Hindawi
Journal of Nanomaterials
Volume 2022, Article ID 6703086, 12 pages
https://doi.org/10.1155/2022/6703086

Research Article

Analytical Approximate Solution of the Fractional Order Biological Population Model by Using Natural Transform

Nasir Ali,[1] Rashid Nawaz,[1] Laiq Zada,[1] Abir Mouldi,[2] Souhail Mohamed Bouzgarrou,[3] and Ndolane Sene[4]

[1]*Department of Mathematics, Abdul Wali Khan University, Mardan, KP, Pakistan*
[2]*Department of Industrial Engineering, College of Engineering, King Khalid University, Abha 61421, Saudi Arabia*
[3]*Department of Civil Engineering, Faculty of Engineering, Jazan University, Saudi Arabia*
[4]*Laboratoire Lmdan, Departement De Mathematiques De Decision, Facultíe des Sciences Economiques et Gestion, Universite Cheikh Anta Diop De Dakar, Fann, BP 5683 Dakar, Senegal*

Correspondence should be addressed to Ndolane Sene; ndolanesene@yahoo.fr

Received 14 January 2022; Revised 6 February 2022; Accepted 24 February 2022; Published 19 March 2022

Academic Editor: Taza Gul

In the present work, the natural transform iterative method (NTIM) is implemented to solve the biological population model (BPM) of fractional order. The method is tested for three nonlinear examples. The NTIM is a combination of a new iterative method and natural transform. We see that the solution pattern converges to the exact solution in a few iterations. The method handles an extensive range of differential equations of both fractional and integer order. The fractional order derivative is considered in Caputo's sense. For mathematical computation, Mathematica 10 is used.

1. Introduction

The globe and our everyday lives have been revolutionized by modern technologies. Technology is being used in a wide range of engineering applications, including aerodynamics, fluid dynamics, medical sciences, and finance. The essence of technology is influenced and designed by mathematical modeling. The modeling might take the form of mathematical models that can be described using differential equations. These differential equations may have been used to represent the transmission of electromagnetic waves, which is at the root of many present technologies. A variety of applications ranging from wireless communications to radar, medical imaging, and remote sensing have played a great role in our life [1, 2]. Mathematics and biosciences have also numerous practical applications related to real life [3, 4]. Several diseases can be modeled through mathematical calculations and can be controlled by collecting data and making precise analysis [5]. There is a strong and interesting relationship between biology and mathematics utilizing differential equations. The noninteger order differential equations are termed as fractional order differential equations (FDEs) [32–35]. The branch of mathematics dealing with FDEs is known as fractional calculus [6]. Depending on the nature of the problem, differential equations can be linear or nonlinear. Simple analytical methods may be used to analyze linear differential equations, but investigators have developed several ways for solving nonlinear differential equations as their exact solutions are not always feasible. The importance of the FDEs can be discussed in many fields of sciences [7, 23–27]. Many operators for fractional derivatives have been given by several researchers. The most famous is Caputo's fractional derivative operator [8]. Li et al. introduced the fractional order integral operator for handling differential equations [9]. Recently, many transformations have been used to solve fractional order differential equations. Some of them are the Laplace transform, Sumudu transform, Elzaki transform, etc. [10–12, 28–31]. In this work, we will deal with the natural transform iterative method (NTIM), a combination of the natural transform and the new iterative method (NIM). The proposed techniques have been recently applied by Nawaz et al. for solving noninteger order differential equations [13]. Many other researchers have applied NIM and natural transform for handling the FDEs [14–17]. In this article, we will consider fractional biological population model (FBPM) as [18]

$$\begin{cases} D_t^\beta \phi = (\phi^2)_{xx} + (\phi^2)_{yy} + f(\phi) = 0, 0 < \beta \leq 1, t > 0, (x, y) \in R^2, \\ f(\phi) = h\phi^a\left(1 - rh^b\right), \qquad \phi(x, y, 0) = g(x, y). \end{cases} \tag{1}$$

In Equation (1), $\phi = \phi(x, y, t)$ is the population density and f is the supply of population due to births and deaths. The h, a, b, and r are the real numbers and $g(x, y)$ is the initial condition. The detailed solution of Equation (1) can be found in [19, 20]. FBPM is a mathematical model of biology which will be thoroughly investigated in this paper. FBPM aids in the understanding of the dynamical procedure of population changes in biological population models, as well as providing useful predictions.

The remaining paper is structured as follows: Preliminary definitions from fractional calculus are contained in Section 2. The notion of NTIM is introduced in Section 3. The NTIM is used to solve three FBPM in Section 4. In Section 5, some results have been discussed. Lastly, a concrete conclusion is given.

2. Preliminaries

Definition 1. Riemann-Liouville (R-L) fractional integral is defined as

$$J_t^\beta f(t) = \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} f(\tau) d\tau, (\beta > 0, t > 0), \qquad J_t^0 f(t) = f(t), \tag{2}$$

where $\Gamma(.)$ is the gamma function.

Definition 2. Caputo's time-fractional derivative operator of order $\beta > 0$is defined as

$$D_t^\beta \phi(\wp,t) = \frac{\partial^\beta \phi(\wp,t)}{\partial t^\beta} = \begin{cases} \frac{1}{\Gamma(n-\beta)}\int_0^t (t-\tau)^{n-\beta-1} \frac{\partial^n \phi(\wp,\tau)}{\partial \tau^n}, & \text{if } n-1 < \beta < n, \\ \frac{\partial^n \phi(\wp,t)}{\partial t^n} & , \text{if } \beta = n \in N. \end{cases} \tag{3}$$

Definition 3. Natural transform of $\phi(t)$ is defined as [21]

$$\mathbb{N}^+(\theta(t)) = R(s, \nu) = \frac{1}{\nu}\int_0^\infty e^{\frac{-st}{\nu}}(\phi(t))dt\,; \quad s, \nu > 0, \tag{4}$$

where s and ν are the transform variables.

Definition 4. The inverse of natural transform of $R(s, \nu)$ is defined as

$$\mathbb{N}^-(R(s, \nu)) = \phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} e^{\frac{st}{\nu}}(R(s, \nu))ds, \tag{5}$$

where $c \in R$ and the integral are taken in the complex plane $s = a + bi$ along $s = c$.

Definition 5. If the nth derivative of $\phi(t)$ is $\phi^n(t)$, then its natural transform is given as

$$\mathbb{N}^+(\phi^n(t)) = R_n(s, \nu) = \frac{s^n}{\nu^n}R(s, \nu) - \sum_{k=0}^{n-1}\frac{s^{n-(k+1)}}{\nu^{n-k}}(\phi^n(0)), \quad n \geq 1. \tag{6}$$

Theorem 6. *If the natural transform of $h(t)$ and $k(t)$ are $h(s, \nu)$ and $k(s, \nu)$ respectively, defined on set A, then*

$$\mathbb{N}[h * k] = \nu H(s, \nu)K(s, \nu), \tag{7}$$

where $\mathbb{N}[h * k]$ is convolution the functions h and k.

