead, the nonlocal integral model is necessary. Due to a limited space, we do not expand the case of cantilever nanoscaled bars in detail. But we can predict the results because the physical mechanism for cantilever nanoscaled bars is the same with that for the cantilever nanobeams. Since the nanoscaled cantilever bar is not included in this paper, the simplification from the nonlocal integral model to the differential model does not affect the results of this paper.

2.2. Equilibrium Equations and Solutions. Considering a simply supported nanoscaled bar subjected to various forces including the horizontal compressive force P , the vertical force Q , and the uniformly distributed lateral forces q as shown in Figure 1, the length of the nanoscaled bar is l and the distance between the action point C of vertical concentrated force Q and the right end B is denoted as c . If we remove the simply supported constraints at both ends, the internal axial force and lateral force at the left side A , as well as the internal lateral force at the right side B , can be determined, respectively, according to the static equilibrium. Note that the internal axial force at the left side is exactly equal to P (equal in magnitude but opposite in direction). There are three situations: (i) When the nanoscaled bar is only subjected to the horizontal tensile force, the bar yields tensile deformation but not bending. The tensile deformation is linear elasticity within the range of small deformation, and then, it undergoes yielding, strengthening, and necking successively within the range of large deformation. (ii) When the nanoscaled bar is only subjected to the compressive force (i.e., the vertical force and uniformly distributed forces in Figure 1 are neglected), the bar yields compression first and then bending deformations, so the issue becomes a buckling problem, and the bar is called the Euler bar. (iii) When the nanoscaled bar is subjected

to the compressive force, vertical force, and uniformly distributed forces as shown in Figure 1, the bar yields compression and bending deformations, but the bending dominates the whole elastic deformation. In particular, a flexible nanoscaled bar is considered herein, and the bending stiffness is small for the flexible nanoscaled bar. On the other hand, the existence of compressive force decreases the bending stiffness or increases the bending deformation. This is why the horizontal compressive force can lead to bending and why the effect of compressive force must be considered in this study. Assuming the deflection of the position x is v according to the Descartes coordinate system in Figure 1, we can derive the bending moment $M(x)$ for AC and BC segments, respectively, as

$$M(x) = \frac{Qc}{l}x - Pv + \frac{1}{2}qlx - \frac{1}{2}qx^2, \quad 0 \leq x \leq l - c, \quad (3a)$$

$$M(x) = \frac{Q(l-c)(l-x)}{l} - Pv + \frac{1}{2}qlx - \frac{1}{2}qx^2, \quad l - c \leq x \leq l, \quad (3b)$$

where the term $-Pv$ represents the contribution of the horizontal compressive force. Because the bending deflection is negative in Figure 1, the term $-Pv$ is positive since the compressive force takes its absolute value in Equations (3a) and (3b). Consequently, the horizontal compressive force increases the bending moment and further increases the bending deflection, which is consistent with the predictions above.

According to the extensive studies on the nonlocal differential model, the relationship between bending moment and deflection for one-dimensional nanostructures in the framework of Eringen's differential constitution can be stated as [3]

$$M - (e_0a)^2 \frac{d^2M}{dx^2} = EI \frac{d^2v}{dx^2}, \quad (4)$$

where e_0 is the nonlocal material parameter, a is the internal characteristic length scale, and EI denotes the bending stiffness. It is noticed that the nonlocal effect is measured by a combination of nonlocal material parameter e_0 and internal characteristic length a . Obviously, if e_0 becomes zero in Equation (4) or the intrinsic internal length scale is negligible when compared with the external characteristic length scale, the bending moment versus deflection relation becomes the corresponding classical continuum counterpart. In fact, this is the assumption of the long-wavelength limit. On the other hand, the nonlocal theory becomes the atomic lattice dynamics under the assumption of short-wavelength limit. In this study, one of the main aims is to reveal the effect of nonlocal small-scale parameter e_0a on the static bending of nanoscaled bars since the scale effect means a great deal to the design and application of nanostructures.

