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Research Article

Estimation of Entropy for Log-Logistic Distribution under Progressive Type II Censoring

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Entropy is a useful indicator of information content that has been used in a number of applications. The Log-Logistic (**LL**) distribution is a probability distribution that is often employed in survival analysis. This paper addresses the problem of estimating multiple entropy metrics for an LL distribution using progressive type II censoring. We derive formulas for six different types of entropy measurements. To obtain the estimators of the proposed entropy measures, the maximum likelihood approach is applied. Approximate confidence intervals are calculated for the entropy metrics under discussion. A numerical evaluation is performed using various censoring methods and sample sizes to characterize the behavior of estimator's measures using relative biases, related mean squared errors, average interval lengths, and coverage probabilities. Numerical analysis revealed that the accuracy measures improve with sample size, and the suggested entropy estimates approach the genuine values as censoring levels decrease. Finally, an actual dataset was evaluated for demonstration purposes.

1. Introduction

Over the last few decades, log-logistic (or Fisk) distribution (**LLD**) has been frequently employed, notably in the areas of survival and reliability. The LLD is a popular alternative to the log-normal distribution because it has a failure rate function that grows with time, peaks after a certain period, and then gradually reduces [1]. Unlike the log-normal, the cumulative distribution function (**CDF**) of LLD has a closed form. For some parameter values, this distribution can have a monotonically declining failure rate function. In economics, the LLD is used to simulate wealth and income [2], while in hydrology, the LLD is used to describe stream flow data [3]. For additional details on the significance and applications of a LLD, see Bennett [4], Ahmad et al. [5], and Robson and Reed [6].

A random variable (**RVr**) X is said to have a LLD with the scale parameter α and the shape parameter β if its CDF is provided via

$$F(y) = \left(1 + \left(\frac{y}{\alpha}\right)^{-\beta}\right)^{-1}, y > 0, \alpha, \beta > 0. \quad (1)$$

Table 1: Removal patterns of units in numerous censoring schemes.

(n, m)	Censoring schemes Scheme I	Scheme II	Scheme III
(60,10)	$(0^{*9}, 50)$	(5^{*10})	$(50, 0^{*9})$
(60,20)	$(0^{*19}, 40)$	(2^{*20})	$(40, 0^{*19})$
(60,30)	$(0^{*29}, 30)$	(1^{*30})	$(30, 0^{*29})$
(60,60)	**Complete sample (0∗60)**		

For contrast, (1∗5, 0) denotes that the censoring scheme used is (1, 1, 1, 1, 1, 0).

The probability density function (**PDF**) corresponding to (1) is then provided via

$$f(y)=\frac{\beta}{\alpha^{\beta}}y^{\beta-1}\left(1+\left(\frac{y}{\alpha}\right)^{\beta}\right)^{-2}, y>0, \alpha, \beta>0. \tag{2}$$

The hazard function of the LLD is either decreasing or inverted bath-tub, and the PDF is either reversed J shaped or unimodal, see Johnson et al. [7].

In reliability studies, researchers want to see how long it takes for units to fail. However, due to time and expense restrictions, as well as a variety of other factors, experimenters are unable to track the lifetime of all units. As a result, filtered data is available. The most frequent types are type I and type II. In the statistical literature, popular censoring techniques are explored. However, in medical/engineering survival analysis, units may be removed at intermediate phases for a variety of causes beyond the experimenter's control. In this case, progressive censoring (PC) system is an acceptable censoring strategy since it permits surviving items to be removed before the test ends. PC has the benefit of quickly terminating the test and including at least some extreme life periods in the sample data. Progressive type I censoring (**PT1C**) happens when the number of survivors reduces to predefined levels, whereas progressive type II (**PT2C)** occurs when the number of survivors drops to specified levels.

The following is how a PT2C sample is carried out: A life testing experiment with n units and the PC method r_i ,$i=1, 2, \cdots, m$ is used. Units are randomly eliminated from the remaining $n-1$ surviving units at the moment of the first failure y_1. Similarly, units from the remaining $n-2-r_1$ units are randomly eliminated after the second failure y_2. The test continues until the m^{th} failure occurs, at which point all remaining $n-m-r_1-r_2-\cdots-r_{m-1}$ units are removed. The number of failures m as well as the progressive censoring design $r_1, r_2, \cdots, r_m$ is preset and fixed. Let $Y_{(1)} \leq Y_{(2):} \leq \cdots \leq Y_{(m)}$ denote such a PT2C sample with $(r_1, \cdots, r_m)$ being the PC scheme. Balakrishnan and Aggrawala [8] provided some historical remarks and a good summary of progressive censoring. It really should be observed that this censorship is limited to the classical type II censoring (**T2C**) when $r_1=r_2=\ldots=r_{m-1}=0$ and $r_m=n-m$; further, it reduces to a complete sample having no censoring for $m=n$ and $r_i=0, i=1, 2, \cdots, n$.