3. Natural Transform Iterative Method (NTIM) [13]

Consider FDE of the form

$$D_t^\beta(\phi(\wp,t)) = f(\wp,t) + L\phi(\wp,t) + \aleph\phi(\wp,t), 0\wp,t > 0, m-1 < \alpha < m, \tag{8}$$

where $\wp = x_1, x_2, \cdots, x_n$ and $m \in N$. The linear operator, nonlinear operator, and the source term are L, $\aleph$, and f, respectively. The initial condition is given as

$$\phi(\wp,0) = g(\wp). \tag{9}$$

By applying the natural transform to Equation (8), we have

$$\mathbb{N}^+\left[D_t^\beta(\phi(\wp,t))\right] = \mathbb{N}^+[f(\wp,t)] + \mathbb{N}^+[L(\phi(\wp,t)) + \aleph(\phi(\wp,t))]. \tag{10}$$

Using the natural transform differentiation property, Equation (10) can be written as

$$\frac{s^\beta}{\nu^\beta}\mathbb{N}^+[\phi(\wp,t)] - \frac{s^{\beta-1}}{\nu^\beta}\phi(\wp,0) = \mathbb{N}^+[f(\wp,t)] + \mathbb{N}^+\left[L\phi(\wp,t) + \aleph\phi\left(\underline{\wp}, t\right)\right]. \tag{11}$$

By rearranging Equation (11), we have

$$\mathbb{N}^+[\phi(\wp,t)] = \frac{g(\wp)}{s} + \frac{\nu^\beta}{s^\beta}(\mathbb{N}^+[f(\wp,t)]) + \frac{\nu^\beta}{s^\beta}(\mathbb{N}^+[L(\phi(\wp,t)) + \aleph(\phi(\wp,t))]). \tag{12}$$

For the NTIM solution, $\phi(\wp,t)$ is expanded as

$$u(\wp,t) = \sum_{i=0}^{\infty}\phi_i(\wp,t), \tag{13}$$

and $\aleph(\phi(\wp,t))$, the nonlinear term, is defined as

$$\aleph\left(\sum_{m=0}^{\infty}\phi_m(\wp,t)\right)=\aleph(\phi_0(\wp,t))+\sum_{m=1}^{\infty}\left\{\aleph\left(\sum_{j=0}^{i}\phi_j(\wp,t)\right)-\aleph\left(\sum_{j=0}^{m-1}\phi_j(\wp,t)\right)\right\}. \tag{14}$$

Using Equation (13) and Equation (14) in Equation (12), we obtain

$$\mathbb{N}^{+}\left[\sum_{i=1}^{\infty}\phi_i\right]=\frac{g(\wp)}{s}+\frac{v^{\beta}}{s^{\beta}}\left(\mathbb{N}^{+}[f(\wp,t)]\right)+\frac{v^{\beta}}{s^{\beta}}\left[\mathbb{N}^{+}\left[\sum_{m=0}^{\infty}L(\phi_m)+\aleph(\phi_0)+\sum_{m=1}^{\infty}\left\{\aleph\left(\sum_{j=0}^{m}\phi_j\right)-\aleph\left(\sum_{j=0}^{m-1}\phi_j\right)\right\}\right]\right]. \tag{15}$$

Using the recursive relation,

$$\begin{aligned}
\mathbb{N}^{+}[\phi_0(\wp,t)]&=\frac{g(\wp)}{s}+\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[f(\wp,t)],\\
\mathbb{N}^{+}[\phi_1(\wp,t)]&=\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[L(\phi_0)+\aleph(\phi_0)],\\
\mathbb{N}^{+}[\phi_2(\wp,t)]&=\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[L(\phi_1(x,t))+\aleph(\phi_0+\phi_1)-\aleph(\phi_0)]\\
&\vdots\\
\mathbb{N}^{+}[\phi_{i+1}(\wp,t)]&=\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[L(\phi_i)+\aleph(\phi_0+\phi_1+\cdots+\phi_i)-\aleph(\phi_0+\phi_1+\cdots+\phi_{i-1})].i\geq 0.
\end{aligned} \tag{16}$$

Now by taking the inverse natural transform of Equation (16), we have

$$\begin{aligned}
\phi_0(\wp,t)&=\mathbb{N}^{-}\left[\frac{g(\wp)}{s}+\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[f]\right],\\
\phi_1(\wp,t)&=\mathbb{N}^{-}\left[\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[L(\phi_0)+\aleph(\phi_0)]\right],\\
\phi_2(\wp,t)&=\mathbb{N}^{-}\left[\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[L(\phi_1)+\aleph(\phi_0+\phi_1)-\aleph(\phi_0)]\right],\\
&\vdots\\
\phi_{i+1}(\wp,t)&=\mathbb{N}^{-}\left[\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}[L(\phi_i)+\aleph(\phi_0+\phi_1+\cdots+\phi_i)-\aleph(\phi_0+\phi_1+\cdots+\phi_{i-1})]\right],\quad i\geq 0.
\end{aligned} \tag{17}$$

Then by adding the components, the approximate solution of Equations (8) and (9) by NITM is given as

$$\phi(\wp,t)=\phi_0(\wp,t)+\phi_1(\wp,t)+\cdots+\phi_{m-1}(\wp,t),\ m\in N. \tag{18}$$

Convergence of NTIM is as a convergence of NIM and is proved by Bhalekar and Daftardar-Gejji [22].

4. Applications of NTIM

In this section, we apply the natural transform iterative method NTIM for handling the three nonlinear cases of FBPM. The method is applied directly to the problems without any discretization by using the given initial conditions. Then, the comparison is made with the help of plots and numerical tables with the existing methods which shows the effectiveness of the proposed method [18].

Problem 7. Consider the population model as [18]

$$D_t^{\beta}\phi=\left(\phi^2\right)_{xx}+\left(\phi^2\right)_{yy}+h\phi,\quad t>0,0<\beta\leq 1, \tag{19}$$

where $\phi=\phi(x,y,t)$ together with initial conditions

$$\phi(x,y,0)=\sqrt{xy}, \tag{20}$$

and the exact solution is

$$\phi(x,y,z,t)=\sqrt{xy}\,e^{ht}. \tag{21}$$

Taking natural transformation of Equation (19), we have

$$\mathbb{N}^{+}\left[D_t^{\beta}\phi\right]=\mathbb{N}^{+}\left[\left(\phi^2\right)_{xx}+\left(\phi^2\right)_{yy}+h\phi\right]. \tag{22}$$

Applying the natural transform differentiation property to Equation (22), we get

$$\frac{s^{\beta}}{v^{\beta}}\phi(x,y,t)-\frac{v^{\beta-1}}{s^{\beta}}\phi(x,y,0)=\mathbb{N}^{+}\left[\left(\phi^2\right)_{xx}+\left(\phi^2\right)_{yy}+h\phi\right]. \tag{23}$$

Taking the inverse natural transform of Equation (23), we have

$$\phi(x,y,t)=\frac{\phi(x,y,0)}{s}+\mathbb{N}^{-}\left[\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[\left(\phi^2\right)_{xx}+\left(\phi^2\right)_{yy}+h\phi\right]\right]. \tag{24}$$