Combining Equations (3a), (3b), and (4), one yields

$$[EI - (e_0 a)^2 P] \frac{d^2 v}{dx^2} + P v = -\frac{1}{2} q x^2 + \frac{1}{2} q l x + \frac{Q c}{l} x + (e_0 a)^2 q, \quad (5a)$$

$$[EI - (e_0 a)^2 P] \frac{d^2 v}{dx^2} + P v = -\frac{1}{2} q x^2 + \frac{1}{2} q l x + \frac{Q(l-c)(l-x)}{l} + (e_0 a)^2 q. \quad (5b)$$

The general solutions for bending deflection can be written as

$$v = A \cos kx + B \sin kx - \frac{q}{2P} x^2 + \left(\frac{ql}{2P} + \frac{Qc}{Pl} \right) x + \frac{EIq}{P^2}, \quad 0 \leq x \leq l-c, \quad (6a)$$

$$v = C \cos kx + D \sin kx - \frac{q}{2P} x^2 + \left[\frac{ql}{2P} - \frac{Q(l-c)}{Pl} \right] x + \frac{Q}{P}(l-c) + \frac{EIq}{P^2}, \quad l-c \leq x \leq l, \quad (6b)$$

where $k = \sqrt{P/(EI - (e_0 a)^2 P)}$, and a nonlinear relation between the deflection and the horizontal compressive force is observed. That is to say, the superposition principle adapts to the vertical force and the uniformly distributed lateral forces but not to the horizontal compressive force. The correlation between the capacity of resisting compression and intrinsic length scale can be determined using the following condition:

$$\frac{P}{EI - (e_0 a)^2 P} > 0. \quad (7)$$

It is implied that the capacity of resisting compression changes significantly with respect to the material rigidity and nonlocal small scale. The capacity of resisting compression decreases with an increase in the nonlocal small scale, while it increases with an increase in bending stiffness. The conclusions can be explained as follows. The stronger nonlocal effect corresponding to a lower capacity of resisting compression means the nonlocal weakening, which is caused by the physical mechanism of the nonlocal differential model with the simply supported boundary constraint. The larger bending stiffness corresponding to a higher capacity of resisting compression means that there must be sufficiently large compression to bend the nanoscaled rod. With the continuous increase in the nonlocal small scale, the sensitivity of the capacity of resisting compression to the rigidity decreases and the capacity of resisting compressions under different values of bending stiffness tends to be nearly a constant.

The Ritz method is employed herein, and the deflection solution should satisfy the displacement boundary conditions. According to the simply supported boundary condition, the well-known displacement boundary conditions require that the deflections of both ends be zero. That is,

$$v = 0 \text{ at } x = 0, l. \quad (8)$$

The unknown coefficients can be determined from Equations (6a), (6b), and (8) as

$$A + \frac{EIq}{P^2} = 0, \quad (9a)$$

$$C \cos kl + D \sin kl + \frac{EIq}{P^2} = 0. \quad (9b)$$

But they are not sufficient because we have four unknown coefficients. On the other hand, the continuity conditions require that the displacement or bending deflection at $x = l - c$ (i.e., the action point of the vertical concentrated force) should be equal from Equations (6a) and (6b) and the rotation at $x = l - c$ should be equal, too. Consequently,

$$(A - C) \cos k(l - c) + (B - D) \sin k(l - c) = 0, \quad (9c)$$

$$(C - A) \sin k(l - c) + (B - D) \cos k(l - c) + \frac{Q}{kP} = 0. \quad (9d)$$

Therefore, the coefficients can be determined as

$$\begin{aligned} A &= -\frac{EIq}{P^2}, \\ B &= \frac{(EIq/P^2)(\cos kl - 1) - (Q/kP) \sin kc}{\sin kl}, \\ C &= -\left[\frac{EIq}{P^2} + \frac{Q}{kP} \sin k(l - c) \right], \\ D &= \frac{(EIq/P^2)(\cos kl - 1) + (Q/kP) \sin k(l - c) \cos kl}{\sin kl}. \end{aligned} \quad (10)$$

As a result, the bending deflection of the nanoscaled bar at the AC section is

$$\begin{aligned} v &= -\frac{EIq}{P^2} \cos kx \\ &+ \frac{(EIq/P^2)(\cos kl - 1) - (Q/kP) \sin kc}{\sin kl} \sin kx - \frac{q}{2P} x^2 \\ &+ \left(\frac{ql}{2P} + \frac{Qc}{Pl} \right) x + \frac{EIq}{P^2}, \end{aligned} \quad (11)$$

and that at the CB section is

$$v = - \left[\frac{EIq}{P^2} + \frac{Q}{kP} \sin k(l-c) \right] \cos kx + \frac{(EIq/P^2)(\cos kl - 1) + (Q/kP) \sin k(l-c) \cos kl}{\sin kl} \sin kx - \frac{q}{2P} x^2 + \left[\frac{ql}{2P} - \frac{Q(l-c)}{Pl} \right] x + \frac{Q}{P} (l-c) + \frac{EIq}{P^2}. \quad (12)$$