In information theory, entropy is a measure of uncertainty in a RVr that gauges the anticipated value of the information embodied in that RVr. Entropies are motivated by how receiving new information decreases uncertainty. Shannon's entropy is one of the earliest and most commonly used measurements of entropy. In the study of communication systems, this measure has proven to be effective. Let Y be a non-negative RVr with a continuous CDF; the formal measure of Shannon's entropy is characterized by:

$$S=-\int_{-\infty}^{\infty} f(y)\log(f(y))dy. \tag{3}$$

One of the most significant disadvantages of Shannon's measure is that it may be negative for particular probability distributions, making it useless as a measure of uncertainty. Rényi [9] developed a new generalized entropy by studying the concepts of uncertainty and randomness. The Rényi entropy (**Ré**) is calculated as follows:

$$R_{\gamma}=(1-\gamma)^{-1}\log\left\{\int_{-\infty}^{\infty} f(y)^{\gamma}dy\right\}, \gamma>0 \text{ and } \gamma\neq 1, \tag{4}$$

where the constant γ is conditional, leading to a positive entropy. Different generalizations of entropy were proposed by Havrda and Charvat [10], Arimoto [11], Awad et al. [12], and Tsallis [13].

Havrda and Charvat's (**HC**) entropy suggested extension of (3). This extension is called **HC** entropy of degree γ and is characterized with

$$HC_{\gamma}=\frac{1}{2^{1-\gamma}-1}\left[\int_{-\infty}^{\infty} f(y)^{\gamma}dy-1\right]. \tag{5}$$

Arimoto's (Ar) entropy measure (see [11]) is characterized with

$$A_{\gamma}=\frac{\gamma}{1-\gamma}\left[\left(\int_{-\infty}^{\infty} f(y)^{\gamma}dy\right)^{\frac{1}{\gamma}}-1\right]. \tag{6}$$

Awad et al. [12] suggested two types of entropy: an extension of Réyni entropy and Havrda and Charvat entropy. The first extension, denoted by$A1_{\gamma}$, and the second extension, denoted by$A2_{\gamma}$, are characterized with

$$A1_{\gamma}=\frac{1}{\gamma-1}\log\left[\int_{-\infty}^{\infty}\frac{f(y)^{\gamma}}{v}dy\right], v=\left[\sup_{0<y<\infty} f(y)\right]^{1-\gamma}, \tag{7}$$

$$A2_{\gamma}=\frac{1}{2^{1-\gamma}-1}\left[\left\{\int_{-\infty}^{\infty}\frac{f(y)^{\gamma}}{v}dy\right\}-1\right]. \tag{8}$$

Table 2

(a) Rbias, MSE, AIL, and CP of different entropy estimates under PT2C schemes at $\gamma = 0.5$, and $(n, m) = (60, 10)$

Scheme		Entropy methods R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
I	**Exact value**	**2.2690**	**5.0930**	**8.6696**	**4.2192**	**1.1987**	**-1.0884**
	Estimate	1.7878	4.3804	9.4136	3.6289	1.9795	-1.2832
	Rbias	0.2121	0.1399	0.0858	0.1399	0.6514	0.1790
	MSE	1.3496	15.0356	166.7630	10.3188	2.5768	0.6486
	AIL	4.1469	14.9484	50.5619	12.3837	5.5007	3.0647
	CP (%)	95.20	94.60	94.70	94.60	95.90	94.00
II	Estimate	1.9068	4.4836	8.8110	3.7143	1.7539	-1.2101
	Rbias	0.1596	0.1197	0.0163	0.1197	0.4632	0.1118
	MSE	0.9131	9.9744	87.5434	6.8453	1.8282	0.5637
	AIL	3.4680	12.1534	36.6908	10.0682	4.8351	2.9056
	CP (%)	95.30	94.50	94.40	94.50	95.60	94.40
III	Estimate	2.1477	5.0985	9.9041	4.2237	1.3879	-1.0378
	Rbias	0.0535	0.0011	0.1424	0.0011	0.1578	0.0465
	MSE	0.5120	7.1138	67.2224	4.8822	1.1105	0.5405
	AIL	2.7655	10.4604	31.7887	8.6656	4.0658	2.8765
	CP (%)	95.90	95.60	94.90	95.60	95.10	95.70

(b) Rbias, MSE, AIL, and CP of different entropy estimates under PT2C schemes at $\gamma = 0.5$, and $(n, m) = (60, 20)$

Scheme		Entropy methods R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
I	**Exact value**	**2.2690**	**5.0930**	**8.6696**	**4.2192**	**1.1987**	**-1.0884**
	Estimate	2.0413	4.6608	8.5353	3.8612	1.5384	-1.1417
	Rbias	0.1004	0.0849	0.0155	0.0849	0.2834	0.0490
	MSE	0.4951	5.7061	42.4241	3.9160	1.1759	0.4370
	AIL	2.6111	9.2138	25.5393	7.6329	4.0389	2.5840
	CP (%)	95.70	96.30	95.00	96.30	95.40	95.70
II	Estimate	2.1097	4.8192	8.7517	3.9923	1.4324	-1.1062
	Rbias	0.0702	0.0538	0.0095	0.0538	0.1950	0.0164
	MSE	0.3686	4.5905	35.1121	3.1504	0.8884	0.3822
	AIL	2.2978	8.3339	23.2371	6.9040	3.5810	2.4236
	CP (%)	95.40	95.70	95.50	95.70	95.40	95.50
III	Estimate	2.1680	4.9539	8.9216	4.1039	1.3719	-1.0937
	Rbias	0.0445	0.0273	0.0291	0.0273	0.1445	0.0049
	MSE	0.2644	3.5581	27.6580	2.4419	0.6971	0.3048
	AIL	1.9773	7.3776	20.6019	6.1118	3.2033	2.1650
	CP (%)	95.60	96.20	95.30	96.20	95.70	96.30

(c) Rbias, MSE, AIL, and CP of different Entropy estimates under PT2C schemes at $\gamma = 0.5$, and $(n, m) = (60, 30)$

Scheme		Entropy methods R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
I	**Exact value**	**2.2690**	**5.0930**	**8.6696**	**4.2192**	**1.1987**	**-1.0884**
	Estimate	2.1433	4.8692	8.6973	4.0338	1.3889	-1.0901
	Rbias	0.0554	0.0439	0.0032	0.0439	0.1587	0.0016

Table 2: Continued.