Using the idea of NTIM and the recursive relation of Equation (16), we obtained the solution components as

$$\left\{\begin{aligned}
\phi_0(x,y,t)&=\mathbb{N}^{-}\left[\frac{\phi(x,y,0)}{s}\right],\\
\phi_1(x,y,t)&=\mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\left(\phi_0^2\right)_{xx}+\left(\phi_0^2\right)_{yy}+h\phi_0\right]\right],\\
\phi_2(x,y,t)&=\mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[(\phi_0+\phi_1)^2{}_{xx}+(\phi_0+\phi_1)^2{}_{yy}+h\phi_1-\left(\left(\phi_0^2\right)_{xx}+\left(\phi_0^2\right)_{yy}\right)\right]\right],\\
\phi_3(x,y,t)&=\mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\begin{array}{c}(\phi_0+\phi_1+\phi_2)^2{}_{xx}+(\phi_0+\phi_1+\phi_2)^2{}_{yy}+h\phi_2\\-(\phi_0+\phi_1)^2{}_{xx}+(\phi_0+\phi_1)^2{}_{yy}\end{array}\right]\right],\\
&\vdots
\end{aligned}\right\} \tag{25}$$

By using the software package, the solution components are obtained as

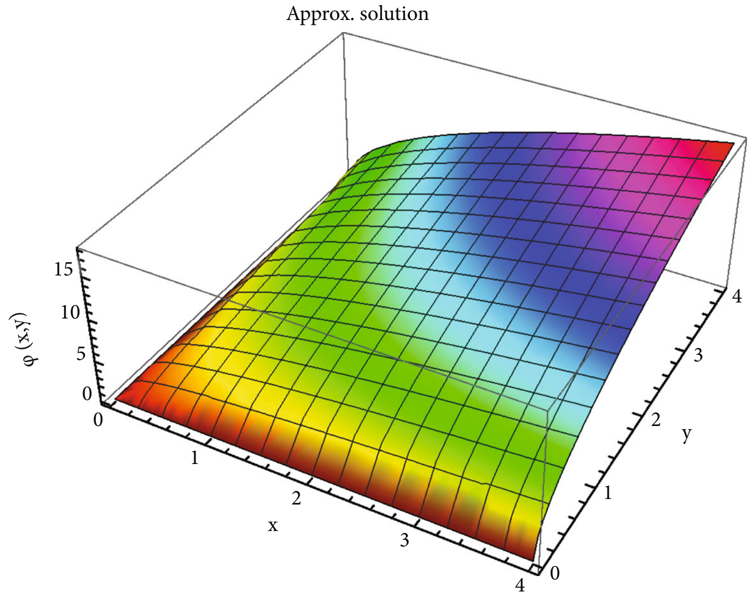

Figure 1: NTIM solution of problem 1 at $h = 1$, $t = 1.5$, and $\beta = 1$.

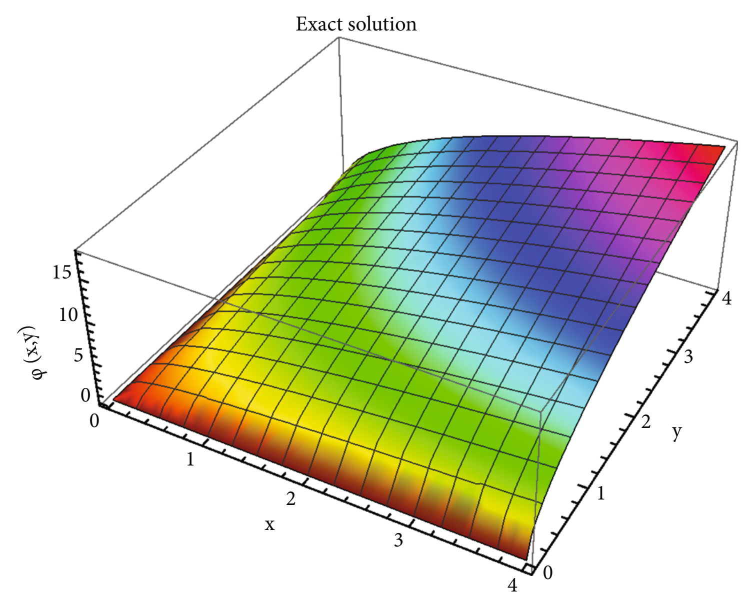

Figure 2: Exact solution of problem 1 at $h = 1$, $t = 1.5$, and $\beta = 1$.

$$\left\{\begin{array}{c} \phi_0(x, y, t) = \sqrt{x y}, \phi_1(x, y, t) = \dfrac{h t^{\beta} \sqrt{x y}}{\Gamma(\beta+1)}, \\ \phi_2(x, y, t) = \dfrac{h^2 t^{2\beta} \sqrt{x y}}{\Gamma(2\beta+1)}, \phi_3(x, y, t) = \dfrac{h^3 t^{3\beta} \sqrt{x y}}{\Gamma(3\beta+1)}, \\ \phi_4(x, y, t) = \dfrac{h^4 t^{4\beta} \sqrt{x y}}{\Gamma(4\beta+1)}, \\ \vdots \end{array}\right\} \tag{26}$$

Combining the components, the 4th order approximate solution is given as

$$\begin{aligned} \phi(x, y, t) &= \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \\ &= \cdots \left\{ \sqrt{xy} + \frac{h t^{\beta} \sqrt{xy}}{\Gamma(\beta+1)} + \frac{h^2 t^{2\beta} \sqrt{xy}}{\Gamma(2\beta+1)} + \frac{h^3 t^{3\beta} \sqrt{xy}}{\Gamma(3\beta+1)} + \frac{h^4 t^{4\beta} \sqrt{xy}}{\Gamma(4\beta+1)} + \cdots \right\}. \end{aligned} \tag{27}$$

Equation (27) can be simplified as

$$\phi(x, y, t) = \sqrt{xy} \left(\frac{h t^{\beta}}{\Gamma(\beta+1)} + \frac{h^2 t^{2\beta}}{\Gamma(2\beta+1)} + \frac{h^3 t^{3\beta}}{\Gamma(3\beta+1)} + \frac{h^4 t^{4\beta}}{\Gamma(4\beta+1)} + \cdots \right). \tag{28}$$

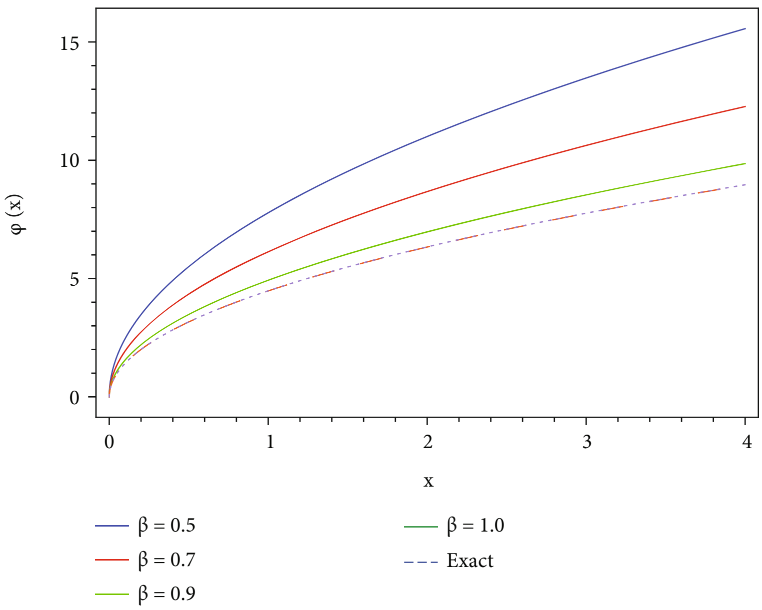

FIGURE 3: Comparison of an approximate solution by NTIM for different values of β at $h = 1$, $t = 1.5$, and $y = 1$ for Problem 7.