In particular, when $c = l/2$, the vertical force Q acts on the midpoint of the nanoscaled bar, and the maximum deflection at the middle section $x = l/2$ can be expressed by

$$v_{\text{mid}} = - \frac{EIq}{P^2 \cos(kl/2)} - \frac{Q}{2kP} \tan \frac{kl}{2} + \frac{ql^2}{8P} + \frac{Ql}{4P} + \frac{EIq}{P^2}. \quad (13)$$

It seems that the horizontal compressive force should not be zero in the solution (13), but it is not a contradiction. This is because we are not studying the ordinary bending of beams (as we know, the governing equation for ordinary bending of nonlocal beams is a fourth-order differential equation), but the bending of a nanoscaled bar (as we see in Equations (5a) and (5b), it is a second-order differential equation), and hence, $P \neq 0$. On the other hand, if $Q = q = 0$, one gets $v = 0$. This is because only the axial tension or compression occurs in the nanoscaled bar but it does not bend (or it is a buckling problem in such a case). The objective of the analyses above is to show the elastic deformation of the nanoscaled bar. Furthermore, we can characterize the internal force in bending of a nanoscaled bar. For example, the relation between the bending moment and the nonlocal scale parameter can be determined by substituting nonlocal deflections shown in Equations (11) and (12) into Equations (3a) and (3b) respectively, as

$$M(x) = \frac{EIq}{P} \cos kx - \frac{(EIq/P)(\cos kl - 1) - (Q/k) \sin kc}{\sin kl} \sin kx - \frac{EIq}{P}, \quad (14a)$$

$$M(x) = \left[\frac{EIq}{P} + \frac{Q}{k} \sin k(l-c) \right] \cos kx - \frac{(EIq/P)(\cos kl - 1) + (Q/k) \sin k(l-c) \cos kl}{\sin kl} \sin kx - \frac{EIq}{P}, \quad (14b)$$

where Equation (14a) represents the bending moment of the AC section ($0 \leq x \leq l-c$) and Equation (14b) represents the bending moment of the BC section ($l-c \leq x \leq l$). When $c = l/2$, we can prove that the midpoint bending moment calculated from the AC section is equal to that from the BC section.

FIGURE 2: The differential element of a nanobeam subjected to the compressive force and uniformly distributed forces.

This is because

$$\begin{aligned} & \left[\frac{EIq}{P} \cos \frac{kl}{2} - \frac{(EIq/P)(\cos kl - 1) - (Q/k) \sin(kl/2)}{\sin kl} \sin \frac{kl}{2} - \frac{EIq}{P} \right] \\ & - \left[\left(\frac{EIq}{P} + \frac{Q}{k} \sin \frac{kl}{2} \right) \cos \frac{kl}{2} - \frac{(EIq/P)(\cos kl - 1) + (Q/k) \sin(kl/2) \cos kl}{\sin kl} \sin \frac{kl}{2} - \frac{EIq}{P} \right] \\ & = \frac{(Q/k) \sin(kl/2)}{\sin kl} \sin \frac{kl}{2} - \frac{Q}{k} \sin \frac{kl}{2} \cos \frac{kl}{2} \\ & + \frac{(Q/k) \sin(kl/2) \cos kl}{\sin kl} \sin \frac{kl}{2} = \frac{Q}{k} \sin \frac{kl}{2} \frac{kl(1 + \cos kl)}{\sin kl} \sin \frac{kl}{2} \\ & - \frac{Q}{k} \sin \frac{kl}{2} \cos \frac{kl}{2} = \frac{Q}{k} \sin \frac{kl}{2} \left(\frac{2 \cos^2(kl/2)}{2 \sin(kl/2) \cos(kl/2)} \sin \frac{kl}{2} \right. \\ & \left. - \cos \frac{kl}{2} \right) = \frac{Q}{k} \sin \frac{kl}{2} \left(\cos \frac{kl}{2} - \cos \frac{kl}{2} \right) = 0. \end{aligned} \quad (15)$$

