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Research Article

A Novel Analysis of Generalized Perturbed Zakharov–Kuznetsov Equation of Fractional-Order Arising in Dusty Plasma by Natural Transform Decomposition Method

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The natural transform decomposition method (NTDM) is a relatively new transformation method for finding an approximate differential equation solution. In the current study, the NTDM has been used for obtaining an approximate solution of the fractional-order generalized perturbed Zakharov–Kuznetsov (GPZK) equation. The method has been tested for three nonlinear cases of the fractional-order GPZK equation. The absolute errors are analyzed by the proposed method and the q-homotopy analysis transform method (q-HATM). 3D and 2D graphs have shown the proposed method's accuracy and effectiveness. NTDM gives a much-closed solution after a few terms.

1. Introduction

The power of applying fractional calculus to physical problems is that, when dealing with the integer order of derivative, which depends on the function's behavior, the fractional derivative produces the whole story of this function. For this reason, studying the behavior of the function fractionally is sometimes called the memory effect. This effect leads to many applications of differential equations. The importance of fractional differential equations (FDEs) cannot be denied in the recent advancement of the real world. There are models which provide a better way of managing the systems through fractional differential equations. These equations may arise in electronic circuits, physics, engineering [1, 2], bioscience, etc. [3, 4]. Finance also deals with the fractional-order differential equations to handle clients in more suitable and affordable packages for dealing with financial crises [5, 6]. Other differential equation (DE) applications are image processing and signal processing [7, 8]. These equations may be linear or nonlinear, depending upon the geometry of the problem. Simple problems can be demonstrated by ordinary differential equations (ODEs), while complex problems can be demonstrated through partial differential equations (PDEs). Most linear problems have the exact solution, but it is hard to find the exact solution to complex nonlinear problems. To handle these problems, the researchers used numerical, analytical, and some homotopy-based methods to approximate such a nonlinear problem. Some famous methods which handle the DEs of fractional and integer order are the residual power series method [9, 10], Laplace decomposition method (LDM) [11], q-homotopy analysis method (q-HAM) [12], Adomian decomposition method (ADM) [13], reduced

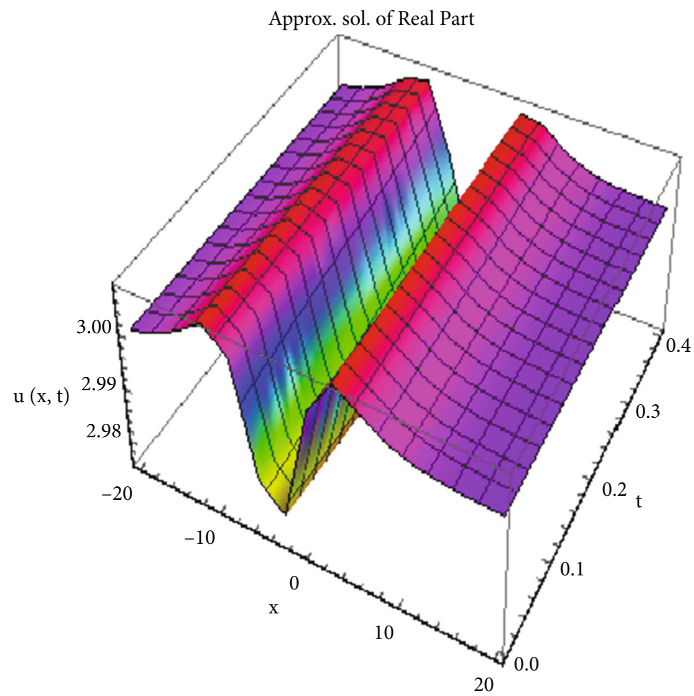

Figure 1: NTDM solution of the real part of problem 1.

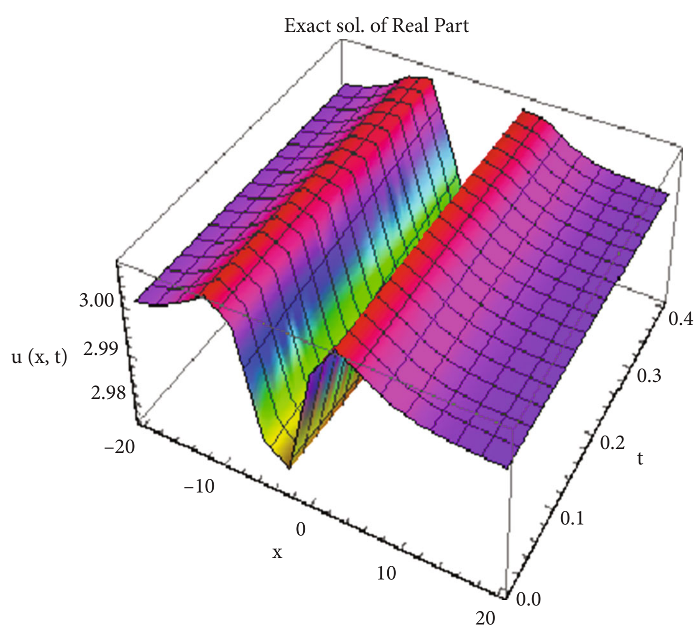

Figure 2: Exact solution of the real part of problem 1.

differential transform method (RDTM) [14], variational iteration method (VIM) [15], optimal homotopy asymptotic method (OHAM) [16, 17], homotopy perturbation method (HPM) [18], homotopy analysis method (HAM) [19], etc. Besides these methods, many researchers have introduced many numerical methods to handle differential equations [20]. Transformations also help for the solution approximation of DEs. In the present study, we apply a relatively new method named the natural transform decomposition method (NTDM). We decompose the nonlinear terms with the help of Adomian polynomials and then apply the natural transformation to obtain the solution to the problem. Many researchers have applied NTDM to handle DEs of fractional order [21, 22]. The time-fractional GPZK equation $(3+1)$ dimension with the following form is taken to analyze NTDM [23].

$$D_t^{\alpha} q + \beta_1 q^{\lambda} q_{\chi} + \beta_2 q_{\chi\chi\chi} + \beta_3 q_{\chi yy} + \beta_3 q_{\chi zz} + \zeta q_{\chi\chi\chi\chi\chi} = 0, 0 < \alpha \leq 1, t > 0, \tag{1}$$

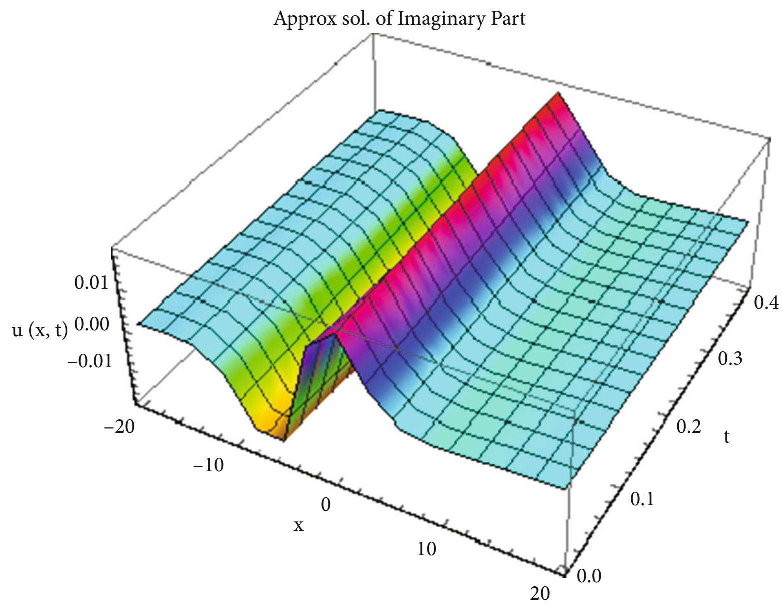

Figure 3: NTDM solution of the imaginary part of problem 1.