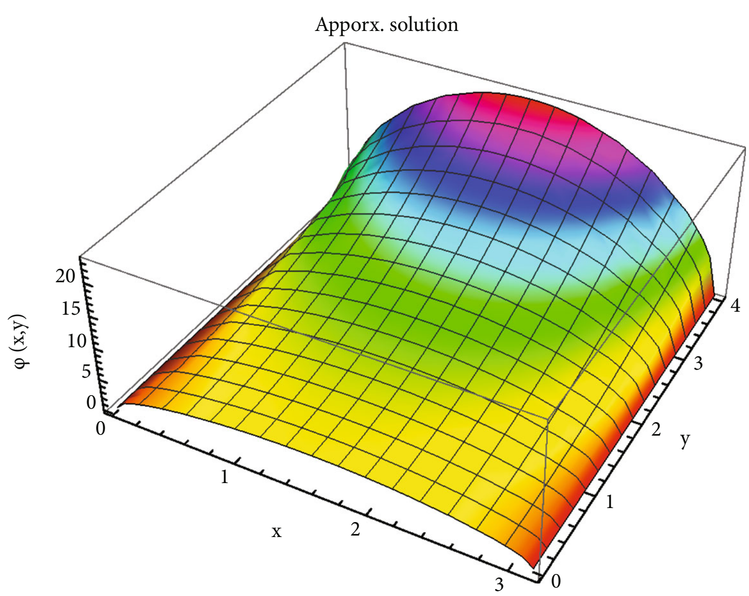

FIGURE 4: NTIM Solution of problem 1 at $t = 1.5$ and $\beta = 1$.

For $\beta = 1$, Equation (28) converges as

$$\phi(x, y, t) = \sqrt{xy}\left(1 + ht + \frac{(h\,t)^2}{2!} + \frac{(h\,t)^3}{3!} + \frac{(h\,t)^4}{4!} + \cdots\right), \quad (29)$$

which converges to the exact solution given by Equation (21).

Problem 8. Consider the biological population model as [18]

$$D_t^\beta \phi = \left(\phi^2\right)_{xx} + \left(\phi^2\right)_{yy} + \phi, t > 0, 0 < \beta \le 1, \quad (30)$$

where $\phi = \phi(x, y, t)$ together with initial conditions

$$\phi(x, y, 0) = \sqrt{\sin\ (x)\ \cosh\ (y)}, \quad (31)$$

and the exact solution is

$$\phi(x, y, t) = \sqrt{\sin\ (x)\ \cosh\ (y)}\, e^t. \quad (32)$$

Taking natural transform of Equation (30), we have

$$\mathbb{N}^+\left[D_t^\beta \phi\right] = \mathbb{N}^+\left[\left(\phi^2\right)_{xx} + \left(\phi^2\right)_{yy} + \phi\right]. \quad (33)$$

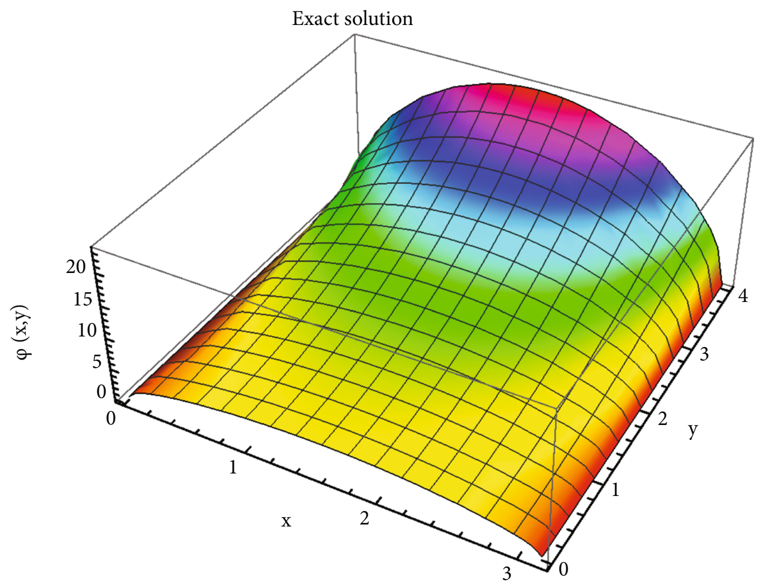

Figure 5: Exact solution of problem 1 at $t = 1.5$ and $\beta = 1$.

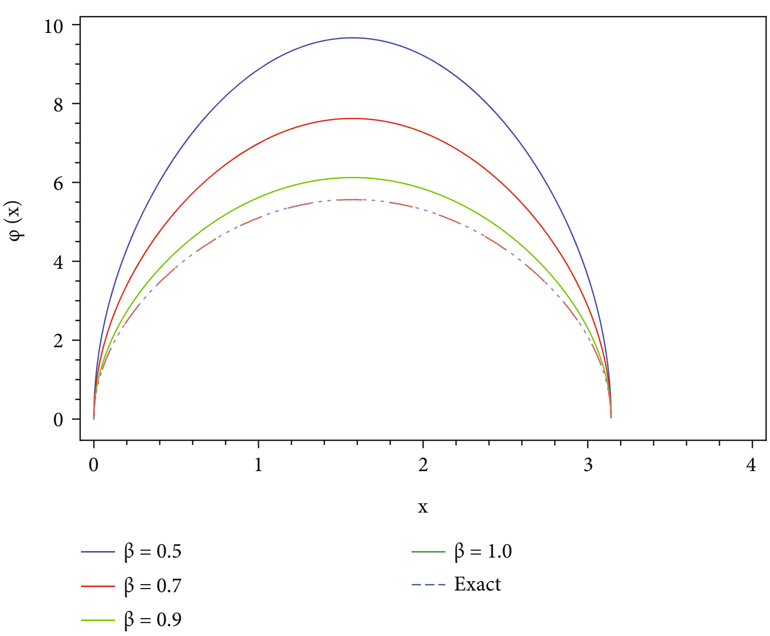

Figure 6: Comparison of the approximate solution by NTIM for different values of β at $t = 1.5$ and $y = 1$ for Problem 8.