Scheme		Entropy methods					
		R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
	MSE	0.2798	3.5222	25.7945	2.4172	0.7902	0.3600
	AIL	2.0151	7.3078	19.9183	6.0540	3.4055	2.3530
	CP (%)	95.30	95.50	95.10	95.50	94.99	95.89
	Estimate	2.1675	4.9167	8.7186	4.0732	1.3572	-1.0943
	Rbias	0.0447	0.0346	0.0057	0.0346	0.1322	0.0054
II	MSE	0.2291	2.9321	21.1112	2.0123	0.6348	0.2711
	AIL	1.8345	6.6799	18.0189	5.5338	3.0622	2.0417
	CP (%)	95.40	95.40	95.40	95.40	94.40	95.40
	Estimate	2.2079	5.0444	9.0077	4.1789	1.2861	-1.0521
	Rbias	0.0269	0.0095	0.0390	0.0095	0.0729	0.0334
III	MSE	0.1969	2.7007	20.2460	1.8535	0.5708	0.2822
	AIL	1.7238	6.4423	17.5969	5.3370	2.9431	2.0787
	CP (%)	94.90	94.80	94.50	94.80	94.90	95.10

Tsallis [13] generalized Shanon's entropy and defined the measure as

$$T_\gamma = \frac{1}{\gamma - 1}\left[1 - \int_{-\infty}^{\infty} f(y)^\gamma dy\right]. \tag{9}$$

Many researchers have worked on entropy estimates for various life distributions. Cramer and Bagh [14] used progressive censoring to investigate entropy in the Weibull distribution. Kang et al. [15] used doubly type II censored data to construct entropy estimators for a double exponential distribution. Using record values from the generalized half-logistic distribution, Seo et al. [16] calculated an entropy estimate. Cho et al. [17] addressed entropy estimates for Rayleigh distribution using doubly generalized type II hybrid censoring. Cho et al. [18] used generalized type II hybrid censored samples to derive estimators for the entropy function of a Weibull distribution. Dey et al. [19] studied the loss of entropy for a truncated Rayleigh distribution using different entropy measures. Chacko and Asha [20] investigated entropy estimation in generalized exponential distribution. Bantan et al. [21] considered the entropy estimators for inverse Lomax via the multiple censored scheme. The dynamic cumulative residual Ré for the Lomax distribution was estimated using Bayesian and maximum likelihood (**ML**) techniques by Al-Babtain et al. [22]. Entropy estimator of Lindley was prepared by Almarashi et al. [23]. Helmy et al. [24] investigated Shannon entropy estimation of Lomax distribution using unified hybrid censored data. Bayesian and non-Bayesian estimation of the Nadarajah–Haghighi distribution using progressive Type-1 censoring scheme studied by Elbatal et al. [25].

In this paper, we are inspired to investigate six possible entropy estimators for log-logistic distributions in the presence of T2PC data. We construct analytical formulas for the entropy measurements proposed. The ML and two-sided approximate confidence intervals of several entropy estimators are calculated. Numerical comparisons for various sample sizes are presented to identify which entropy estimator outperforms the others.

The paper is broken down into five sections. Section 2 presents expressions for the recommended entropy measures based on LLD. PT2C is used in Section 3 to give several entropy estimators as well as their estimated confidence intervals. In Section 4, numerical comparisons of different entropy estimators and data analysis are examined. Finally, in Section 5, there are some conclusions and a summary of the study.

2. Expressions of Entropy Measures

Statistical entropy measures the amount of uncertainty or variability in a RVr. The higher the value of entropy leads to more variability in the data. This section focuses on obtaining the expression for various entropy measurements of LLD.

2.1. Rényi Entropy. The Ré of LLD is obtained by replacing (1) in (3) as follows:

$$R_\gamma = (1-\gamma)^{-1} \log\left\{\frac{\beta^\gamma}{\alpha^{\beta\gamma}}\int_0^\infty y^{\beta\gamma-\gamma}\left(1+\left(\frac{y}{\alpha}\right)^\beta\right)^{-2\gamma} dy\right\}. \tag{10}$$

Write $I = \beta^\gamma/\alpha^{\beta\gamma}\int_0^\infty y^{\beta\gamma-\gamma}(1+(y/\alpha)^\beta)^{-2\gamma}dy$, and assume that $z=(y/\alpha)^\beta$, $y=\alpha z^{1/\beta}dy = (\alpha/\beta)z^{1/\beta-1}dz$, thus the integral I will be written as

$$\begin{aligned} I &= \frac{\beta^{\gamma-1}}{\alpha^{\beta\gamma-1}}\int_0^\infty \alpha^{\beta\gamma-\gamma} z^{\gamma-\frac{\gamma}{\beta}+\frac{1}{\beta}-1}(1+z)^{-2\gamma}dz \\ &= \beta^{\gamma-1}\alpha^{1-\gamma}B\left(\gamma-\frac{\gamma}{\beta}+\frac{1}{\beta}, \gamma+\frac{\gamma}{\beta}-\frac{1}{\beta}\right), \end{aligned} \tag{11}$$

