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Research Article

New Iterative Method for Solving a Coupled System of Fractional-Order Drinfeld–Sokolov–Wilson (FDSW) and Fractional Shallow Water (FSW) Equations

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In this study, the new iterative method has been applied to a coupled system of fractional-order Drinfeld–Sokolov–Wilson (FDSW) and fractional shallow water (FSW) equations. The fractional-order derivatives are taken in the Caputo sense whose order is between 0 and 1. The suggested method is capable to handle the FDEs without any transformation and discretization. The obtained results have been compared with the exact solution and with the q-homotopy analysis transform method. The outcomes show the efficiency and effectiveness of NIM by comparing through tables and graphs.

1. Introduction

Differential equations (DEs) can be used to model the majority of physical occurrences on the planet. The DEs are divided into many categories. They may be in the form of ordinary differential equations (ODEs) or partial differential equations (PDEs). Due to significant advancements in mathematics, a new discipline known as fractional calculus was introduced which has new concepts and operations for handling derivatives and integrations. Fractional calculus deals with the DEs of noninteger order known as fractional-order differential equations (FDEs). Linear differential equations model the simple phenomenon while nonlinear equations are used in a variety of research and engineering applications including plasma physics, hydrodynamics, fluid dynamics, solid-state physics, acoustics, and optical fibers. In many fields of engineering and biosciences, the DEs occur in the form of coupled systems. The solution of a differential equation may depend on the linearity of the DE. The coupled systems in linear cases may be solved using basic analytical methods. However, due to the higher degree of nonlinearity, solving nonlinear differential equations by simple methods is not always practicable. As a result of the complexity for obtaining a solution of nonlinear DEs, researchers initiated some new approaches for approximating the solution of nonlinear DEs. They may be perturbation methods [1, 2], numerical methods [3, 4], iterative methods [5, 6], etc. Sometimes these techniques apply some transformation to reduce the equations into more simple equations or even a system of equations while some other techniques offer the solution in the form of series that converges to the exact solution [7, 8]. Besides, some other techniques which employ a trial function in an iterative scheme converge quickly. The concept of homotopy from topology and conventional perturbation methods introduced new methods such as the optimal homotopy asymptotic method, homotopy perturbation method, homotopy analysis method suggest a general analytic solution [9–11]. Therefore, these techniques are independent of the availability of a small parameter. On the other hand, a relatively new method known as the new iterative method (NIM) [12, 13] is a modified form of the Adomian decomposition method (ADM) [14] in which the Adomian polynomials are replaced by DJ polynomials in the nonlinear terms.

In the present work, coupled system of fractional-order differential equations will be solved using NIM. FDEs have been solved using NIM with the help of fractional derivative and integral operators [13, 15–17]. In this paper, we will find the solution of the Fractional Drinfeld–Sokolov–Wilson (FDSW) coupled system and fractional shallow water (FSW) coupled system. The fractional-order DSW equation is used to add memory effects and genetic consequences into the system, and these features let us grasp important physical properties of complex issues. The fractional DSW coupled system is of the form [18]

$$D_t^{\beta}\varphi(x,t) + 3\psi(x,t)\psi(x,t)_x = 0, \tag{1}$$

$$D_t^{\beta}\psi(x,t) + 2\psi(x,t)_{xxx} + 2\varphi(x,t)\psi(x,t)_x + \varphi(x,t)_x\psi(x,t) = 0, \tag{2}$$

where $0 < \beta \le 1$ is the fractional order of derivative of the system and is defined with Caputo's fractional derivative operator.

The second coupled system presented in this paper is the fractional-order shallow water (FSW) equation which describes a thin layer of fluid in hydrostatic equilibrium with a constant density. The equivalent wave motion is the coupled SW equation. The time-fractional SW coupled system is of the form [19]

$$D_t^{\beta}\varphi(x,t) = -\psi(x,t)\varphi(x,t)_x - \varphi(x,t)\psi(x,t)_x, \tag{3}$$

$$D_t^{\beta}\psi(x,t) = -\psi(x,t)\psi(x,t)_x - \varphi(x,t)_x, \tag{4}$$

where $\varphi(x,t)$ and $\psi(x,t)$ denote the free surface and the horizontal velocity component.

Many researchers pay attention to fractional DSW and fractional SW equations by using different approaches [20–23].