Using the natural transform differentiation property, we obtain

$$\frac{s^{\beta}}{v^{\beta}}\phi(x,y,t) - \frac{v^{\beta-1}}{s^{\beta}}\phi(x,y,0) = \mathbb{N}^{+}\left[\left(\phi^{2}\right)_{xx} + \left(\phi^{2}\right)_{yy} + \phi\right]. \tag{34}$$

Taking the inverse natural transform, we have

$$\phi(x,y,t) = \frac{\phi(x,y,0)}{s} + \mathbb{N}^{-}\left[\frac{v^{\beta}}{s^{\beta}}\mathbb{N}^{+}\left[\left(\phi^{2}\right)_{xx} + \left(\phi^{2}\right)_{yy} + \phi\right]\right]. \tag{35}$$

Using the recursive relation, the solution components can be obtained as

$$\left\{\begin{array}{c} \phi_0(x,y,t) = \mathbb{N}^{-}\left[\frac{\phi(x,y,0)}{s}\right], \\ \phi_1(x,y,t) = \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\left(\phi_0^2\right)_{xx} + \left(\phi_0^2\right)_{yy} + \phi_0\right]\right], \\ \phi_2(x,y,t) = \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\left(\phi_0+\phi_1\right)^2_{xx} + \left(\phi_0+\phi_1\right)^2_{yy} + \phi_1 - \left(\left(\phi_0^2\right)_{xx} + \left(\phi_0^2\right)_{yy}\right)\right]\right], \\ \phi_3(x,y,t) = \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[\begin{array}{c}\left(\phi_0+\phi_1+\phi_2\right)^2_{xx} + \left(\phi_0+\phi_1+\phi_2\right)^2_{yy} + \phi_2 \\ -\left(\phi_0+\phi_1\right)^2_{xx} + \left(\phi_0+\phi_1\right)^2_{yy}\end{array}\right]\right], \\ \vdots \end{array}\right\} \tag{36}$$

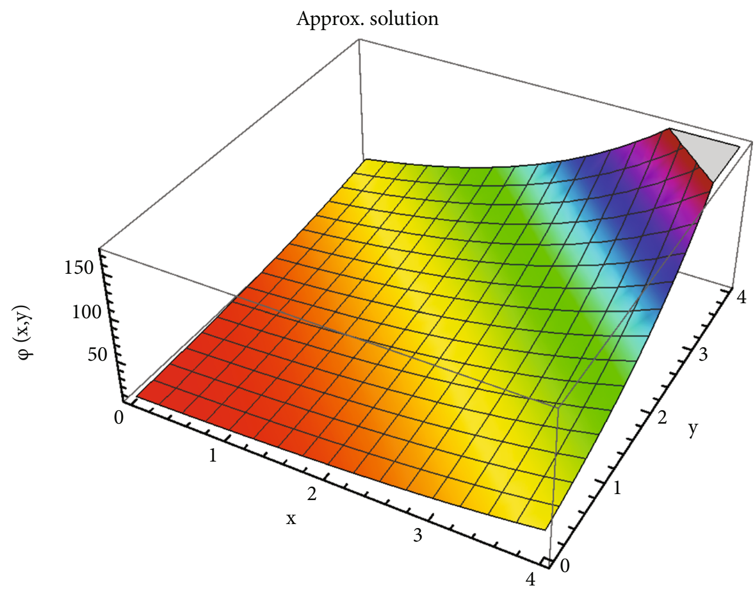

Figure 7: NTIM solution of Problem 7 at $h = 1$, $t = 1.5$, and $\beta = 1$.

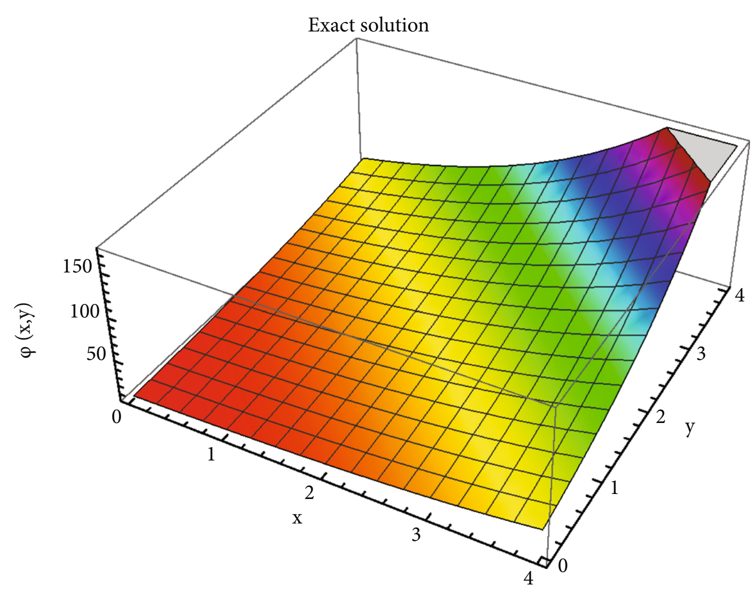

Figure 8: Exact solution of Problem 7 at $h = 1$, $t = 1.5$, and $\beta = 1$.

By using the software package, the solution components are obtained as

$$\left\{\begin{array}{c} \phi_0(x,y,t) = \sqrt{\sin(x)\cosh(y)}, \phi_1(x,y,t) = \dfrac{t^{\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(\beta+1)}, \\ \phi_2(x,y,t) = \dfrac{t^{2\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(2\beta+1)}, \phi_3(x,y,t) = \dfrac{t^{3\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(3\beta+1)}, \\ \phi_4(x,y,t) = \dfrac{t^{4\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(4\beta+1)}, \\ \vdots \end{array}\right\} \quad (37)$$

Combining the components, the 4th order approximate solution is given as

$$\phi(x,y,t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \cdots$$
$$= \left\{\begin{array}{c} \sqrt{\sin(x)\cosh(y)} + \dfrac{t^{\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(\beta+1)} + \dfrac{t^{2\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(2\beta+1)} \\ + \dfrac{t^{3\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(3\beta+1)} + \dfrac{t^{4\beta}\sqrt{\sin(x)\cosh(y)}}{\Gamma(4\beta+1)} + \cdots, \end{array}\right\} \quad (38)$$

Equation (38) can be simplified as

$$\phi(x,y,t) = \sqrt{\sin(x)\cosh(y)}\left(\frac{t^{\beta}}{\Gamma(\beta+1)} + \frac{t^{2\beta}}{\Gamma(2\beta+1)} + \frac{t^{3\beta}}{\Gamma(3\beta+1)} + \frac{t^{4\beta}}{\Gamma(4\beta+1)} + \cdots\right). \quad (39)$$

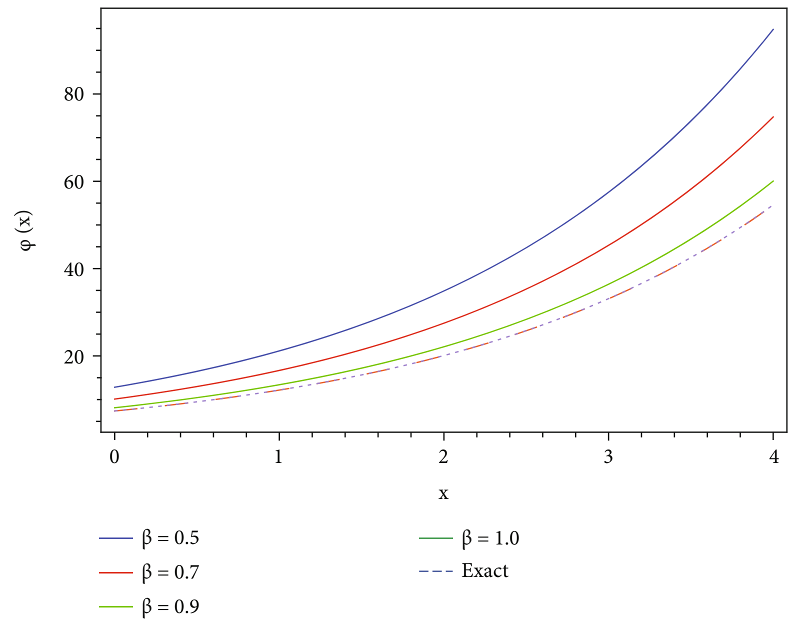

Figure 9: Comparison of the approximate solution by NTIM for different values of β at $h = 1$, $t = 1.5$, and $y = 1$ for Problem 9.