Finally, the midpoint bending moment is given as

$$M_{\text{mid}} = \frac{EIq}{P} \left[\left(\cos \frac{kl}{2} \right)^{-1} - 1 \right] + \frac{Q}{2k} \tan \frac{kl}{2}. \quad (16)$$

In order to validate the present theoretical model, a comparative analysis is necessary. However, the previous studies on a nanoscaled bar with vertical and horizontal forces were rarely seen before. During the past years, the nanobeams have attracted the most researchers' attention, and bending of nanobeams has been fully studied. For a nanobeam subjected to the compressive force P and uniformly distributed forces q , we can derive the nonlocal differential equilibrium formulation as shown in Figure 2.

Selecting a differential element with length dx , as shown in Figure 2, we can gain the classical equilibrium equation according to the static force equilibrium analyses as

$$V + \frac{dV}{dx} dx + P \left(\theta + \frac{d\theta}{dx} dx \right) = V + q dx + P \theta, \quad (17a)$$

$$V = \frac{dM}{dx}, \quad (17b)$$

where the internal force V is the shear force, θ is the angle of rotation, and the higher-order items with respect to dx^2 are omitted. Therefore, we have

$$\frac{d^2 M}{dx^2} + P \frac{d^2 w}{dx^2} = q, \quad (18)$$

where the relationship between the deflection and angle of rotation (slope) $\theta = dw/dx$ is adopted. Subsequently, the nonlocal differential equilibrium formulation is obtained by using Equation (4) and (18) as

$$[EI - (e_0 a)^2 P] \frac{d^4 w}{dx^4} + P \frac{d^2 w}{dx^2} = q, \quad (19)$$

where the deflection symbol v in Equation (4) was replaced with w . It is seen that Equation (19) is similar to Equations (5a) and (5b) in form. It proves the correctness of the present theoretical equations for a nanoscaled bar with vertical and horizontal forces indirectly. In summary, the present problem is different from a beam under bending, and the latter has been examined widely but the nanoscaled bar with vertical and horizontal forces is seldom a problem. So it is difficult to find existing numerical results for comparison directly. However, some qualitative comparisons can be carried out in the following numerical examples.

3. Numerical Examples

The classical solution of the midpoint deflection is recovered with $e_0 = 0$ or $k = \sqrt{P/EI}$ in Equation (13), which means that the classical counterpart can be obtained without the nonlocal effect. Assuming $P = 5$ N, $Q = 1.5$ N, $q = 2.5 \times 10^7$ N/m, and $l = 20$ nm, the relationship between the deflection of the midpoint and the nonlocal small scale as well as the material rigidity is shown in Figure 3. From Figure 3(a), it is observed that the bending deflection increases with increasing nonlocal scale parameter; namely, the equivalent stiffness is reduced in such a numerical example. This is consistent with the widespread nonlocal weakening model [3–5]. Hence, the observation validates the theoretical model in this paper. Subsequently, as a whole, it is observed that the deflection increases by 0.0235% when the nonlocal scale parameter increases from 0 to 0.15 nm with $EI = 10^{-15}$ Nm², while it decreases by 0.1030% and 1.1181% when the scale parameter increases from 0 to 0.15 nm with $EI = 10^{-16}$ Nm² and $EI = 10^{-17}$ Nm², respectively. The deformation is related to not only the nonlocal small scale but also the bending stiffness. Hence, the coupling effects of the nonlocal small scale and the bending stiffness on static deformation are observed. It is also demonstrated that the bending deflection may increase or decrease with increasing nonlocal effect. Therefore, the numerical result validates both the nonlocal stiffness-weakening and stiffness-strengthening models, which is consistent with the previous work [30]. This also verifies the correctness of the proposed model in this paper