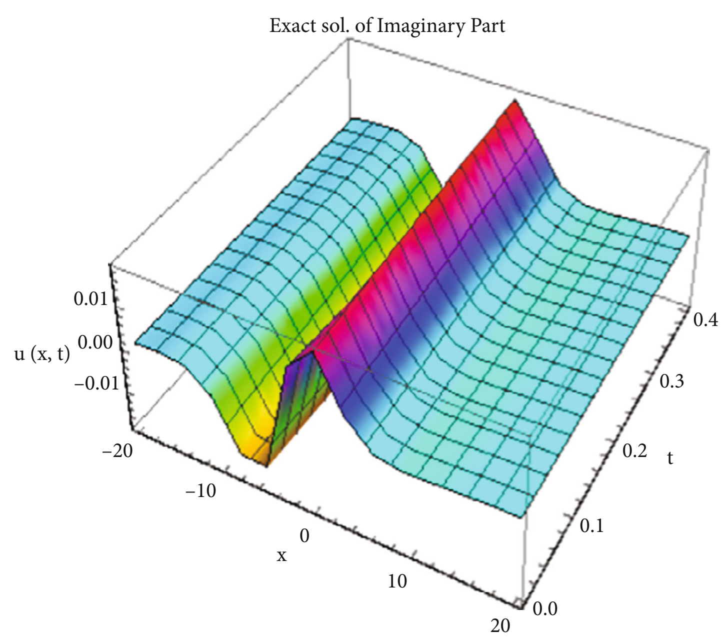

Figure 4: Exact solution of the imaginary part of problem 1.

where α is the fractional-order of the Caputo's derivative, q is the electrostatic potential, ζ represents the smallness parameter, λ is a positive number, and β_1, β_2, and β_3 are constants. Equation (1) describes the nonlinear dust-ion-acoustic waves in the magnetized plasmas [23]. The study of ion-acoustic waves and structures in dense quantum plasmas has sparked much interest in recent years.

The remaining paper is organized as follows: The preliminary definitions are given in Section 2 contains. Section 3 introduces the core concept of NTDM. In Section 4, NTDM is applied to three fractional-order ZK equations. There is a conclusion to the method applications, and in the end, the bibliography is given.

2. Preliminaries

This section introduces some basic definition including the fractional Caputo's definition, the fractional order Riemann-Liouville (R-L) integral and some basic definitions of transformation regarding derivative and integration.

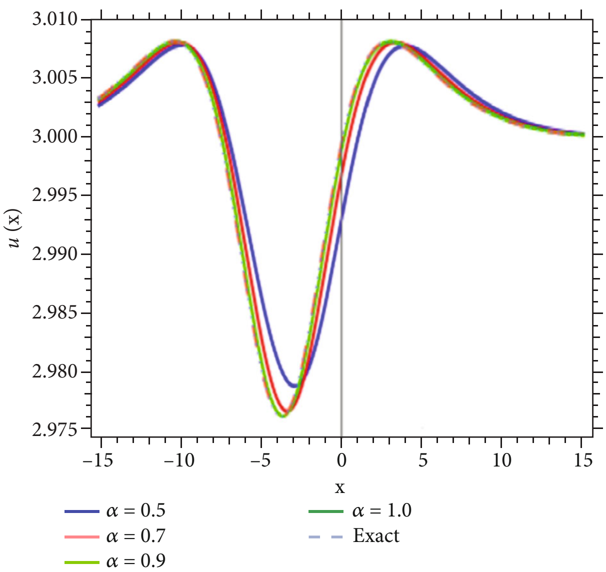

Figure 5: Real part solution comparison for fractional values of α for problem 1.

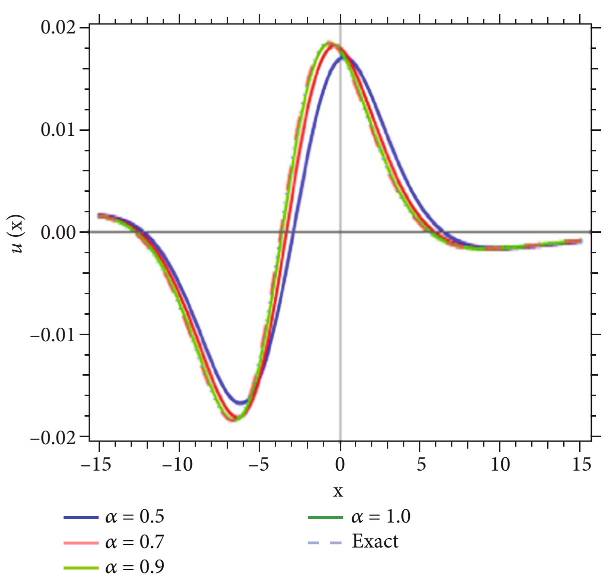

Figure 6: Imaginary part solution comparison for fractional values of α for problem 1.

Definition 1. The R-L fractional integral (I_t^α) of order $\alpha \geq 0$, $f(t)$ is defined as follows:

$$I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t (t-\eta)^{\alpha-1} f(\eta) d\eta, (\alpha > 0, t > 0),$$
$$I_t^0 f(t) = f(t). \tag{2}$$

Definition 2. Caputo's fractional derivative of order $\alpha > 0$ is defined as follows:

$$D_t^\alpha q(x,t) = \frac{\partial^\alpha q(\wp,t)}{\partial t^\alpha} = \begin{cases} \dfrac{1}{\Gamma(n-\alpha)} \displaystyle\int_0^t (t-\eta)^{n-\alpha-1} \frac{\partial^n q(\wp,\eta)}{\partial \eta^n}, & \text{if } n-1 < \alpha < n, \\ \dfrac{\partial^n q(\wp,t)}{\partial t^n}, & \text{if } \alpha = n \in N. \end{cases} \tag{3}$$

Definition 3. Natural transform of $q(t)$ is given as [17]

$$\mathbb{N}^+(q(t)) = \mathbb{R}(s,v) = \frac{1}{v} \int_0^\infty e^{-st/v}(q(t))dt\,; s, v > 0, \tag{4}$$

s and v are the transform variables.

Definition 4. Inverse natural transform of $\mathbb{R}(s,v)$ is written as follows:

$$\mathbb{N}^-(\mathbb{R}(s,v)) = q(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} e^{st/v}(\mathbb{R}(s,v))ds, \tag{5}$$

the integral is along the complex plane.

Definition 5. If $q^n(t)$ is the nth derivative of the function $q(t)$, then its natural transform is as follows:

$$\mathbb{N}^+(q^n(t)) = \mathbb{R}_n(s,v) = \frac{s^n}{v^n}\mathbb{R}(s,v) - \sum_{k=0}^{n-1} \frac{s^{n-(k+1)}}{v^{n-k}}(q^n(0)), n \geq 1. \tag{6}$$

Theorem 6. *If natural transform of $g(t)$ and $k(t)$are, respectively, $G(s,v)$ and $K(s,v)$ and are defined in set A, then*

$$\mathbb{N}[g * k] = vG(s,v)K(s,v), \tag{7}$$

where $[g * k]$ is a convolution of g and k.