Table 3

(a) Rbias, MSE, AIL, and CP of different Entropy estimates under PT2C schemes at $\gamma = 1.5$, and $(n, m) = (60, 10)$

Scheme		Entropy methods					
		R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
I	**Exact value**	**0.5601**	**0.8339**	**0.5109**	**0.4885**	**2.9076**	**-11.1963**
	Estimate	0.3186	0.3469	0.2329	0.2032	3.3739	-17.4037
	Rbias	0.4312	0.5840	0.5441	0.5840	0.1604	0.5544
	MSE	1.0201	1.0552	0.3681	0.3621	1.5820	16.4038
	AIL	3.8461	3.5471	2.1151	2.0778	4.5814	8.7875
	CP (%)	99.90	95.30	95.40	95.30	98.00	94.90
II	Estimate	0.3901	0.4935	0.3156	0.2891	3.2303	-15.4706
	Rbias	0.3034	0.4082	0.3823	0.4082	0.1110	0.3818
	MSE	0.6745	0.6767	0.2438	0.2322	1.1372	9.5542
	AIL	3.1511	2.9370	1.7784	1.7205	3.9863	6.8951
	CP (%)	99.70	95.20	95.30	95.20	98.30	94.50
III	Estimate	0.4853	0.6819	0.4254	0.3994	3.0255	-12.8624
	Rbias	0.1335	0.1823	0.1674	0.1823	0.0406	0.1488
	MSE	0.1656	0.3220	0.1247	0.1105	0.4050	3.0465
	AIL	1.5689	2.1440	1.3435	1.2559	2.4526	4.1274
	CP (%)	95.10	95.10	95.20	95.10	94.80	96.30

(b) Rbias, MSE, AIL, and CP of different Entropy estimates under PT2C schemes at $\gamma = 1.5$, and $(n, m) = (60, 20)$

Scheme		Entropy methods					
		R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
I	**Exact value**	**0.5601**	**0.8339**	**0.5109**	**0.4885**	**2.9076**	**-11.1963**
	Estimate	0.4796	0.6946	0.4292	0.4069	3.0801	-13.3805
	Rbias	0.1437	0.1671	0.1600	0.1671	0.0593	0.1951
	MSE	0.1048	0.2064	0.0801	0.0708	0.4507	3.6689
	AIL	1.2295	1.6960	1.0624	0.9935	2.5447	4.4314
	CP (%)	95.40	95.60	95.90	95.60	95.00	95.30
II	Estimate	0.4936	0.7208	0.4442	0.4223	3.0513	-12.8807
	Rbias	0.1188	0.1356	0.1306	0.1356	0.0494	0.1505
	MSE	0.0810	0.1554	0.0609	0.0533	0.3152	2.3936
	AIL	1.0853	1.4813	0.9317	0.8677	2.1286	3.6029
	CP (%)	95.30	95.30	95.10	95.30	95.20	95.40
III	Estimate	0.4997	0.7270	0.4485	0.4259	3.0106	-12.4840
	Rbias	0.1079	0.1282	0.1221	0.1282	0.0354	0.1150
	MSE	0.0864	0.1642	0.0645	0.0563	0.2711	1.9416
	AIL	1.1280	1.5327	0.9652	0.8978	2.0016	3.3053
	CP (%)	94.60	94.70	94.80	94.70	94.30	95.80

(c) Rbias, MSE, AIL, and CP of different entropy estimates under PT2C schemes at $\gamma = 1.5$, and $(n, m) = (60, 30)$

Scheme		Entropy methods					
		R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
I	**Exact value**	**0.5601**	**0.8339**	**0.5109**	**0.4885**	**2.9076**	**-11.1963**
	Estimate	0.5081	0.7498	0.4606	0.4392	3.0397	-12.7520
	Rbias	0.0928	0.1008	0.0985	0.1008	0.0454	0.1389

Table 3: Continued.

Scheme		Entropy methods R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
	MSE	0.0510	0.0944	0.0375	0.0324	0.2928	2.2382
	AIL	0.8622	1.1591	0.7337	0.6790	2.0582	3.5045
	CP (%)	95.20	95.40	95.40	95.40	94.90	96.10
II	Estimate	0.5195	0.7656	0.4704	0.4485	2.9842	-12.1793
	Rbias	0.0725	0.0819	0.0792	0.0819	0.0263	0.0878
	MSE	0.0484	0.0873	0.0350	0.0299	0.2182	1.4672
	AIL	0.8479	1.1271	0.7163	0.6602	1.8073	2.9039
	CP (%)	95.60	96.50	96.10	96.50	94.60	95.70
III	Estimate	0.5081	0.7498	0.4606	0.4392	3.0397	-12.7520
	Rbias	0.0928	0.1008	0.0985	0.1008	0.0454	0.1389
	MSE	0.0510	0.0944	0.0375	0.0324	0.2928	2.2382
	AIL	0.8622	1.1591	0.7337	0.6790	2.0582	3.5045
	CP (%)	95.20	95.40	95.40	95.40	95.00	96.00

Table 4: Rbias, MSE, AIL, and CP of different Entropy estimates under PT2C schemes at $\gamma = (0.5, 1.5)$, and $(n, m) = (60, 30)$.