qualitatively. Additionally, it is indicated that when the bending stiffness is greater than 10^{-18} Nm², the midpoint deflection varies monotonically and gently with increasing nonlocal small scale or intrinsic length scale; i.e., the deflection decreases with an increase in the intrinsic length scale. When the bending stiffness is equal or less than 10^{-18} Nm², the midpoint deflection tends to fluctuate up and down with increasing intrinsic length scale. Therefore, under the conditions of $P = 5$ N, $Q = 1.5$ N, $q = 2.5 \times 10^7$ N/m, and $l = 20$ nm, there must exist a certain value that can be defined as the threshold bending stiffness of the present material. On the other hand, when the bending stiffness is less than 10^{-15} Nm², the midpoint deflection turns from negative to positive, which means that the bending direction is reversed under a certain value of bending stiffness. It also implies the existence of a threshold bending stiffness. When the bending stiffness is below the threshold value, or the bending stiffness is small enough for a flexible bar, the nanoscaled bar may become unstable. For example, Figures 3(d) and 3(e) show that the nanoscaled bar is unstable with increasing nonlocal scale parameter under the condition of $P = 5$ N, $Q = 1.5$ N, $q = 2.5 \times 10^7$ N/m, $l = 20$ nm, and $EI = 10^{-18}$ Nm² and 10^{-19} Nm², respectively. However, it is stable under $EI = 10^{-17}$ Nm², 10^{-16} Nm², or 10^{-15} Nm² with the same range of nonlocal scale parameter and the same force and geometric parameters. Therefore, it is the flexible bending stiffness instead of the nonlocal small scale that makes the nanoscaled bar unstable. This is because the nonlocal scale parameter has a value range and its peak value cannot be chosen freely [34, 35]. In most previous studies, the range of the nonlocal scale parameter is from 0 to 0.2 nm. Even so, there is a coupling phenomenon between the nonlocal scale parameter and nanostructural bending stiffness. For example, the combinations of different nonlocal scale parameters and different bending stiffness result in different trends of midpoint deflection from Figures 3(d) and 3(e).

Here, we explain the selection of parameters used in the examples. As we know, a nanostructure means that the structural size of at least one dimension is no more than 100 nm. Regarding a nanoscaled bar, the length is much larger than the width and thickness. Hence, we can assume that width is 100 nm and thickness is from 10 nm to 100 nm. Of course, the length may reach the micron level. For example, we choose width to be 100 nm and thickness to be 60 nm, and the area moment of inertia I is 1.8×10^{-30} m⁴. Meanwhile, Young's modulus of nanomaterials can reach the order of TPa. Taking the single-walled carbon nanotube as an example, since the nanoscaled bar can be regarded as its continuum model, Young's modulus is about 6.85 TPa [36]. The bending stiffness is therefore 1.2×10^{-17} Nm² or so.

The midpoint deflection versus bending stiffness is shown in Figure 4, where $e_0 a = 0.1$ nm and the logarithmic coordinate is adopted for bending stiffness. With increasing bending stiffness, jumping in the midpoint deflection is observed. It should be pointed out that a flexible bar is focused mostly in this paper, so the bending stiffness is relatively small. If so, we should pay particular attention to the left half of Figure 4(a) where the bending stiffness is not large,

FIGURE 3: Effects of nonlocal small scale and bending stiffness on midpoint deflection.

FIGURE 4: The midpoint deflection versus bending stiffness.

TABLE 1: The deflection versus the length of the nanoscaled bar ($e_0 = 0.39$, $a = 0.54$ nm).

l (nm)	20	25	30	35	40	45	50	55	60
$P = 0.5$ N, $Q = 1.5$ N, $q = 2 \times 10^7$ N/m, $EI = 10^{-15}$ Nm ²									
v_{mid} (nm)	-0.308	-0.636	-1.16	-1.949	-3.082	-4.657	-6.796	-9.648	-13.4
$P = 1$ N, $Q = 1.5$ N, $q = 2.5 \times 10^7$ N/m, $EI = 10^{-15}$ Nm ²									
v_{mid} (nm)	-0.315	-0.657	-1.218	-2.085	-3.377	-5.254	-7.942	-11.767	-17.225
$P = 0.5$ N, $Q = 1$ N, $q = 2.5 \times 10^7$ N/m, $EI = 10^{-15}$ Nm ²									
v_{mid} (nm)	-0.223	-0.468	-0.866	-1.473	-2.357	-3.601	-5.308	-7.606	-10.654
$P = 0.5$ N, $Q = 1.5$ N, $q = 2 \times 10^7$ N/m, $EI = 10^{-15}$ Nm ²									
v_{mid} (nm)	-0.298	-0.609	-1.105	-1.844	-2.9	-4.36	-6.33	-8.944	-12.367
$P = 0.5$ N, $Q = 1.5$ N, $q = 2.5 \times 10^7$ N/m, $EI = 10^{-14}$ Nm ²									
v_{mid} (nm)	-0.03	-0.062	-0.111	-0.184	-0.286	-0.423	-0.602	-0.831	-1.172