3. Basic Idea of NTDM

Consider the fractional order PDE of the form

$$D_t^\alpha(q(\wp,t)) = f(\wp,t) + Lq(\wp,t) + \aleph q(\wp,t), 0 < t \leq 1, m-1 < \alpha < m, m \in N, \tag{8}$$

where α is fractional derivative, L represents linear, $\aleph$ shows the nonlinear operator, and $f(\wp,t)$ is known function. The initial condition is given as follows:

$$u(\wp,0) = g(\wp). \tag{9}$$

When we use the natural transform on Equation (8), we

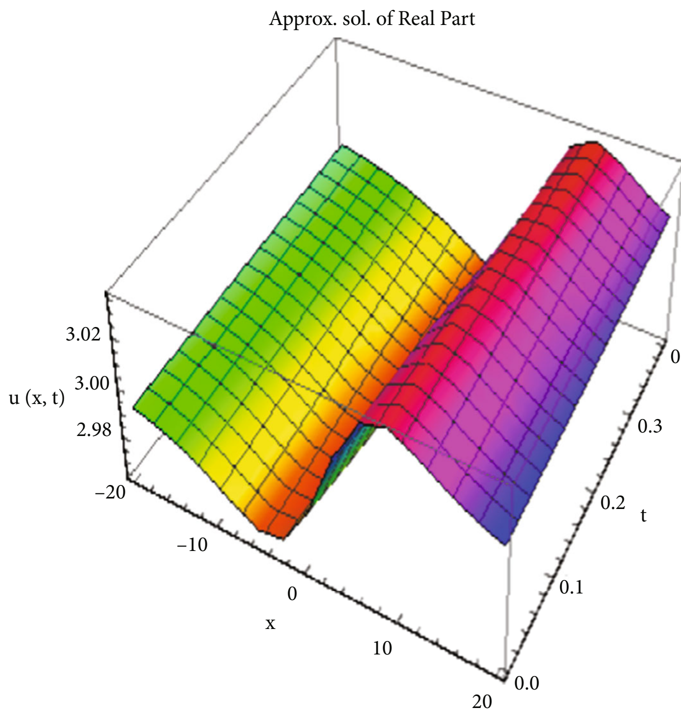

Figure 7: NTDM solution of the real part of problem 2.

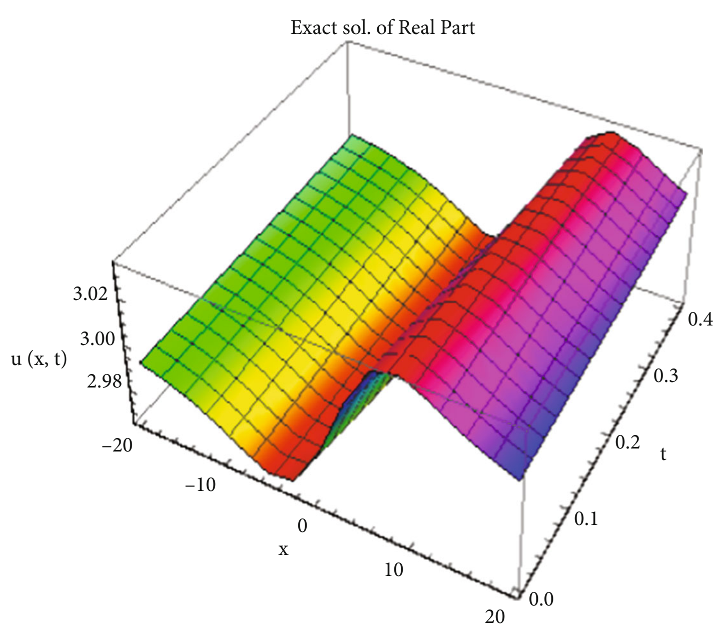

Figure 8: Exact solution of the real part of problem 2.

get

$$\mathbb{N}^{+}[D_t^{\alpha}(q(\wp,t))] = \mathbb{N}^{+}[f(\wp,t)] + \mathbb{N}^{+}[L(q(\wp,t)) + \aleph(q(\wp,t))]. \tag{10}$$

Using the natural transform's differentiation characteristic to Equation (10), we have

$$\begin{aligned}\frac{s^{\alpha}}{\nu^{\alpha}}\mathbb{N}^{+}[q(\wp,t)] - \frac{s^{\alpha-1}}{\nu^{\alpha}}q(\wp,0) \\ = \mathbb{N}^{+}[f(\wp,t)] + \mathbb{N}^{+}\left[Lq(\wp,t) + \aleph q\left(\underline{\wp},t\right)\right].\end{aligned} \tag{11}$$

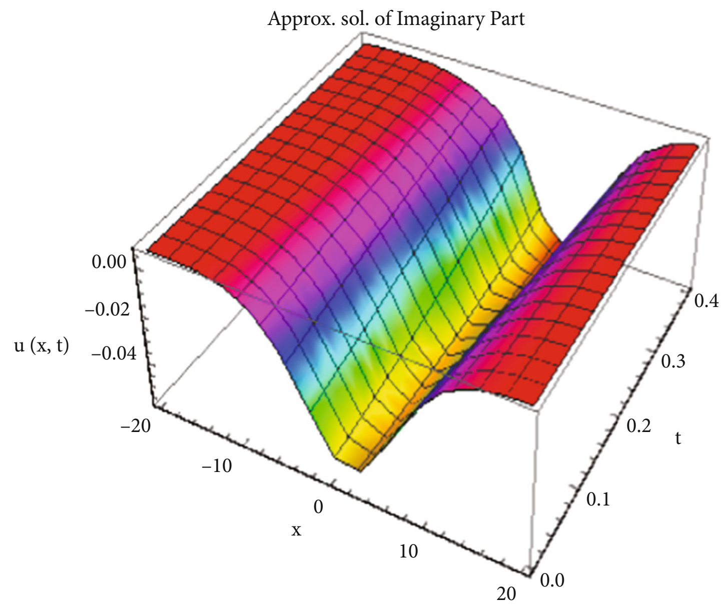

Figure 9: NTDM solution of the imaginary part of problem 2.

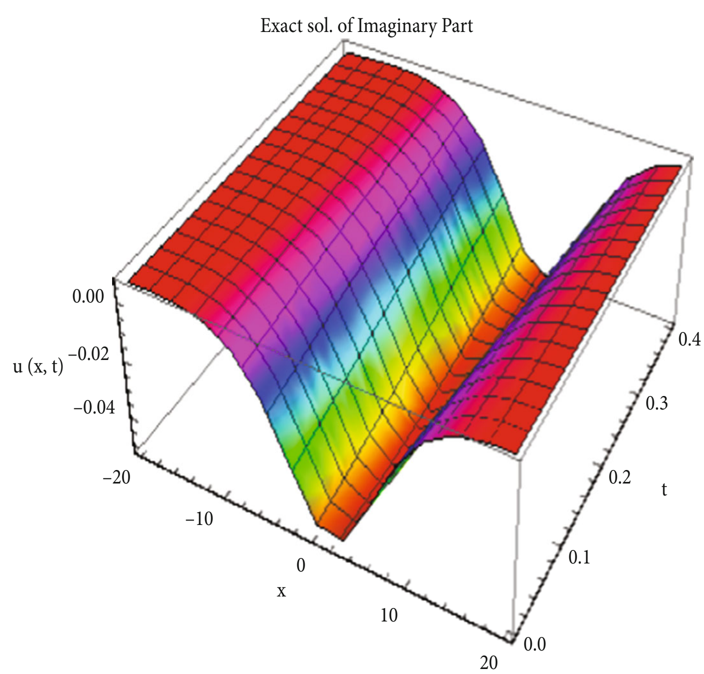

Figure 10: Exact solution of the imaginary part of problem 2.