Table 1: Comparison of the different values of β and the absolute error of the 6th order NTIM solution with the 6th order MGTFSM solution for Problem 7.

x, y	t	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 1.0$	Exact	NTIM error	Error [18]
	0.1	0.778064	0.6135	0.492898	0.110517	0.110517	2.009212×10^{-12}	1.4090×10^{-10}
	0.3	0.778064	0.6135	0.492898	0.134986	0.134986	4.507600×10^{-9}	1.0576×10^{-7}
0.1	0.5	0.778064	0.6135	0.492898	0.164872	0.164872	1.652645×10^{-7}	2.3354×10^{-6}
	0.7	0.778064	0.6135	0.492898	0.201373	0.201375	1.788941×10^{-6}	1.8129×10^{-5}
	0.9	0.778064	0.6135	0.492898	0.24595	0.24596	1.067487×10^{-5}	8.4486×10^{-5}
	0.1	2.33419	1.8405	1.47869	0.331551	0.331551	6.027678×10^{-12}	4.2269×10^{-10}
	0.3	2.33419	1.8405	1.47869	0.404958	0.404958	1.352280×10^{-8}	3.1727×10^{-7}
0.3	0.5	2.33419	1.8405	1.47869	0.494616	0.494616	4.957934×10^{-7}	7.0062×10^{-6}
	0.7	2.33419	1.8405	1.47869	0.60412	0.604126	5.366824×10^{-6}	5.4387×10^{-5}
	0.9	2.33419	1.8405	1.47869	0.737849	0.737881	3.202460×10^{-5}	2.5346×10^{-4}
	0.1	3.89032	3.0675	2.46449	0.552585	0.552585	1.004596×10^{-11}	7.0449×10^{-10}
	0.3	3.89032	3.0675	2.46449	0.674929	0.674929	2.253800×10^{-8}	5.2879×10^{-7}
0.5	0.5	3.89032	3.0675	2.46449	0.82436	0.824361	8.263223×10^{-7}	1.1677×10^{-5}
	0.7	3.89032	3.0675	2.46449	1.00687	1.00688	8.944707×10^{-6}	9.0645×10^{-5}
	0.9	3.89032	3.0675	2.46449	1.22975	1.2298	5.337433×10^{-5}	4.2243×10^{-4}

For $\beta = 1$, Equation (39) converges as

$$\phi(x, y, t) = \sqrt{\sin(x) \cosh(y)} \left(1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \frac{t^4}{4!} + \cdots\right), \quad (40)$$

which yields the exact solution given by Equation (32).

Problem 9. Consider the biological population model as [18]

$$D_t^\beta \phi = \left(\phi^2\right)_{xx} + \left(\phi^2\right)_{yy} + h\, \phi\, (1 - r\, \phi),\ t > 0, 0 < \beta \le 1, \quad (41)$$

where $\phi = \phi(x, y, t)$ subject to the initial condition

$$\phi(x, y, 0) = e^{\left(\sqrt{hr/8}\right)(x+y)}, \quad (42)$$

and the exact solution is

TABLE 2: Comparison of the different values of β and the absolute error of the 6th order NTIM solution with the 6th order MGTFSM solution for Problem 8.

x, y	t	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 1.0$	Exact	NTIM error	Error [18]
	0.1	2.46455	1.94329	1.56127	0.350067	0.350067	6.364298×10^{-12}	1.4090×10^{-10}
	0.3	2.46455	1.94329	1.56127	0.427573	0.427573	1.427800×10^{-8}	1.0576×10^{-7}
0.1	0.5	2.46455	1.94329	1.56127	0.522238	0.522239	5.234815×10^{-7}	2.3354×10^{-6}
	0.7	2.46455	1.94329	1.56127	0.637858	0.637864	5.666541×10^{-6}	1.8129×10^{-5}
	0.9	2.46455	1.94329	1.56127	0.779055	0.779089	3.381305×10^{-5}	8.4486×10^{-5}
	0.1	4.32451	3.40986	2.73955	0.614259	0.614259	1.116729×10^{-11}	4.2268×10^{-10}
	0.3	4.32451	3.40986	2.73955	0.750258	0.750258	2.505345×10^{-8}	3.1726×10^{-7}
0.3	0.5	4.32451	3.40986	2.73955	0.916366	0.916367	9.185473×10^{-7}	7.0059×10^{-6}
	0.7	4.32451	3.40986	2.73955	1.11924	1.11925	9.943018×10^{-6}	5.4385×10^{-5}
	0.9	4.32451	3.40986	2.73955	1.367	1.36706	5.933139×10^{-5}	2.5345×10^{-4}
	0.1	5.72082	4.51084	3.6241	0.812592	0.812592	1.477307×10^{-11}	7.0425×10^{-10}
	0.3	5.72082	4.51084	3.6241	0.992502	0.992502	3.314275×10^{-8}	5.2860×10^{-7}
0.5	0.5	5.72082	4.51084	3.6241	1.21224	1.21224	1.21513×10^{-6}	1.1673×10^{-5}
	0.7	5.72082	4.51084	3.6241	1.48063	1.48064	1.315344×10^{-5}	9.0614×10^{-5}
	0.9	5.72082	4.51084	3.6241	1.80838	1.80846	7.848841×10^{-5}	4.2228×10^{-4}

TABLE 3: Comparison of the different values of β and the absolute error of the 6th order NTIM solution with the 6th order MGTFSM solution for Example 1.

x, y	t	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 1.0$	Exact	NTIM error	Error [18]
	0.1	8.59894	6.78023	5.44736	1.2214	1.2214	2.220490×10^{-11}	1.5572×10^{-9}
	0.3	8.59894	6.78023	5.44736	1.49182	1.49182	4.981669×10^{-8}	1.1688×10^{-6}
0.1	0.5	8.59894	6.78023	5.44736	1.82212	1.82212	1.826455×10^{-6}	2.5810×10^{-5}
	0.7	8.59894	6.78023	5.44736	2.22552	2.22554	1.977086×10^{-5}	2.0036×10^{-4}
	0.9	8.59894	6.78023	5.44736	2.71816	2.71828	1.179755×10^{-4}	9.3372×10^{-4}
	0.1	10.5028	8.28139	6.65342	1.49182	1.49182	2.712142×10^{-11}	1.9019×10^{-9}
	0.3	10.5028	8.28139	6.65342	1.82212	1.82212	6.084624×10^{-8}	1.4276×10^{-6}
0.3	0.5	10.5028	8.28139	6.65342	2.22554	2.22554	2.230837×10^{-6}	3.1525×10^{-5}
	0.7	10.5028	8.28139	6.65342	2.71826	2.71828	2.414818×10^{-5}	2.4472×10^{-4}
	0.9	10.5028	8.28139	6.65342	3.31997	3.32012	1.440956×10^{-4}	1.1404×10^{-3}
	0.1	12.8281	10.1149	8.12651	1.82212	1.82212	3.312617×10^{-11}	2.3230×10^{-9}
	0.3	12.8281	10.1149	8.12651	2.22554	2.22554	7.431777×10^{-8}	1.7436×10^{-6}
0.5	0.5	12.8281	10.1149	8.12651	2.71828	2.71828	2.724750×10^{-6}	3.8504×10^{-5}
	0.7	12.8281	10.1149	8.12651	3.32009	3.32012	2.949466×10^{-5}	2.9890×10^{-4}
	0.9	12.8281	10.1149	8.12651	4.05502	4.0552	1.759988×10^{-4}	1.3929×10^{-3}