The remaining article is planned as follows: Section 2 contains some basics from fractional calculus relevant to our study. The essential concepts of the NIM are presented in Section 3. The proposed approach is used to solve the fractional DSW and fractional SW coupled systems in Section 4. The numerical results by NIM are presented in Section 5 with the help of graphs and Tables 1–4. The final paragraph contains the conclusion.

2. Fractional Calculus

We will present some essential definitions from fractional calculus that are relevant to our work [24].

Definition 1. The fractional integral operator in Riemann-Liouville (R-L) sense is defined as

$$I_t^{\beta} = \begin{cases} \dfrac{1}{\Gamma(\beta)}\displaystyle\int_0^t (t-\gamma)^{\beta-1} f(\gamma)\,d\gamma & \text{if } \beta > 0,\ t > 0, \\ f(\gamma) & \text{if } \beta = 0, \end{cases} \tag{5}$$

where Γ is the gamma function.

Definition 2. The Caputo fractional derivative operator of order β is described as follows:

$$D_t^{\beta}\varphi(t) = \frac{1}{\Gamma(n-\beta)}\left[\int_0^t (t-\gamma)^{n-\beta-1} f^n(\gamma)\,d\gamma\right] \text{ if } n-1 < \beta \le n,\ n \in N. \tag{6}$$

Definition 3. Relationship of the Caputo's fractional derivative and the R-L integral is defined as follows:

If $m-1 < \beta \le m$, $m \in N$, then

$$I_t^{\beta}\left[D_t^{\beta}\varphi(t)\right] = \varphi(t) + \sum_{j=0}^{m-1}\varphi^{(j)}(\gamma)\frac{(t-\gamma)^j}{\Gamma(j+1)},\ t > 0,\ \varphi \in C_{\beta}^{\mu},\ \mu \ge -1. \tag{7}$$

3. New Iterative Method

Assume a nonlinear equation of the form [12]

$$\psi\left(\underline{\chi},t\right) = h\left(\underline{\chi},t\right) + L\psi\left(\underline{\chi},t\right) + \aleph\psi\left(\underline{\chi},t\right), \tag{8}$$

where $\underline{\chi} = \chi_1, \chi_2, \cdots, \chi_n$, $h(\underline{\chi},t)$, L, and $\aleph$ indicate the source term, linear operator, and nonlinear operator, respectively. The solution of Equation (8), according to NIM, can be expanded as follows:

$$\psi\left(\underline{\chi},t\right) = \sum_{m=0}^{\infty}\psi_m\left(\underline{\chi},t\right). \tag{9}$$

Due to the linearity of L, $\psi(\underline{\chi},t)$ is expressed as

$$L\left(\sum_{m=0}^{\infty}\psi_m\left(\underline{\chi},t\right)\right) = \sum_{m=0}^{\infty}L\left(\psi_m\left(\underline{\chi},t\right)\right). \tag{10}$$

The nonlinear operator N presented by Daftardar-Geiji and Jaffari is expressed as

$$\aleph\left(\sum_{m=0}^{\infty}\psi_m\left(\underline{\chi},t\right)\right) = \aleph\left(\psi_0\left(\underline{\chi},t\right)\right) + \sum_{m=1}^{\infty}\left\{\aleph\left(\sum_{j=0}^{i}\psi_j\left(\underline{\chi},t\right)\right) - \aleph\left(\sum_{j=0}^{m-1}\psi_j\left(\underline{\chi},t\right)\right)\right\}. \tag{11}$$

Table 1: The absolute error of the 2[nd]-order NIM and 2[nd]-order q-HATM solution for $\varphi(x, t)$ the FDSW coupled system.