Scheme		Entropy methods R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
0.5	**Exact value**	**2.2690**	**5.0930**	**8.6696**	**4.2192**	**1.1987**	**-1.0884**
	Estimate	2.2430	5.0925	8.9237	4.2188	1.2379	-1.0622
	Rbias	0.0115	0.0001	0.0293	0.0001	0.0327	0.0240
	MSE	0.1039	1.4883	11.0349	1.0214	0.3110	0.1533
	AIL	1.2601	4.7846	12.9899	3.9637	2.1816	1.5323
	CP (%)	95.50	95.50	95.00	95.50	95.80	94.80
1.5	**Exact value**	**0.5601**	**0.8339**	**0.5109**	**0.4885**	**2.9076**	**-11.1963**
	Estimate	0.5461	0.8069	0.4955	0.4727	2.9603	-11.8045
	Rbias	0.0249	0.0324	0.0302	0.0324	0.0181	0.0543
	MSE	0.0277	0.0478	0.0195	0.0164	0.1186	0.7083
	AIL	0.6510	0.8507	0.5441	0.4983	1.3346	2.0323
	CP (%)	95.40	95.50	95.30	95.50	94.90	95.10

where $B(.,.)$ is the beta function. As a result of inserting (11) into (10), LLD's Ré entropy takes the form

$$R_\gamma = (1-\gamma)^{-1} \log\left(\beta^{\gamma-1}\alpha^{1-\gamma} B\left(\gamma - \frac{\gamma}{\beta} + \frac{1}{\beta}, \gamma + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\right), \tag{12}$$

Hence, (12) is the necessary formulation of LLD Ré entropy.

2.2. Havrda and Charvat Entropy. The HC of the LLD is obtained by replacing (1) in (4) as follows:

$$HC_\gamma = \frac{1}{2^{1-\gamma}-1}\left[\frac{\beta^\gamma}{\alpha^{\beta\gamma}}\int_0^\infty y^{\beta\gamma-\gamma}\left(1+\left(\frac{y}{\alpha}\right)^\beta\right)^{-2\gamma} dy - 1\right]. \tag{13}$$

As a result, we get the HC of LLD by inserting (11) in (4) as follows

$$HC_\gamma = \frac{1}{2^{1-\gamma}-1}\left[\beta^{\gamma-1}\alpha^{1-\gamma} B\left(\gamma - \frac{\gamma}{\beta} + \frac{1}{\beta}, \gamma + \frac{\gamma}{\beta} - \frac{1}{\beta}\right) - 1\right]. \tag{14}$$

Hence, (14) is the necessary formulation of LLD HC entropy.

2.3. Arimoto Entropy. The Ar entropy of LLD is obtained by replacing (1) in (5) as follows:

$$A_\gamma = \frac{\gamma}{1-\gamma}\left[\left(\frac{\beta^\gamma}{\alpha^{\beta\gamma}}\int_0^\infty y^{\beta\gamma-\gamma}\left(1+\left(\frac{y}{\alpha}\right)^\beta\right)^{-2\gamma} dy\right)^{\frac{1}{\gamma}} - 1\right]. \tag{15}$$

Table 5: Estimates of different entropy measures under PT2C schemes for the given real data, where $\gamma = (0.5, 1.5)$, and $(n, m) = (128{,}30)$.

Scheme	$(\widehat{\alpha}, \widehat{\beta})$	Entropy methods R_γ	HC_γ	A_γ	T_γ	$A1_\gamma$	$A2_\gamma$
$\gamma = 0.5$							
I	$\widehat{\alpha} = 6.9409 \widehat{\beta} = 1.4379$	4.6318	22.0524	101.7061	18.2688	1.3603	-1.1913
II	$\widehat{\alpha} = 6.0531 \widehat{\beta} = 1.6856$	4.2481	17.7813	68.9778	14.7305	1.9920	-1.5225
III	$\widehat{\alpha} = 6.2489 \widehat{\beta} = 1.5729$	4.3961	19.3326	80.1414	16.0157	1.7126	-1.3888
Complete	$\widehat{\alpha} = 6.0962 \widehat{\beta} = 1.7160$	4.2227	17.5263	67.2218	14.5193	2.0659	-1.5548
$\gamma = 1.5$							
I	$\widehat{\alpha} = 6.9409 \widehat{\beta} = 1.4379$	3.2140	2.7297	1.9723	1.5990	2.7781	-10.2809
II	$\widehat{\alpha} = 5.3452 \widehat{\beta} = 1.6388$	2.8755	2.6034	1.8496	1.5251	3.1745	-13.2824
III	$\widehat{\alpha} = 4.7569 \widehat{\beta} = 1.6371$	2.7595	2.5550	1.8042	1.4967	3.1715	-13.2575
Complete	$\widehat{\alpha} = 6.0962 \widehat{\beta} = 1.7160$	2.9766	2.6434	1.8877	1.5484	3.3120	-14.4713

We obtain the Ar entropy of LLD by putting (11) into (15) as follows

$$A_\gamma = \frac{\gamma}{1-\gamma}\left[\beta^{1-\frac{1}{\gamma}}\alpha^{\frac{1}{\gamma}-1}\left(B\left(\gamma - \frac{\gamma}{\beta} + \frac{1}{\beta}, \gamma + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\right)^{\frac{1}{\gamma}} - 1\right]. \quad (16)$$

Thus, the formula for Ar entropy found in equation (16).