Equation (11), after rearranging, is

$$\mathbb{N}^{+}[q(\wp,t)] = \frac{g(\wp)}{s} + \frac{\nu^{\alpha}}{s^{\alpha}}\left(\mathbb{N}^{+}[f(\wp,t)]\right) + \frac{\nu^{\alpha}}{s^{\alpha}}\left(\mathbb{N}^{+}[L(q(\wp,t)) + \aleph(q(\wp,t))]\right). \tag{12}$$

For the NTDM solution, $q(\wp,t)$ expands as the infinite series

$$q(\wp,t) = \sum_{i=0}^{\infty} q_i(\wp,t). \tag{13}$$

The infinite series defines the nonlinear terms as follows:

$$\aleph q(\wp,t) = \sum_{i=0}^{\infty} A_i, \tag{14}$$

$$A_i = \frac{1}{i!}\left[\frac{d^i}{d\lambda^i}\left[\aleph\left(\sum_{i=0}^{\infty}\lambda^i q_i\right)\right]\right]_{\lambda=0}, i = 0, 1, 2 \cdots \tag{15}$$

The Adomian polynomials are represented by A_i. Equation (12) is modified by substituting Equations (13) and

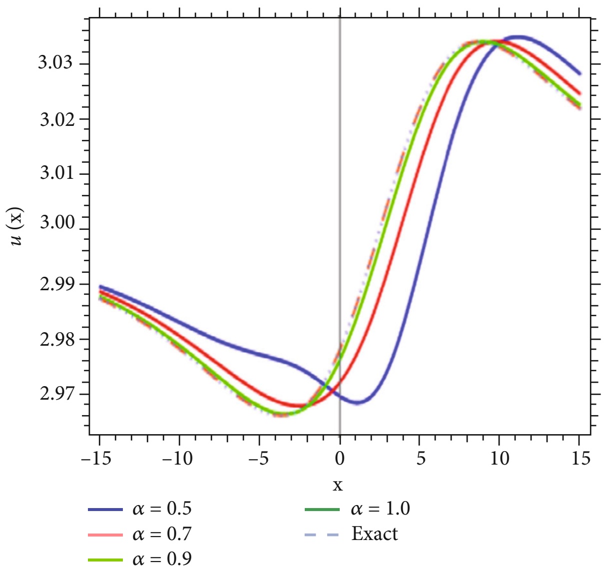

FIGURE 11: Real part solution comparison for fractional values of α for problem 2.

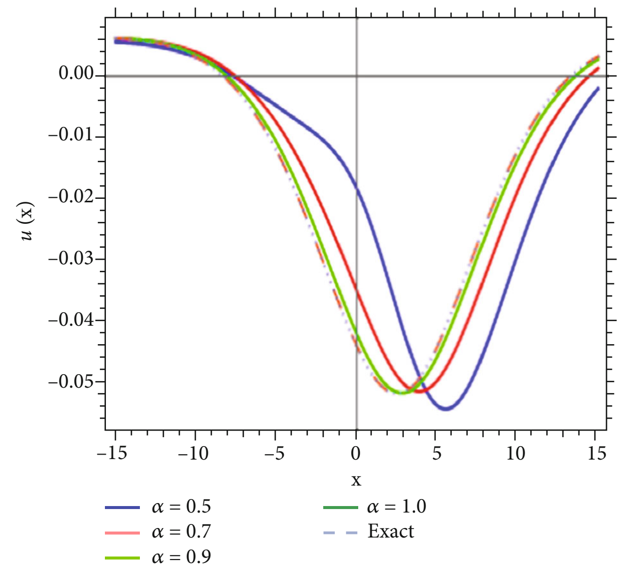

FIGURE 12: Imaginary part solution comparison for fractional values of α for problem 2.

(14).

$$\mathbb{N}^+\left[\sum_{i=0}^{\infty} q(\wp,t)\right] = \frac{g(\wp)}{s} + \frac{v^\alpha}{s^\alpha}\mathbb{N}^+[f(\wp,t)] + \frac{v^\alpha}{s^\alpha}\mathbb{N}^+\left[L\left(\sum_{i=0}^{\infty} q(\wp,t)\right) + \sum_{i=0}^{\infty} A_i\right], \tag{16}$$

using the natural transform's linearity as follows:

$$\begin{aligned} \mathbb{N}^+[q_0(\wp,t)] &= \frac{q(\wp,0)}{s} + \frac{v^\alpha}{s^\alpha}\mathbb{N}^+[f(\wp,t)], \\ \mathbb{N}^+[q_1(\wp,t)] &= \frac{v^\alpha}{s^\alpha}\mathbb{N}^+[L(q_0(\wp,t)) + A_0], \\ &\vdots \\ \mathbb{N}^+[q_{i+1}(\wp,t)] &= \frac{v^\alpha}{s^\alpha}\mathbb{N}^+[Lq_i(\wp,t) + A_i], i \geq 0. \end{aligned} \tag{17}$$

The solution components of Equation (28) are obtained by taking the inverse natural transform as follows:

$$\begin{aligned} q_0(\wp,t) &= g(\wp,t), \\ q_{i+1}(\wp,t) &= \mathbb{N}^-\left[\frac{v^\alpha}{s^\alpha}\mathbb{N}^+[Lq_i(\wp,t) + A_i]\right], \end{aligned} \tag{18}$$

where the term $g(\wp,t)$ is derived from the specified source term and initial condition.

The approximate m-term solution of Equations (8) and (9) is

$$q(\wp,t) = q_0(\wp,t) + q_1(\wp,t) + q_2(\wp,t) + \cdots + q_{m-1}(\wp,t). \tag{19}$$

4. Convergence Analysis of NTDM

Theorem 7. *Let H be the Hilbert space defined by $H = L^2((\alpha, \beta)X[0, T])$ the set of applications*

$$q = (\alpha, \beta)X[0, T] \longrightarrow R \, with \int_{(\alpha,\beta)X[0,T]} q^2(x, s)dsd\vartheta < +\infty. \tag{20}$$

In light of the above assumptions, we now consider the GPZK equation of fractional order and denote

$$L(q) = \frac{\partial^\alpha}{\partial t^\alpha}(q). \tag{21}$$

The GPZK equation is then written in operator form

$$L(q) = -\beta_1 q^k u_x - \beta_2 q_{xxx} - \beta_3 q_{xyy} - \beta_3 q_{xzz} - \zeta q_{xxxxx}. \tag{22}$$

If the following hypotheses are true, the NTDM is convergent:

$$(H1)(L(q) - L(\eta), q - \eta) \geq k\|q - \eta\|^2 ; k > 0, \forall q, \eta \in H. \tag{23}$$

$H(2)$ whatever maybe $M > 0$, there exists a constant $C(M) > 0$, such that for $q, \eta \in H$ with $\|q\| \leq M$, $\|\eta\| \leq M$, we have for every $(L(q) - L(\eta), q - \eta) \leq C(M)\|q - \eta\|\|w\|$ for every $w \in H$.

5. Applications of NTDM

5.1. Problem 1. Consider the following $(3 + 1)$ dimension GPZK equation [23]

$$D_t^\alpha q + \beta_1 q q_{\chi\chi} + \beta_2 q_{\chi\chi\chi} + \beta_3 q_{\chi yy} + \beta_3 q_{\chi zz} + \zeta q_{\chi\chi\chi\chi\chi} = 0, \tag{24}$$

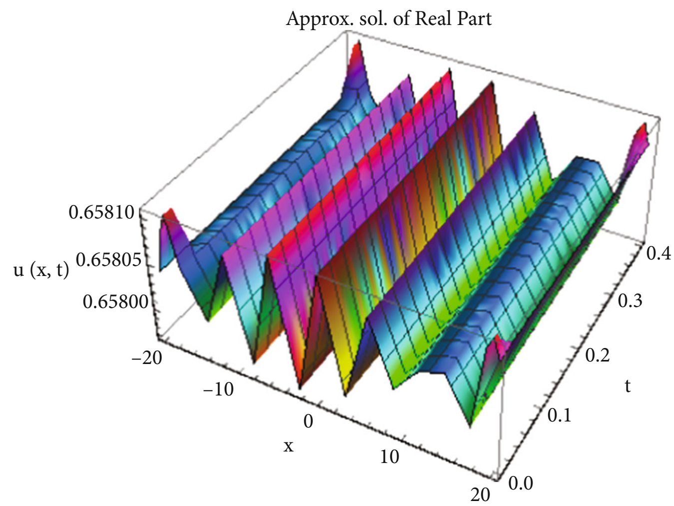

FIGURE 13: NTDM solution of the real part of problem 3.