x	t	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 1.0$	Exact sol.	NIM error	Error [18]
2.5	0.025	0.0880367	0.0880367	0.0880367	0.0880367	0.0880429	6.13333×10^{-6}	1.23436×10^{-5}
	0.05	0.0971	0.0971	0.0971	0.0971	0.0971513	5.13081×10^{-5}	1.00990×10^{-4}
	0.075	0.107004	0.107004	0.107004	0.107004	0.107185	1.809556×10^{-4}	3.48632×10^{-4}
	0.1	0.117785	0.117785	0.117785	0.117785	0.118233	4.479405×10^{-4}	8.45395×10^{-4}
5	0.025	0.0880367	0.0880367	0.0880367	0.0880367	0.0880429	6.13333×10^{-6}	9.30466×10^{-8}
	0.05	0.0971	0.0971	0.0971	0.0971	0.0971513	5.13081×10^{-5}	7.63637×10^{-7}
	0.075	0.107004	0.107004	0.107004	0.107004	0.107185	1.809556×10^{-4}	2.64499×10^{-6}
	0.1	0.117785	0.117785	0.117785	0.117785	0.118233	4.479405×10^{-4}	6.43687×10^{-6}
7.5	0.025	4.05657×10^{-6}	4.05657×10^{-6}	4.05657×10^{-6}	4.05657×10^{-6}	4.05689×10^{-6}	3.215065×10^{-10}	6.27408×10^{-10}
	0.05	4.48085×10^{-6}	4.48085×10^{-6}	4.48085×10^{-6}	4.48085×10^{-6}	4.48356×10^{-6}	2.702048×10^{-9}	5.14926×10^{-9}
	0.075	4.94552×10^{-6}	4.94552×10^{-6}	4.94552×10^{-6}	4.94552×10^{-6}	4.95510×10^{-6}	9.576419×10^{-9}	1.78358×10^{-8}
	0.1	5.45240×10^{-6}	5.4524×10^{-6}	5.4524×10^{-6}	5.45240×10^{-6}	5.47623×10^{-6}	2.382852×10^{-8}	4.34062×10^{-8}
10	0.025	5.37770×10^{-8}	3.49043×10^{-8}	2.87695×10^{-8}	2.7333×10^{-8}	2.73351×10^{-8}	2.166307×10^{-12}	4.22746×10^{-12}
	0.05	7.37214×10^{-8}	4.34152×10^{-8}	3.28010×10^{-8}	3.01918×10^{-8}	3.02100×10^{-8}	1.820637×10^{-11}	3.46956×10^{-11}
	0.075	9.27727×10^{-8}	5.2151×10^{-8}	3.71170×10^{-8}	3.33227×10^{-8}	3.33872×10^{-8}	6.452584×10^{-11}	1.20177×10^{-10}
	0.1	1.11597×10^{-7}	6.12942×10^{-8}	4.17686×10^{-8}	3.67380×10^{-8}	3.68986×10^{-8}	1.605564×10^{-10}	2.92470×10^{-10}

Table 2: The absolute error of the 2[nd]-order NIM and 2[nd]-order q-HATM solution for $\psi(x, t)$ the FDSW coupled system.

x	t	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 1.0$	Exact sol.	NIM error	Error [18]
2.5	0.025	0.474805	0.385607	0.351313	0.342621	0.342623	1.595560×10^{-6}	5.67484×10^{-6}
	0.05	0.558666	0.429127	0.374743	0.359897	0.35991	1.308247×10^{-5}	4.57167×10^{-5}
	0.075	0.63306	0.470474	0.398436	0.377993	0.378038	4.519525×10^{-5}	1.55336×10^{-4}
	0.1	0.703019	0.511305	0.422738	0.396935	0.397044	1.095138×10^{-4}	3.70588×10^{-4}
5	0.025	0.0392622	0.0319312	0.0290606	0.0283316	0.0283322	5.654244×10^{-7}	5.67870×10^{-7}
	0.05	0.0459425	0.0355452	0.0310238	0.0297801	0.0297847	4.581037×10^{-6}	4.60060×10^{-6}
	0.075	0.051697	0.038937	0.0330033	0.0312959	0.0313116	1.565947×10^{-5}	1.57255×10^{-5}
	0.1	0.0569726	0.0422418	0.0350246	0.0328792	0.0329168	3.759878×10^{-5}	3.77553×10^{-5}
7.5	0.025	0.00322299	0.00262123	0.00238556	0.00232572	0.00232577	4.667055×10^{-8}	4.66719×10^{-8}
	0.05	0.00377121	0.0029179	0.00254673	0.00244463	0.00244501	3.781143×10^{-7}	3.78125×10^{-7}
	0.075	0.00424335	0.00319632	0.00270924	0.00256908	0.00257037	1.292494×10^{-6}	1.29253×10^{-6}
	0.1	0.00467611	0.00346756	0.00287517	0.00269905	0.00270215	3.103259×10^{-6}	3.10335×10^{-6}
10	0.025	0.000264559	0.000215164	0.000195819	0.000190907	0.000190911	3.831095×10^{-9}	3.83110×10^{-9}
	0.05	0.00030956	0.000239516	0.000209049	0.000200668	0.000200699	3.103867×10^{-8}	3.10387×10^{-8}
	0.075	0.000348315	0.00026237	0.000222388	0.000210883	0.000210989	1.060983×10^{-7}	1.06098×10^{-7}
	0.1	0.000383838	0.000284635	0.000236009	0.000221552	0.000221806	2.547404×10^{-7}	2.54740×10^{-7}