2.4. A-entropies. To get (6) and (7), we must obtain $\sup_{0<y<\infty} f(y)$, by getting the maximum value of $f(y)$ as below:

$$y^{\beta-2}\left(1 + \left(\frac{y}{\alpha}\right)^{\beta}\right)^{-3}\left\{(\beta-1)\left(1 + \left(\frac{y}{\alpha}\right)^{\beta}\right) - 2\frac{\beta}{\alpha^{\beta}}y^{\beta}\right\} = 0. \quad (17)$$

After simplification, then (17) is written as follows

$$\frac{\beta+1}{\alpha^{\beta}}y^{\beta} = \beta - 1, \quad (18)$$

which leads to the following

$$y = \alpha\left(\frac{\beta-1}{\beta+1}\right)^{\frac{1}{\beta}}. \quad (19)$$

Using (17), we get $\sup_{0<y<\infty} f(y)$, as

$$\sup_{0<y<\infty} f(y) = \frac{\beta}{\alpha^{\beta}}\left\{\alpha\left(\frac{\beta-1}{\beta+1}\right)^{\frac{1}{\beta}}\right\}^{\beta-1}\left(1 + \left(\left(\frac{\beta-1}{\beta+1}\right)^{\frac{1}{\beta}}\right)^{\beta}\right)^{-2} = \frac{(\beta-1)^{1-1/\beta}(\beta+1)^{1+(1/\beta)}}{4\alpha\beta}. \quad (20)$$

Using (11) and (19), hence the A-entropies can be expressed as

$$A1_\gamma = \frac{1}{\gamma-1}\log\left[\left[\frac{(\beta-1)^{1-1/\beta}(\beta+1)^{1+(1/\beta)}}{4\alpha\beta}\right]^{1-\gamma}\beta^{\gamma-1}\alpha^{1-\gamma}B \cdot \left(\gamma - \frac{\gamma}{\beta} + \frac{1}{\beta}, \gamma + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\right], \quad (21)$$

$$A2_\gamma = \frac{1}{2^{1-\gamma}-1}\left[\left\{\left[\frac{(\beta-1)^{1-1/\beta}(\beta+1)^{1+(1/\beta)}}{4\alpha\beta}\right]^{1-\gamma}\beta^{\gamma-1}\alpha^{1-\gamma}B \cdot \left(\gamma - \frac{\gamma}{\beta} + \frac{1}{\beta}, \gamma + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\right\} - 1\right]. \quad (22)$$

Thus, (20) and (21) provide essential expressions of A-entropy.

2.5. Tsallis Entropy. The Tsallis entropy of LLD is obtained by replacing (1) in (9) as follows:

$$T_\gamma = \frac{1}{\gamma-1}\left[1 - \frac{\beta^{\gamma}}{\alpha^{\beta\gamma}}\int_0^{\infty} y^{\beta\gamma-\gamma}\left(1 + \left(\frac{y}{\alpha}\right)^{\beta}\right)^{-2\gamma}dy\right]. \quad (23)$$

Thus, using (11) in (23), then the Tsallis entropy of LLD is obtained as follows

$$T_\gamma = \frac{1}{\gamma-1}\left[1 - \beta^{\gamma-1}\alpha^{1-\gamma}B\left(\gamma - \frac{\gamma}{\beta} + \frac{1}{\beta}, \gamma + \frac{\gamma}{\beta} - \frac{1}{\beta}\right)\right]. \quad (24)$$

Thus, the formula of Tsallis entropy of LLD is provided in equation (24).

3. Estimation of Different Entropies

Using the ML method and T2PC data, we get estimators for the various entropies metrics provided in the preceding section. To construct entropies estimators, we start with the ML estimator of population parameters. The invariance property of ML estimators may then be used to determine the ML of the recommended entropies measurements. Furthermore, we obtain the approximate confidence intervals of the suggested entropy measures.

Assume that $Y_{(1)} \leq Y_{(2)} \leq \ldots \leq Y_{(m)}$ be a PT2C sample of size m from a sample of size n drawn from CDF (1) and PDF (2) with censoring scheme $r_1, r_2, \cdots, r_m$. The likelihood (L) function based on the PT2C sample is given by

$$L(\alpha, \beta) = c\prod_{i=1}^{m} f\left(y_{(i)}\right)\left[1 - F\left(y_{(i)}\right)\right]^{r_i} = c\prod_{i=1}^{m} \frac{\beta}{\alpha^{\beta}} y_{(i)}{}^{\beta-1} \cdot \left(1 + \left(\frac{y_{(i)}}{\alpha}\right)^{\beta}\right)^{-2}\left[1 - \left(1 + \left(\frac{y_{(i)}}{\alpha}\right)^{-\beta}\right)^{-1}\right]^{r_i}, \tag{25}$$

where $c = n(n - r_1 - 1)(n - r_1 - r_2 - 2) \cdots n - m + 1 - \sum_{i=1}^{m-1} r_i$. Thus, the constant is the number of ways in which the m PT2C order statistics may occur if the observed failure times are $Y_{(1)}, Y_{(2)}, \cdots, Y_{(m)}$. The log-$L$ function of (25), say $\ln L^{\bullet}$, is then provided via

$$\ln L^{\bullet} \propto m(\ln \beta - \beta \ln \alpha) + (\beta - 1)\sum_{i=1}^{m} \ln y_i - 2\sum_{i=1}^{m} \ln\left(1 + z_i^{\beta}\right) + \sum_{i=1}^{m} r_i \ln\left[1 - \left(1 + z_i^{-\beta}\right)^{-1}\right]. \tag{26}$$