and further this part is shown in Figure 4(b) separately. The instability phenomenon is thus observed again. It further verifies the existence of threshold bending stiffness. In this example with $e_0 a = 0.1$ nm, the threshold bending stiffness reads 2.026×10^{-16} Nm² from Figure 4. When the bending stiffness is less than 2.026×10^{-16} Nm², the nanoscaled bar deviates away from its equilibrium position and loses the stability. Another reason for the instability is that the horizontal compressive force may approach its critical value in this example.

In the case of the vertical concentrated force acting at the middle point of the nanoscaled bar, we study the effect of the external length scale on the deflection at the middle section. Taking a nanoscaled silicon bar as an example, we adopt $e_0 = 0.39$ [2] and $a = 0.54$ nm. The relationship between the midpoint deflection and the length of the nanoscaled bar under different vertical concentrated forces, vertical distributed forces, horizontal concentrated forces, and material rigidity is shown in Table 1. It is demonstrated that the midpoint deflection increases with increasing length. An increase in external vertical or horizontal forces results in an increase in the deflection too. A larger bend-

ing stiffness leads to lower bending deflection. Such conclusions are consistent with the classical continuum results qualitatively. This is because the nonlocal small-scale parameter $e_0 a$ is a constant in this case, and the physical mechanism of deformation-external length relation with given conditions of external forces and material rigidity is identical for both macroscale and nanoscale. Still, we can consider the ratio of the internal characteristic length scale to the external length scale. It is implied that the deflection decreases with an increase in the ratio of internal to external length scales since a larger external length scale (or a smaller ratio) leads to a higher midpoint deflection under given parameters. This is consistent with the conclusion in previous studies [9, 11, 13]. Hence, the validity of the present model is confirmed once again.

Finally, it is worth mentioning that research interest has been focused on deflection rather than internal forces for bending nanostructures in previous studies. The internal forces are also important in the design and optimization of nanostructures. Hence, the nonlocal bending moment in bending is examined. The effect of the nonlocal scale parameter on midpoint bending moment is plotted in Figure 5,

FIGURE 5: Effects of nonlocal small scale on bending moment.

where $P = 5 \text{ N}$, $Q = 1.5 \text{ N}$, $q = 2.5 \times 10^7 \text{ N/m}$, and $l = 20 \text{ nm}$. It is shown that the nonlocal small scale plays a significant role in bending moment. For example, the value of bending moment increases by 28.43% when the nonlocal scale parameter increases from 0 to 0.15 nm with $EI = 5 \times 10^{-18} \text{ Nm}^2$. On the other hand, the bending moment is monotonically changing with increasing nonlocal scale parameter with relative larger bending stiffness. Nevertheless, when the bending stiffness is relatively small, the bending moment may increase or decrease. The observations are similar to those predicted from other work [34].

4. Conclusions

Based on the nonlocal continuum theory, the paper investigates the compression bending of the nanoscaled bar via Eringen's nonlocal differential law. The nonlocal equilibrium equation and corresponding analytical solutions are derived. It concludes that the bending deflection of the nanoscaled bar is influenced by both the nonlocal scale and the bending stiffness of nanoscaled bars, and the coupling effects on static deformation are presented in detail. The increase in both linear vertical and horizontal forces results in larger deflection. The significant scale effect for the stability is observed in nanoscaled bars. The capacity of resisting compression decreases with the increase in nonlocal small scale. Additionally, the correlations between the material rigidity and the midpoint deflection or midpoint bending moment are examined, respectively. It is observed that the bending stiffness has a threshold value and the nanoscaled bar becomes unstable with material rigidity below the threshold value. Some qualitative comparisons are implemented in order to explain the observations in the numerical examples, and they can be regarded as indirect evidences to prove the present theoretical model. In addition, the present work verifies both nonlocal stiffness-weakening and stiffness-strengthening models and may provide useful information for modeling the current research hotspot in nanomechanics.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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