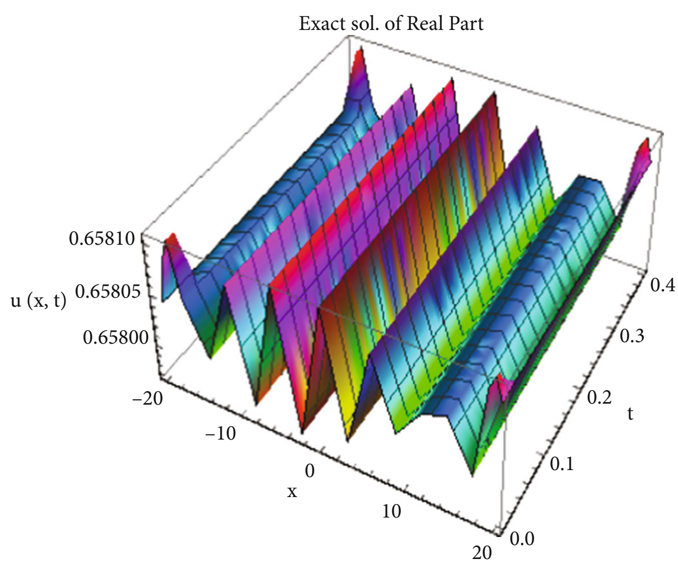

FIGURE 14: Exact solution of the real part of problem 3.

where $q = q(\chi, y, z, t)$ together with initial conditions

$$q(\chi, y, z, 0) = e_0 - \frac{1680\zeta\rho^4}{\beta_1\left(-z\sqrt{-(\beta_2\rho^2/\beta_3) - \kappa^2} + \rho\chi + \kappa y + \phi\right)^4}. \tag{25}$$

The exact solution is given as [23]

$$q(\chi, y, z, t) = e_0 - \frac{1680\zeta\rho^4}{\beta_1\left(-z\sqrt{-(\beta_2\rho^2/\beta_3) - \kappa^2} + \rho\chi + \kappa y - \beta_1 e_0\rho t + \phi\right)^4}. \tag{26}$$

The (24) is rearranged as follows:

$$D_t^\alpha q = -\beta_1 q q_\chi - \beta_2 q_{\chi\chi\chi} - \beta_3 q_{\chi yy} - \beta_3 q_{\chi zz} - \zeta q_{\chi\chi\chi\chi\chi}. \tag{27}$$

Apply natural transform to Equation (27), we have

$$\mathbb{N}^+[D_t^\alpha q] = \mathbb{N}^+\left[-\beta_1 q q_\chi - \beta_2 q_{\chi\chi\chi} - \beta_3 q_{\chi yy} - \beta_3 q_{\chi zz} - \zeta q_{\chi\chi\chi\chi\chi}\right]. \tag{28}$$

Use the natural transform's differentiation characteristic to Equation (28), we have

$$\frac{s^\alpha}{\nu^\alpha} q(\chi, y, z, t) - \frac{\nu^{\alpha-1}}{s^\alpha} q(\chi, y, z, 0) = \mathbb{N}^+\left[-\beta_1 q q_\chi - \beta_2 q_{\chi\chi\chi} - \beta_3 q_{\chi yy} - \beta_3 q_{\chi zz} - \zeta q_{\chi\chi\chi\chi\chi}\right]. \tag{29}$$

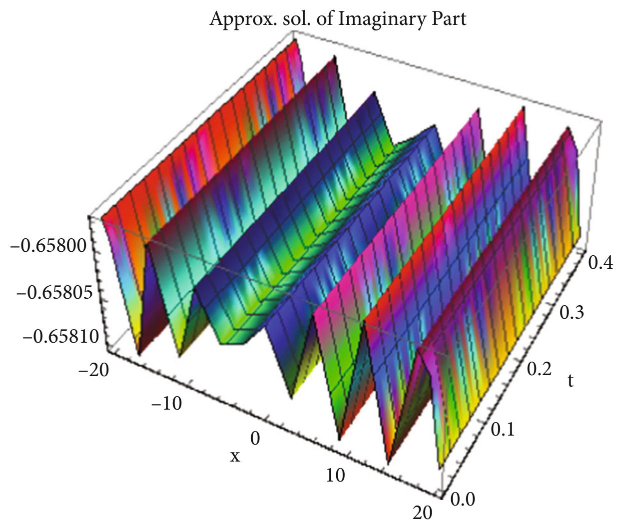

Figure 15: NTDM solution of the imaginary part of problem 3.

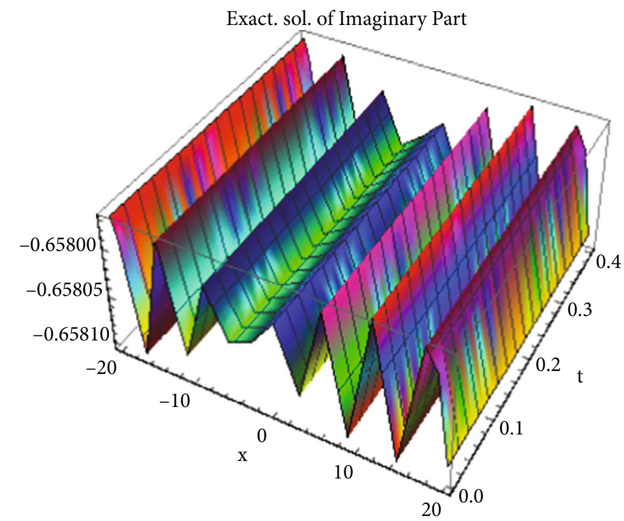

Figure 16: Exact solution of the imaginary part of problem 3.

After the application of inverse transform, we have

$$q(\chi, y, z, t) = \frac{q(\chi, y, z, 0)}{s} + \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[-\beta_1 q q_{\chi} - \beta_2 q_{\chi\chi\chi} - \beta_3 q_{\chi yy} - \beta_3 q_{\chi zz} - \zeta q_{\chi\chi\chi\chi\chi}\right]\right]. \tag{30}$$

Using the recursive relation and replacing the nonlinear term qq_x by Adomian polynomials, Equation (30) yields

$$q(\chi, y, z, t) = \frac{q(\chi, y, z, 0)}{s} + \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[-\beta_1 A_i - \beta_2 q_{\chi\chi\chi} - \beta_3 q_{\chi yy} - \beta_3 q_{\chi zz} - \zeta q_{\chi\chi\chi\chi\chi}\right]\right], i = 0, 1, 2, \cdots \tag{31}$$

We got the solution components as follows by using the NTDM concept

$$\left\{\begin{array}{l} q_0(\chi, y, z, t) = \mathbb{N}^{-}\left[\frac{q(\chi, y, z, 0)}{s}\right] \\ q_1(\chi, y, z, t) = \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[-\beta_1 A_0 - \beta_2 q_{0_{\chi\chi\chi}} - \beta_3 q_{0_{\chi yy}} - \beta_3 q_{0_{\chi zz}} - \zeta q_{0_{\chi\chi\chi\chi\chi}}\right]\right] \\ q_2(\chi, y, z, t) = \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[-\beta_1 A_1 - \beta_2 q_{1_{\chi\chi\chi}} - \beta_3 q_{1_{\chi yy}} - \beta_3 q_{1_{\chi zz}} - \zeta q_{1_{\chi\chi\chi\chi\chi}}\right]\right] \\ q_3(\chi, y, z, t) = \mathbb{N}^{-}\left[\frac{v^{\alpha}}{s^{\alpha}}\mathbb{N}^{+}\left[-\beta_1 A_2 - \beta_2 q_{2_{\chi\chi\chi}} - \beta_3 q_{2_{\chi yy}} - \beta_3 q_{2_{\chi zz}} - \zeta q_{2_{\chi\chi\chi\chi\chi}}\right]\right] \\ \vdots \end{array}\right\}, \tag{32}$$