where we write $z_i = (y_i/\alpha)$, and $z_i = z_{(i)}$ for simplified forms. Form (26), we derive the L equation for α and β as

$$\frac{\partial \ln L^{\bullet}}{\partial \alpha} = -\frac{m\beta}{\alpha} + \frac{2\beta}{\alpha}\sum_{i=1}^{m} \frac{z_i^{\beta}}{1 + (z_i)^{\beta}} + \alpha\beta\sum_{i=1}^{m} r_i \frac{\left(1 + (z_i)^{-\beta}\right)^{-2}(z_i)^{-\beta}}{1 - \left(1 + (z_i)^{-\beta}\right)^{-1}}, \tag{27}$$

$$\frac{\partial \ln L^{\bullet}}{\partial \beta} = \frac{m}{\beta} - m\ln\alpha + \sum_{i=1}^{m} \ln y_i - 2\sum_{i=1}^{m} \frac{(z_i)^{\beta}\ln(z_i)}{1 + (z_i)^{\beta}} - \sum_{i=1}^{m} r_i \frac{\left(1 + (z_i)^{-\beta}\right)^{-2}(z_i)^{-\beta}\ln(z_i)}{1 - \left(1 + (z_i)^{-\beta}\right)^{-1}}. \tag{28}$$

The ML estimator of α and β can be obtained using the numerical method by solving the non-linear Equations (27) and (28) after setting them with zero. Once the ML estimator of α and β, say $\hat{\alpha}$and $\hat{\beta}$, is computed, we can obtain the ML estimator of entropy measures provided in (12), (14), (16), (20), (21), and (24). Consequently, the ML estimator of Ré entropy, denoted by $\hat{R}_\gamma$, is obtained by inserting $\hat{\alpha}$ and $\hat{\beta}$ in (11) as follows

$$\hat{R}_\gamma = (1 - \gamma)^{-1} \log\left(\hat{\beta}^{\gamma-1}\hat{\alpha}^{1-\gamma} B\left(\gamma - \frac{\gamma}{\hat{\beta}} + \frac{1}{\hat{\beta}}, \gamma + \frac{\gamma}{\hat{\beta}} - \frac{1}{\hat{\beta}}\right)\right). \tag{29}$$

The other entropy estimators, indicated by $H\hat{C}_\gamma, \hat{A}_\gamma, \hat{T}_\gamma$, $\hat{A}1_\gamma$, and $\hat{A}2_\gamma$, may be obtained in a similar fashion.

The preceding theoretical results may be specialized in two cases; firstly, ML estimators of $\alpha, \beta, R_\gamma, HC_\gamma, A_\gamma, T_\gamma, A1_\gamma$, and $A2_\gamma$ are produced when $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = n - m$, via T2C. Second, for $r_1 = r_2 = \cdots = r_{m-1} = 0$ and $r_m = 0$, we get the ML estimators of population parameters as well as the proposed entropy measures.

The asymptotic normality of ML estimation may be used to determine the asymptotic 100(1-v) confidence intervals (**CIn**) for the parameters as

$$\hat{\alpha} \pm Z_{v/2}\sqrt{\operatorname{var}(\hat{\alpha})}, \hat{\beta} \pm Z_{v/2}\sqrt{\operatorname{var}\left(\hat{\beta}\right)}. \tag{30}$$

Also, the asymptotic 100(1-v) CIns for entropy measures are given by

$$\hat{R}_\gamma \pm Z_{v/2}\sqrt{\operatorname{var}(\hat{R}_\gamma)}, H\hat{C}_\gamma \pm Z_{v/2}\sqrt{\operatorname{var}\left(H\hat{C}_\gamma\right)}, \hat{A}_\gamma \pm Z_{v/2}\sqrt{\operatorname{var}\left(\hat{A}_\gamma\right)}, \hat{T}_\gamma \pm Z_{v/2}\sqrt{\operatorname{var}\left(\hat{T}_\gamma\right)},$$
$$\hat{A}1_\gamma \pm Z_{v/2}\sqrt{\operatorname{var}\left(\hat{A}1_\gamma\right)}, \hat{A}2_\gamma \pm Z_{v/2}\sqrt{\operatorname{var}\left(\hat{A}2_\gamma\right)}, \tag{31}$$

where $Z_{V/2}$ is standard normal and (1-v) is the confidence coefficient.

4. Simulation and Real Data Outcomes

The challenge in this section is to analyze the outcomes of the numerous entropy estimations stated before. To evaluate the behavior of the suggested entropy measures and to analyze the statistical performances of the estimators under PT2C, a Monte Carlo study is used. An actual data is also examined for demonstration purposes. For calculations, the statistical programming language R will be used in this study.

4.1. Simulation Study. The effectiveness of the approaches recommended of entropy estimation using Monte Carlo is compared using a simulated exercise. Six entropy estimates are calculated using the Monte Carlo process. For the ML estimates (**MLEs**), one may generate 1000 data from the LLD with the following assumptions:

(1) Presume the parameters of the LLD in the coming situations:$\alpha = 0.5, \beta = 1.5$

(2) Assume two values for the constant $\gamma = (0.5, 1.5)$

(3) Sample size is $n=60$ and number of observed failures are $m=10, 20, 30$.

(4) Removed items r_j are assume the accompanying:

Scheme I: $r_1 = r_2 - \cdots = r_{m-1} = 0, r_m = n - m$.
Scheme II: $r_1 = r_2 - \cdots = r_{m-1} = r_m = n - m/m$.
Scheme III: $r_1 = n - m, r_2 - \cdots = r_m = 0$.