$$\phi(x, y, z, t) = e^{(\sqrt{hr/8})(x+y)+ht}. \tag{43}$$

Using the same procedure as for Problems 7 and 8, we obtain the solution as

$$\left\{\begin{array}{c} \phi_0(x, y, t) = e^{\sqrt{\frac{hr}{8}}(x+y)}, \phi_1(x, y, t) = \dfrac{ht^{\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(\beta+1)}, \\ \phi_2(x, y, t) = \dfrac{h^2 t^{2\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(2\beta+1)}, \phi_3(x, y, t) = \dfrac{h^3 t^{3\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(3\beta+1)}, \\ \phi_4(x, y, t) = \dfrac{h^4 t^{4\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(4\beta+1)}, \\ \vdots \end{array}\right\} \tag{44}$$

Combining the components, the 3rd order approximate solution is given as

$$\phi(x, y, t) = \phi_0 + \phi_1 + \phi_2 + \phi_3 + \phi_4 + \cdots$$
$$= \left\{\begin{array}{c} e^{\frac{\sqrt{hr}(x+y)}{2\sqrt{2}}} + \dfrac{ht^{\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(\beta+1)} + \dfrac{h^2 t^{2\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(2\beta+1)} + \dfrac{h^3 t^{3\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(3\beta+1)} + \dfrac{h^4 t^{4\beta} e^{\sqrt{hr}(x+y)/2\sqrt{2}}}{\Gamma(4\beta+1)} \\ \vdots \end{array}\right\}. \tag{45}$$

Equation (27) can be simplified as

$$\phi(x, y, t) = e^{\frac{\sqrt{hr}(x+y)}{2\sqrt{2}}} \left(\frac{ht^{\beta}}{\Gamma(\beta+1)} + \frac{h^2 t^{2\beta}}{\Gamma(2\beta+1)} + \frac{h^3 t^{3\beta}}{\Gamma(3\beta+1)} + \frac{h^4 t^{4\beta}}{\Gamma(4\beta+1)} + \cdots \right). \tag{46}$$

For $\beta = 1$, Equation (29) converges as

$$\phi(x, y, t) = e^{(\sqrt{hr}(x+y))/(2\sqrt{2})} \left(1 + h\,t + \frac{(h\,t)^2}{2!} + \frac{(h\,t)^3}{3!} + \frac{(h\,t)^4}{4!} + \cdots \right), \tag{47}$$

which converges to the exact solution given by Equation (43).

5. Results and Discussions

The biological population model of fractional order has been investigated in the present work. We observe that the solution pattern for Examples 1-3 converges very rapidly to the exact solution in a few iterations. The results have been compared through graphs and tables which confirms the convergence of NTIM. Figure 1 is the NTIM approximate solution, and Figure 2 is the exact solution for $\beta = 1$ of Example 1. The 6th order approximate solution for different fractional values of β for Example 1 is depicted in Figure 3. Furthermore, Figures 4 and 5 show the 6th order NTIM solution and the exact solution, respectively, for Example 2 by mean of 3D plots. The comparison for different fractional values of β and exact solution is made in Figure 6 for Example 2. Similarly Figures 7 and 8 show, respectively, the 6th order NTIM solution and exact solution for Example 3. Figure 9 is the comparison of fractional values of β and exact solution for Example 3. In Table 1–3, the approximate solution has been compared in tabular form for $\beta = 0.5$, $\beta = 0.7$, $\beta = 0.9$, and $\beta = 1.0$, with the exact solution. The value $\beta = 1.0$ converts the FDE to the classical PDE. The absolute error by the proposed NTIM in Tables 1–3 has been compared with the absolute error obtained by the modified generalized Taylor fractional series method (MGTFSM) for Examples 1-3. It is concluded from the results of Examples 1-3 that as the fractional value of β reaches 1, the NTIM approximate solution meets with the exact solution. We also observe that NTIM yields an excellent approximate solution.

6. Conclusion

Three nonlinear problems of fractional order biological population model have been investigated by the natural transform iterative method in the current study. The method is applied to nonlinear problems without any discretization. We found that NTIM converges very rapidly to the exact solution. The advantage of the method is that it is free of any large or small parameter assumptions or to find any constant at the end of the solution. The obtained results of the FBPM have been compared through 3D and 2D plots, and also, the numerical values have been compared in tabular form for different values of β. The comparison between absolute errors of the NTIM approximate solution and modified generalized Taylor fractional series method solution is done with the help of tables. In each case, NTIM reveals an efficient approximate solution as compared with other methods in the literature.

Data Availability

No data were generated or analyzed during the study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

Acknowledgments

The authors extend their appreciation to the Deanship of Scientific Research at King Khalid University for funding this work through research groups under grant number R.G.P.2/14/43.

References

[1] T. Rabczuk, H. Ren, and X. Zhuang, “A nonlocal operator method for partial differential equations with application to electromagnetic waveguide problem,” *Computers, Materials & Continua*, vol. 59, no. 1, pp. 31–55, 2019.

[2] D. H. Staelin, A. W. Morgenthaler, and J. A. Kong, *Electromagnetic Waves*, Pearson Education, India, 1994.

[3] D. Tripathi and O. A. Bég, “Mathematical modelling of peristaltic propulsion of viscoplastic bio-fluids,” *Proceedings of the Institution of Mechanical Engineers, Part H: Journal of Engineering in Medicine*, vol. 228, no. 1, pp. 67–88, 2014.

[4] A. Dumas, J. Dijkstra, and J. France, "Mathematical modelling in animal nutrition: a centenary review," *The Journal of Agricultural Science*, vol. 146, no. 2, pp. 123–142, 2008.

[5] D. Baleanu, H. Mohammadi, and S. Rezapour, "A mathematical theoretical study of a particular system of Caputo–Fabrizio fractional differential equations for the Rubella disease model," *Advances in Difference Equations*, vol. 2020, no. 1, 2020.

[6] R. Hilfer, *Applications of Fractional Calculus in Physics*, World scientific, Covent Garden, London, 2000.

[7] R. Almeida, "A Caputo fractional derivative of a function with respect to another function," *Communications in Nonlinear Science and Numerical Simulation*, vol. 44, pp. 460–481, 2017.