where A_0,, A_1, and A_2 are the Adomian polynomials and can be calculated using Equation (15). By solving, we get the solution as follows:

$$\left\{\begin{array}{l} q_0(\chi, y, z, t) = e_0 - \frac{1680\zeta\rho^4}{\beta_1\left(-z\sqrt{-(\beta_2\rho^2/\beta_3) - \kappa^2} + \rho\chi + \kappa y + \phi\right)^4} \\ q_1(\chi, y, z, t) = -\frac{6720 e_0 \zeta \rho^5 t^{\alpha}}{\Gamma(\alpha+1)\left(-z\sqrt{-(\beta_2\rho^2/\beta_3) - \kappa^2} + \rho\chi + \kappa y + \phi\right)^5} \\ q_2(\chi, y, z, t) = -\frac{33600\beta_1 e_0^2 \zeta \rho^6 t^{2\alpha}}{\Gamma(2\alpha+1)\left(-z\sqrt{-(\beta_2\rho^2/\beta_3) - \kappa^2} + \rho\chi + \kappa y + \phi\right)^6} \\ q_3(\chi, y, z, t) = \frac{201600\beta_1 e_0^2 \zeta p^7 \Gamma(2\alpha+1) t^{3\alpha}\left(-\left(\beta_1 e_0\left(-z\sqrt{-(\beta_2\rho^2/\beta_3) - \kappa^2} + \rho\chi + \kappa y + \phi\right)^4\right)/\left(\Gamma(2\alpha+1) + \left(1120\zeta\rho^4/\Gamma(\alpha+1)^2\right) - \left(2240\zeta\rho^4/\Gamma(2\alpha+1)\right)\right)\right)}{\Gamma(3\alpha+1)\left(-z\sqrt{-(\beta_2\rho^2/\beta_3) - \kappa^2} + \rho\chi + \kappa y + \phi\right)^{11}} \end{array}\right\}. \tag{33}$$

Table 1: Real part solution comparison for fractional values of α for problem 1 at $t = 0.01$.

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$	Exact	Abs. error NTDM	Abs. error [27]
-15	3.0034	3.0036	3.00369	3.00371	3.00371	2.007727×10^{-12}	2.007727×10^{-12}
-10	3.00812	3.00805	3.00798	3.00796	3.00796	1.639577×10^{-12}	1.639577×10^{-12}
-5	2.98107	2.97965	2.97917	2.97907	2.97907	5.8479×10^{-11}	5.8479×10^{-11}
0	2.99902	3.00045	3.00097	3.00108	3.00108	4.721068×10^{-11}	4.721068×10^{-11}
5	3.0066	3.00634	3.00623	3.00621	3.00621	6.424195×10^{-12}	6.424195×10^{-12}
10	3.00164	3.00152	3.00148	3.00147	3.00147	5.728751×10^{-14}	5.728751×10^{-14}
15	3.00023	3.0002	3.0002	3.00019	3.00019	1.092459×10^{-13}	1.092459×10^{-13}

Table 2: Imaginary part solution comparison for fractional values of α for problem 1 at $t = 0.01$.

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$	Exact	Abs. error NTDM	Abs. error [38]
-15	0.00151571	0.00146344	0.00143688	0.00143011	0.00143011	1.789912×10^{-12}	1.789912×10^{-12}
-10	-0.00722543	-0.00801857	-0.00835063	-0.0084304	-0.0084304	2.285489×10^{-11}	2.285489×10^{-11}
-5	-0.0123648	-0.0108878	-0.0102153	-0.0100505	-0.0100505	6.996747×10^{-11}	6.996747×10^{-11}
0	0.0175886	0.0171401	0.016905	0.0168458	0.0168458	6.948674×10^{-12}	6.948674×10^{-12}
5	0.000776644	0.000419639	0.000295525	0.000267266	0.000267266	4.853387×10^{-13}	4.853387×10^{-13}
10	-0.00160753	-0.00158678	-0.0015772	-0.00157485	-0.00157485	7.746282×10^{-13}	7.746282×10^{-13}
15	-0.000870554	-0.00084329	-0.000832837	-0.000830384	-0.000830384	5.348467×10^{-14}	5.348467×10^{-14}

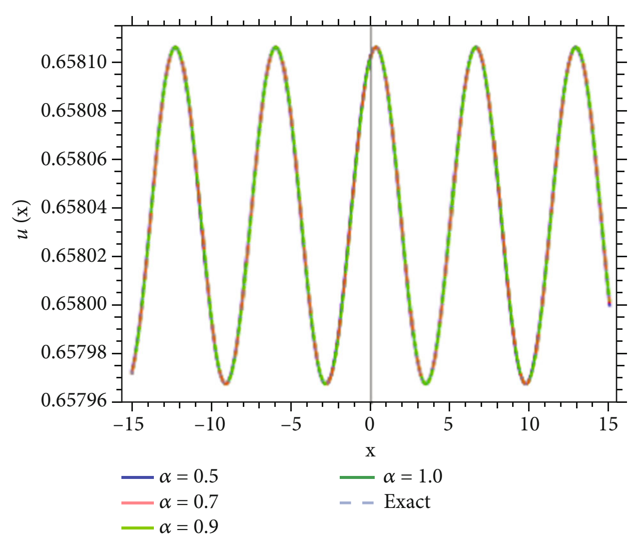

Figure 17: Real part solution comparison for fractional values of α for problem 3.

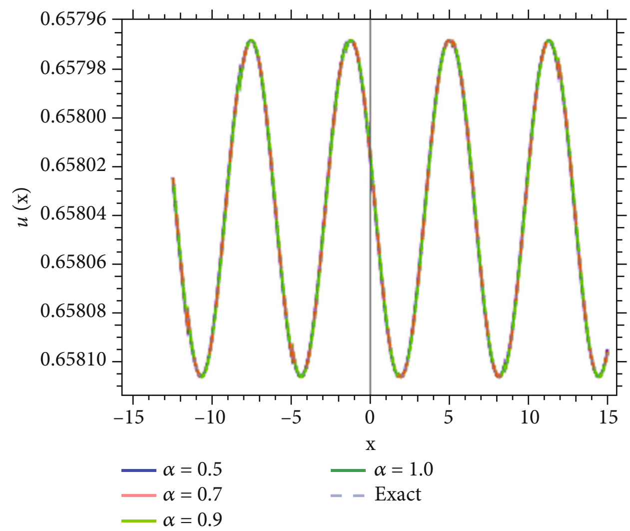

Figure 18: Imaginary part solution comparison for different fractional values of α for problem 3.

Table 3: Real part solution comparison for fractional values of α for problem 2 at $t = 0.01$.

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$	Exact	Abs. error
-15	2.9874	2.98657	2.98621	2.98612	2.98612	1.55851×10^{-9}
-10	2.97838	2.97696	2.97638	2.97624	2.97624	9.833575×10^{-9}
-5	2.96771	2.96659	2.96637	2.96633	2.96633	5.097527×10^{-8}
0	2.97778	2.98204	2.984	2.98448	2.98448	8.531288×10^{-8}
5	3.02044	3.02508	3.02652	3.02683	3.02683	4.077737×10^{-9}
10	3.03291	3.03204	3.03161	3.0315	3.0315	3.67115×10^{-8}
15	3.02214	3.0207	3.02016	3.02003	3.02003	2.767871×10^{-9}

Table 4: Imaginary part solution comparison for fractional values of α for problem 2 at $t = 0.01$.