Table 1 shows the patterns that were eliminated for each suggested schemes. Note that the first scheme (Scheme I) is a particular instance of PT2C, which is the most common T2C. Another, particular case is considered when $n = m = 60$ which leads to a complete sample.

According to the generated data, MLEs are computed under the above assumptions using PT2C. When getting MLEs, keep in mind that the initial estimate values are treated the same as the real parameter values. These values, MLEs, are then plugged-in to calculate the desired entropy estimates.

All the average entropy estimates, relative biases (**Rbias**), associated mean squared errors (**MSEs**), corresponding average interval lengths (**AIL**), and coverage probabilities (**CPs**) for all six entropy methods are reported in Table 2 for$\gamma = 0.5$ and in Table 3 for$\gamma = 1.5$. Also, the results of the complete sample case are reported in Table 4 for both values of γ.

From tabulated values, it can be noticed that:

(i) Higher values of m lead to a decrease in Rbias, MSE, and AIL for all different schemes of removing items and all different entropy methods.

(ii) All CPs are greater than 93% for all different schemes of removing items and all different entropy methods.

(iii) The increase in the constant term γ leads to a decrease in estimates of all different entropy methods.

4.2. Real Data Application. A real data set is analyzed for illustrative purposes as well as to assess the statistical performances of the MLEs for different entropy estimates in the case of the LLD under different PT2C schemes.

The uncensored data set below corresponds to the remission periods (in months) of a random sample of 128 bladder cancer patients reported in Lee and Wang [26]. The following are the bladder cancer remission times:

0.08, 2.09, 3.48, 4.87, 6.94, 8.66, 13.11, 23.63, 0.20, 2.23, 3.52, 4.98, 6.97, 9.02, 13.29, 0.40, 2.26, 3.57, 5.06, 7.09, 9.22, 13.80, 25.74, 0.50, 2.46, 3.64, 5.09, 7.26, 9.47, 14.24, 25.82, 0.51, 2.54, 3.70, 5.17, 7.28, 9.74, 14.76, 26.31, 0.81, 2.62, 3.82, 5.32, 7.32, 10.06, 14.77, 32.15, 2.64, 3.88, 5.32, 7.39, 10.34, 14.83, 34.26, 0.90, 2.69, 4.18, 5.34, 7.59, 10.66, 15.96, 36.66, 1.05, 2.69, 4.23, 5.41, 7.62, 10.75, 16.62, 43.01, 1.19, 2.75, 4.26, 5.41, 7.63, 17.12, 46.12, 1.26, 2.83, 4.33, 7.66, 11.25, 17.14, 79.05, 1.35, 2.87, 5.62, 7.87, 11.64, 17.36, 1.40, 3.02, 4.34, 5.71, 7.93, 11.79, 18.10, 1.46, 4.40, 5.85, 8.26, 11.98, 19.13, 1.76, 3.25, 4.50, 6.25, 8.37, 12.02, 2.02, 3.31, 4.51, 6.54, 8.53, 12.03, 20.28, 2.02, 3.36, 6.76, 12.07, 21.73, 2.07, 3.36, 6.93, 8.65, 12.63, 22.69, 5.49.

We first check whether the LLD is suitable for analyzing this data set. We report the MLEs of the parameters and the value of the Kolmogorov–Smirnov (**K–S**) test statistic to judge the goodness of fit. The calculated K-S distance between the empirical and fitted LLD distribution is 0.0440 and its p-value is 0.9667 where $\widehat{\alpha} = 6.0978$ and $\widehat{\beta} = 1.7160$ which indicate that this distribution can be considered an adequate model for the given data set.

From the original data, one can generate, e.g., three PT2C samples with a number of stages $m = 30$ and removed items r_j are assumed to be as follows:

Scheme I: $r_1 = r_2 - \cdots = r_{m-1} = 0, r_m = n - m$.
Scheme II: $r_1 = r_2 - \cdots = r_{m-1} = r_m = n - m + 1/m$.
Scheme III: $r_1 = n - m, r_2 - \cdots = r_m = 0$.

Also, we consider the complete case as $n = m = 128$. Two different values of the constant are proposed: 0.5 and 1.5. In Table 5, the MLEs of the parameters have been calculated and then plugged into different entropy methods in the proposed schemes for PT2C samples as in the given real data set.

5. Summary and Conclusion

In this paper, we look at the estimation problem of certain entropy measures for log-logistic distribution under the PT2C scheme. Rényi entropy, Havrda and Charvat entropy, Arimoto entropy, A-entropies, and Tsallis entropy are the six entropy measures considered. The recommended entropy measurement expressions are computed. The point and two-sided approximate confidence intervals for the recommended entropy measures are obtained using the maximum likelihood procedure. To describe and compare the behavior of estimator's measures, a numerical evaluation is done in terms of relative biases, associated mean squared errors, average interval lengths, and coverage probabilities under different censoring schemes as well as different sample sizes. The suggested entropy estimates approach the real values with decreasing censoring levels, as well as the accuracy of measurements increases with sample sizes. Finally, an actual dataset was analyzed for demonstration purposes.

Data Availability

If you would like to get the numerical dataset used to conduct the study reported in the publication, please contact the appropriate author.

Conflicts of Interest

The authors state that they have no conflicts of interest to disclose in relation to this work.

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