[8] N. H. Tuan, H. Mohammadi, and S. Rezapour, "A mathematical model for COVID-19 transmission by using the Caputo fractional derivative," *Chaos, Solitons & Fractals*, vol. 140, article 110107, 2020.

[9] C. Li, D. Qian, and Y. Chen, "On Riemann-Liouville and Caputo derivatives," *Discrete Dynamics in Nature and Society*, vol. 2011, 15 pages, 2011.

[10] L. Kexue and P. Jigen, "Laplace transform and fractional differential equations," *Applied Mathematics Letters*, vol. 24, no. 12, pp. 2019–2023, 2011.

[11] D. S. Bodkhe and S. K. Panchal, "On Sumudu transform of fractional derivatives and its applications to fractional differential equations," *Asian Journal of Mathematics and Computer Research*, vol. 11, no. 1, pp. 69–77, 2016.

[12] Y. Singh, V. Gill, S. Kundu, and D. Kumar, "On the Elzaki transform and its applications in fractional free electron laser equation," *Acta Universitatis Sapientiae, Mathematica*, vol. 11, no. 2, pp. 419–429, 2019.

[13] R. Nawaz, N. Ali, L. Zada, K. S. Nisar, M. R. Alharthi, and W. Jamshed, "Extension of natural transform method with Daftardar-Jafari polynomials for fractional order differential equations," *Alexandria Engineering Journal*, vol. 60, no. 3, pp. 3205–3217, 2021.

[14] R. Nawaz, N. Ali, L. Zada, Z. Shah, A. Tassaddiq, and N. A. Alreshidi, "Comparative analysis of natural transform decomposition method and new iterative method for fractional foam drainage problem and fractional order modified regularized long-wave equation," *Fractals*, vol. 28, no. 7, article 2050124, 2020.

[15] L. Zada, R. Nawaz, S. Ahsan et al., "New iterative approach for the solutions of fractional order inhomogeneous partial differential equations," *AIMS Mathematics*, vol. 6, no. 2, pp. 1348–1365, 2021.

[16] S. Bhalekar and V. Daftardar-Gejji, "New iterative method: application to partial differential equations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 778–783, 2008.

[17] H. Eltayeb, Y. T. Abdalla, I. Bachar, and M. H. Khabir, "Fractional telegraph equation and its solution by natural transform decomposition method," *Symmetry*, vol. 11, no. 3, p. 334, 2019.

[18] A. Khalouta, A. Kadem, and Laboratory of Fundamental and Numerical Mathematics, Department of Mathematics, Faculty of Sciences, Ferhat Abbas University of Setif 1, Sétif 19000, Algeria, "A new numerical technique for solving Caputo time-fractional biological population equation," *AIMS Mathematics*, vol. 4, no. 5, pp. 1307–1319, 2019.

[19] M. Zellal and K. Belghaba, "An accurate algorithm for solving biological population model by the variational iteration method using He's polynomials," *Arab Journal of Basic and Applied Sciences*, vol. 25, no. 3, pp. 142–149, 2018.

[20] A. M. A. El-Sayed, S. Z. Rida, and A. A. M. Arafa, "Exact solutions of fractional-order biological population model," *Communications in Theoretical Physics*, vol. 52, no. 6, pp. 992–996, 2009.

[21] R. Shah, H. Khan, S. Mustafa, P. Kumam, and M. Arif, "Analytical solutions of fractional-order diffusion equations by natural transform decomposition method," *Entropy*, vol. 21, no. 6, p. 557, 2019.

[22] S. Bhalekar and V. Daftardar-Gejji, "Convergence of the new iterative method," *International journal of differential equations*, vol. 2011, 10 pages, 2011.

[23] A. Khan, R. Zarin, U. W. Humphries, A. Akgül, A. Saeed, and T. Gul, "Fractional optimal control of COVID-19 pandemic model with generalized Mittag-Leffler function," *Advances in Difference Equations*, vol. 2021, no. 1, 2021.

[24] A. Saeed, M. Bilal, T. Gul, P. Kumam, A. Khan, and M. Sohail, "Fractional order stagnation point flow of the hybrid nanofluid towards a stretching sheet," *Scientific Reports*, vol. 11, no. 1, pp. 1–15, 2021.

[25] T. Gul, M. Altaf Khan, A. Khan, and M. Shuaib, "Fractional-order three-dimensional thin-film nanofluid flow on an inclined rotating disk," *The European Physical Journal Plus*, vol. 133, no. 12, article 500, 2018.

[26] T. Gul, H. Anwar, M. A. Khan, I. Khan, and P. Kumam, "Integer and non-integer order study of the GO-W/GO-EG nanofluids flow by means of Marangoni convection," *Symmetry*, vol. 11, no. 5, p. 640, 2019.

[27] T. Gul, M. Khan, W. Noman, I. Khan, T. Abdullah Alkanhal, and I. Tlili, "Fractional order forced convection carbon nanotube nanofluid flow passing over a thin needle," *Symmetry*, vol. 11, no. 3, p. 312, 2019.

[28] W. Jamshed, N. A. A. M. Nasir, S. S. P. M. Isa et al., "Thermal growth in solar water pump using Prandtl–Eyring hybrid nanofluid: a solar energy application," *Scientific Reports*, vol. 11, no. 1, pp. 1–21, 2021.

[29] W. Jamshed, K. S. Nisar, R. W. Ibrahim, F. Shahzad, and M. R. Eid, "Thermal expansion optimization in solar aircraft using tangent hyperbolic hybrid nanofluid: a solar thermal application," *Journal of Materials Research and Technology*, vol. 14, pp. 985–1006, 2021.

[30] W. Jamshed, S. U. Devi, and K. S. Nisar, "Single phase based study of ag-cu/EO Williamson hybrid nanofluid flow over a stretching surface with shape factor," *Physica Scripta*, vol. 96, no. 6, article 065202, 2021.

[31] W. Jamshed, K. S. Nisar, R. W. Ibrahim, F. Shahzad, and M. R. Eid, "Thermal expansion optimization in solar aircraft using tangent hyperbolic hybrid nanofluid: a solar thermal application," *Journal of Materials Research and Technology*, vol. 14, pp. 985–1006, 2021.

[32] X.-H. Zhang, A. Ali, M. A. Khan, M. Y. Alshahrani, T. Muhammad, and S. Islam, "Mathematical analysis of the TB model with treatment via Caputo-type fractional derivative," *Discrete Dynamics in Nature and Society 2021*, vol. 2021, pp. 1–15, 2021.

[33] A. Ali, F. S. Alshammari, S. Islam, M. A. Khan, and S. Ullah, "Modeling and analysis of the dynamics of novel coronavirus (COVID-19) with Caputo fractional derivative," *Results in Physics*, vol. 20, article 103669, 2021.

[34] A. Ali, Q. Iqbal, J. K. K. Asamoah, and S. Islam, "Mathematical modeling for the transmission potential of Zika virus with optimal control strategies," *The European Physical Journal Plus*, vol. 137, no. 1, pp. 1–30, 2022.

[35] Y. M. Chu, A. Ali, M. A. Khan, S. Islam, and S. Ullah, "Dynamics of fractional order COVID-19 model with a case study of Saudi Arabia," *Results in Physics*, vol. 21, article 103787, 2021.