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$	Exact	Abs. error
-15	0.00640835	0.00637269	0.00632898	0.0063165	0.0063165	5.566582×10^{-9}
-10	0.00329588	0.00252435	0.00211105	0.00200535	0.00200536	1.470755×10^{-8}
-5	-0.0117248	-0.0148465	-0.0162419	-0.0165841	-0.0165841	2.044629×10^{-8}
0	-0.0429444	-0.0469602	-0.0481431	-0.0483942	-0.0483943	7.870128×10^{-8}
5	-0.0448139	-0.0411819	-0.0396028	-0.0392184	-0.0392183	9.6700550×10^{-8}
10	-0.0128468	-0.0095154	-0.00836997	-0.00811094	-0.00811095	6.485631×10^{-9}
15	0.00309416	0.00400363	0.00429329	0.00435689	0.00435688	1.14261×10^{-8}

Adding the components, we obtain the 3rd order solution as follows:

$$q(\chi,y,z,t)=q_0(\chi,y,z,t)+q_1(\chi,y,z,t)+q_2(\chi,y,z,t)+q_3(\chi,y,z,t)$$
$$\left\{\begin{array}{c} -\dfrac{33600\beta_1 e_0^2\zeta p^6 t^{2\alpha}}{\Gamma(2\alpha+1)\left(-z\sqrt{-(\beta_2p^2/\beta_3)-\kappa^2}+p\chi+\kappa y+\phi\right)^6-\left(6720e_0\zeta p^5t^\alpha/\left(\Gamma(\alpha+1)\left(-z\sqrt{-(\beta_2p^2/\beta_3)-\kappa^2}+p\chi+\kappa y+\phi\right)^5\right)\right)} \\ +\dfrac{201600\beta_1 e_0^2\zeta p^7\Gamma(2\alpha+1)t^{3\alpha}\left(\begin{array}{c}-\left(\left(\beta_1e_0\left(-z\sqrt{-(\beta_2p^2/\beta_3)-\kappa^2}+p\chi+\kappa y+\phi\right)^4\right)/\Gamma(2\alpha+1)\right)\\ +\left(1120\zeta p^4/\Gamma(\alpha+1)^2\right)-\left(2240\zeta p^4/\Gamma(2\alpha+1)\right)\end{array}\right)}{\Gamma(3\alpha+1)\left(-z\sqrt{-(\beta_2p^2/\beta_3)-\kappa^2}+p\chi+\kappa y+\phi\right)^{11}+e_0-\left(1680\zeta p^4/\left(\beta_1\left(-z\sqrt{-(\beta_2p^2/\beta_3)-\kappa^2}+p\chi+\kappa y+\phi\right)^4\right)\right)} \end{array}\right\}. \tag{34}$$

5.2. Problem 2. Consider the following $(3+1)$ dimension GPZK equation [23]

$$D_t^\alpha q+\beta_1q^2q_\chi+\beta_2q_{\chi\chi\chi}+\beta_3q_{\chi yy}+\beta_3q_{\chi zz}+\zeta q_{\chi\chi\chi\chi\chi}=0,\ 0<\alpha\le 1,\ t>0, \tag{35}$$

and initial condition as follows:

$$q(\chi,y,z,0)=e_0+\frac{6\sqrt{10}i\sqrt{\zeta}\rho^2}{\sqrt{\beta_1}\left(-\left(\sqrt{\sqrt{10}(-i)\sqrt{\beta_1}e_0\sqrt{\zeta}\rho^2-\beta_2\rho^2-\beta_3\kappa^2}/\sqrt{\beta_3}\right)z+\rho\chi+\kappa y+\phi\right)^2}. \tag{36}$$

The exact solution is given as [24]

$$q(\chi,y,z,t)=e_0+\frac{6\sqrt{10}i\sqrt{\zeta}\rho^2}{\sqrt{\beta_1}\left(-\left(\sqrt{\sqrt{10}(-i)\sqrt{\beta_1}e_0\sqrt{\zeta}\rho^2-\beta_2\rho^2-\beta_3\kappa^2}/\sqrt{\beta_3}\right)z-\beta_1e_0^2\rho t+\rho\chi+\kappa y+\phi\right)^2}. \tag{37}$$

By using basics concepts of NTDM, the solution components obtained as follows:

$$\left\{\begin{array}{l} q_0(\chi,y,z,t)=e_0+\dfrac{6\sqrt{10}i\sqrt{\zeta}\rho^2}{\sqrt{\beta_1}\left(-\left(\sqrt{\sqrt{10}(-i)\sqrt{\beta_1}e_0\sqrt{\zeta}\rho^2-\beta_2\rho^2-\beta_3\kappa^2}/\sqrt{\beta_3}\right)z+\rho\chi+\kappa y+\phi\right)^2} \\ q_1(\chi,y,z,t)=\dfrac{12i\sqrt{10}\sqrt{\beta_1}e_0^2\sqrt{\zeta}\rho^3t^\alpha}{\Gamma(\alpha+1)\left(-\left(\sqrt{\sqrt{10}(-i)\sqrt{\beta_1}e_0\sqrt{\zeta}\rho^2-\beta_2\rho^2-\beta_3\kappa^2}/\sqrt{\beta_3}\right)z+\rho\chi+\kappa y+\phi\right)^3} \\ q_2(\chi,y,z,t)=\dfrac{36i\sqrt{10}\beta_1^{3/2}e_0^4\sqrt{\zeta}\rho^4t^{2\alpha}}{\Gamma(2\alpha+1)\left(-\left(z\sqrt{\sqrt{10}(-i)\sqrt{\beta_1}e_0\sqrt{\zeta}\rho^2-\beta_2\rho^2-\beta_3\kappa^2}/\sqrt{\beta_3}\right)+\rho\chi+\kappa y+\phi\right)^4} \end{array}\right\}. \tag{38}$$

Table 5: Real part solution comparison for fractional values of α for problem 3 at $t = 0.01$.

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$	Exact	Abs. error
-15	0.657972	0.6579720	0.6579720	0.6579720	0.6579720	1.354472×10^{-14}
-10	0.657995	0.6579950	0.657996	0.6579960	0.657996	3.119727×10^{-14}
-5	0.658078	0.658078	0.658078	0.658079	0.658079	3.108624×10^{-14}
0	0.658102	0.658102	0.658102	0.658102	0.658102	1.354472×10^{-14}
5	0.658033	0.658033	0.658032	0.658032	0.658032	3.885781×10^{-14}
10	0.65797	0.657969	0.657969	0.657969	0.657969	8.437695×10^{-15}
15	0.658002	0.658003	0.658003	0.658003	0.658003	3.397282×10^{-14}

Table 6: Imaginary part solution comparison for fractional values of α for problem 3 at $t = 0.01$.

x	$\alpha = 0.5$	$\alpha = 0.7$	$\alpha = 0.9$	$\alpha = 1.0$	Exact	Abs. error
-15	-0.658012	-0.6580130	-0.658013	-0.658013	-0.658013	$3.6526340 \times 10^{-14}$
-10	-0.658092	-0.6580920	-0.658093	-0.658093	-0.658093	2.331468×10^{-14}
-5	-0.6580930	-0.6580930	-0.6580930	-0.6580930	-0.658093	$2.3425710 \times 10^{-14}$
0	-0.658014	-0.658013	-0.658013	-0.658013	-0.6580130	3.663736×10^{-14}
5	-0.657968	-0.657968	-0.6579680	-0.657968	-0.657968	2.775558×10^{-15}
10	-0.6580210	-0.6580220	-0.658022	-0.6580220	-0.658022	3.808065×10^{-14}
15	-0.658097	-0.658097	-0.658098	-0.658098	-0.658098	1.876277×10^{-14}

Adding the components, we get the solution as follows:

$$q(\chi, y, z, t) = q_0(\chi, y, z, t) + q_1(\chi, y, z, t) + q_2(\chi, y, z, t) = \left\{ \begin{array}{l} e_0 + \dfrac{6\sqrt{10} i \sqrt{\zeta} \rho^2}{\sqrt{\beta_1}\left(-\left(\sqrt{\sqrt{10}(-i)\sqrt{\beta_1} e_0 \sqrt{\zeta}\rho^2 - \beta_2\rho^2 - \beta_3\kappa^2 / \sqrt{\beta_3}}\right) z + \rho\chi + \kappa y + \phi\right)^2} \\ + \dfrac{12 i \sqrt{10}\sqrt{\beta_1} e_0^2 \sqrt{\zeta}\rho^3 t^\alpha}{\Gamma(\alpha+1)\left(-\left(\sqrt{\sqrt{10}(-i)\sqrt{\beta_1} e_0 \sqrt{\zeta}\rho^2 - \beta_2\rho^2 - \beta_3\kappa^2 / \sqrt{\beta_3}}\right) z + \rho\chi + \kappa y + \phi\right)^3} \\ + \dfrac{36 i \sqrt{10}\beta_1^{3/2} e_0^4 \sqrt{\zeta}\rho^4 t^{2\alpha}}{\Gamma(2\alpha+1)\left(-\left(z\sqrt{\sqrt{10}(-i)\sqrt{\beta_1} e_0 \sqrt{\zeta}\rho^2 - \beta_2\rho^2 - \beta_3\kappa^2 / \sqrt{\beta_3}}\right) + \rho\chi + \kappa y + \phi\right)^4} \end{array} \right\}. \tag{39}$$

5.3. Problem 3. Consider the following $(3+1)$ dimension GPZK equation [23]

$$D_t^\alpha q + \beta_1 q^4 q_\chi + \beta_2 q_{\chi\chi\chi} + \beta_3 q_{\chi yy} + \beta_3 q_{\chi zz} + \zeta q_{\chi\chi\chi\chi\chi} = 0, \tag{40}$$

and initial condition as follows:

$$q(\chi, y, z, 0) = \frac{\left(2^{3/4}\sqrt[4]{-15}\sqrt[4]{\zeta}\rho\right)}{\sqrt[4]{\beta_1}} \tan\left(\rho\chi + \kappa y - \frac{\sqrt{-20\zeta\rho^4 - \beta_2\rho^2 - \beta_3\kappa^2}}{\sqrt{\beta_3}} z\right). \tag{41}$$

The exact solution is given as [18]

$$q(\chi, y, z, t) = \frac{\left(2^{3/4}\sqrt[4]{-15}\sqrt[4]{\zeta}\rho\right)}{\sqrt[4]{\beta_1}} \tan \cdot \left(\rho\chi + \kappa y - \frac{\sqrt{-20\zeta\rho^4 - \beta_2\rho^2 - \beta_3\kappa^2}}{\sqrt{\beta_3}} z + 24\rho^5\zeta t\right). \tag{42}$$

Using NTDM concept, the solution components are obtained as follows:

$$\left\{\begin{array}{c} q_0(\chi,y,z,t)=\dfrac{\left(2^{3/4}\sqrt[4]{-15}\sqrt[4]{\zeta}\rho\right)}{\sqrt[4]{\beta_1}}\tan\left(\rho\chi+\kappa y-\dfrac{\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}}{\sqrt{\beta_3}}z\right), q_1(\chi,y,z,t)=\dfrac{(24+24i)\sqrt[4]{30}\zeta^{5/4}\rho^6t^\alpha\sec^2\left(-\left(z\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}/\sqrt{\beta_3}\right)+\rho\chi+\kappa y\right)}{\sqrt[4]{\beta_1}\Gamma(\alpha+1)} \\ q_2(\chi,y,z,t)=\dfrac{(1152+1152i)\sqrt[4]{30}\zeta^{9/4}\rho^{11}t^{2\alpha}}{\sqrt[4]{\beta_1}\Gamma(2\alpha+1)}\tan\left(-\dfrac{z\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}}{\sqrt{\beta_3}}+\rho\chi+\kappa y\right)\times\sec^2\left(-\dfrac{z\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}}{\sqrt{\beta_3}}+\rho\chi+\kappa y\right) \end{array}\right\}. \tag{43}$$

Adding the components, we get the solution as follows:

$$q(\chi,y,z,t)=q_0(\chi,y,z,t)+q_1(\chi,y,z,t)+q_2(\chi,y,z,t)=\left\{\begin{array}{c} \dfrac{\left(2^{3/4}\sqrt[4]{-15}\sqrt[4]{\zeta}\rho\right)}{\sqrt[4]{\beta_1}}\tan\left(\rho\chi+\kappa y-\dfrac{\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}}{\sqrt{\beta_3}}z\right)+ \\ +\dfrac{(24+24i)\sqrt[4]{30}\zeta^{5/4}\rho^6t^\alpha\sec^2\left(-\left(z\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}/\sqrt{\beta_3}\right)+\rho\chi+\kappa y\right)}{\sqrt[4]{\beta_1}\Gamma(\alpha+1)} \\ +\dfrac{(1152+1152i)\sqrt[4]{30}\zeta^{9/4}\rho^{11}t^{2\alpha}}{\sqrt[4]{\beta_1}\Gamma(2\alpha+1)}\tan\left(-\dfrac{z\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}}{\sqrt{\beta_3}}+\rho\chi+\kappa y\right)\times \\ \sec^2\left(-\dfrac{z\sqrt{-20\zeta\rho^4-\beta_2\rho^2-\beta_3\kappa^2}}{\sqrt{\beta_3}}+\rho\chi+\kappa y\right) \end{array}\right\}. \tag{44}$$

6. Results and Discussion

This section presents the application of NTDM for solving fractional-order nonlinear GPZK equations. The values for the arbitrary constants have been taken as $\beta_1=1$, $\beta_2=2$, $\beta_3=0.1$, $y=2$, $z=2$, $\zeta=0.1$, $e_o=3$, $\rho=0.5$, $\boxtimes=0.5$, and $\phi=1$ for problem 1-3. Problem 1 is approximated up to 3rd order. Problem 2 and problem 3 are approximated up to 2nd order by the NTDM algorithm. The approximate and exact real part of the solution is shown by Figures 1 and 2, respectively, by 3D plots for problem 1. The 3rd order approximate and exact solution of the imaginary part of problem 1 has been displayed by Figures 3 and 4, respectively. The solution obtained by NTDM has been compared by plots in Figures 5 and 6 for fractional values of α. Figures 7 and 8 display the 3D plots of the 2nd order real part solution and exact solution, respectively, for problem 2. Figures 9 and 10 show the 2nd order imaginary part solution and exact solution, respectively, for problem 2. The solution gained by NTDM has been compared by 2D plots in Figures 11 and 12 for fractional values of α.

Similarly, Figures 13 and 14 display the real part solution, and Figures 15 and 16 display the imaginary part solution for problem 3. In Tables 1 and 2, the absolute error is compared with the q-HATM solution, which shows the convergence of NTDM. Figure 17 shows the real part solution comparison for fractional values of α for problem 3. Figure 18 shows the imaginary part solution comparison for different fractional values of α for problem 3. In Tables 1 and 2, the absolute error is compared with the q-HATM solution, which shows the convergence of NTDM. Similarly, Tables 3 and 4 are compared for problem 2, while Tables 5 and 6 show the comparison for problem 3.

The above conversation summarizes that NTDM is appropriate for solving DEs of fractional order.

7. Conclusion

The GPZK equation of fractional order has been solved by NTDM in the current study. Three nonlinear cases have been shown convergent by comparing the results with the exact solution and q-HATM solution. The effectiveness of NTDM has been shown by showing numerical results. The plots for different numerical values of α confirm the convergence of NTDM as α tend 1 the NTDM solution overlaps the exact solution. This discussion summarizes that the NTDM is suitable for approximating the complex nonlinear PDEs and ODEs of integer and fractional order.

Data Availability

No data available for this study.

Conflicts of Interest

The authors have declared no conflict of interest.

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