Table 3: Numerical comparison of different values of β and absolute error of the 2^{nd}-order NIM solution of $\varphi(x, t)$ for the FSW system.

x	t	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 1.0$	Exact sol.	NIM error
2.5	0.025	0.380492	0.298803	0.269985	0.262974	0.262985	1.092004×10^{-5}
	0.05	0.461865	0.337071	0.289279	0.276917	0.277008	9.164358×10^{-5}
	0.075	0.536682	0.374877	0.309367	0.291859	0.292184	3.24701×10^{-4}
	0.1	0.608701	0.413306	0.330496	0.307833	0.308642	8.08642×10^{-4}
5	0.025	2.70572	2.12482	1.91989	1.87004	1.87011	7.765359×10^{-5}
	0.05	3.28437	2.39695	2.05709	1.96919	1.96984	6.516877×10^{-4}
	0.075	3.81641	2.66579	2.19994	2.07544	2.07775	2.308985×10^{-3}
	0.1	4.32854	2.93907	2.3502	2.18904	2.19479	5.750343×10^{-3}
7.5	0.025	7.1448	5.61085	5.06972	4.93807	4.93827	2.05054×10^{-4}
	0.05	8.6728	6.32945	5.43201	5.19988	5.2016	1.720863×10^{-3}
	0.075	10.0777	7.03935	5.80922	5.48047	5.48657	6.097163×10^{-3}
	0.1	11.43	7.76097	6.20598	5.78043	5.79561	1.51845×10^{-2}
10	0.025	13.6977	10.7569	9.71947	9.46706	9.46746	3.931213×10^{-4}
	0.05	16.6271	12.1346	10.414	9.969	9.9723	3.299169×10^{-3}
	0.075	19.3206	13.4956	11.1372	10.5069	10.5186	1.168923×10^{-2}
	0.1	21.9132	14.879	11.8979	11.082	11.1111	2.911111×10^{-2}

Table 4: Numerical comparison of different values of β and absolute error of the 2^{nd}-order NIM solution of $\psi(x, t)$ for the FSW system.

x	t	$\beta = 0.5$	$\beta = 0.7$	$\beta = 0.9$	$\beta = 1.0$	Exact sol.	NIM error
2.5	0.025	-1.23094	-1.09261	-1.03916	-1.02563	-1.02564	1.255342×10^{-5}
	0.05	-1.35945	-1.16031	-1.07566	-1.05253	-1.05263	1.038012×10^{-4}
	0.075	-1.47213	-1.22437	-1.11257	-1.08072	-1.08108	3.623311×10^{-4}
	0.1	-1.57702	-1.28731	-1.15038	-1.11022	-1.11111	8.888889×10^{-4}
5	0.025	-3.2825	-2.91362	-2.7711	-2.73501	-2.73504	3.347578×10^{-5}
	0.05	-3.62521	-3.09415	-2.86844	-2.80674	-2.80702	2.768031×10^{-4}
	0.075	-3.92569	-3.26498	-2.96685	-2.88192	-2.88288	9.662162×10^{-4}
	0.1	-4.20538	-3.43283	-3.06769	-2.96059	-2.96296	2.37037×10^{-3}
7.5	0.025	-5.33406	-4.73464	-4.50304	-4.44439	-4.44444	5.439815×10^{-5}
	0.05	-5.89096	-5.028	-4.66121	-4.56095	-4.5614	4.498051×10^{-4}
	0.075	-6.37925	-5.30559	-4.82113	-4.68311	-4.68468	1.570101×10^{-3}
	0.1	-6.83374	-5.57835	-4.98499	-4.81096	-4.81481	3.851852×10^{-3}
10	0.025	-7.38562	-6.55566	-6.23498	-6.15377	-6.15385	7.532051×10^{-5}
	0.05	-8.15671	-6.96184	-6.45399	-6.31517	-6.31579	6.22807×10^{-4}
	0.075	-8.83281	-7.3462	-6.67542	-6.48431	-6.48649	2.173986×10^{-3}
	0.1	-9.4621	-7.72387	-6.9023	-6.66133	-6.66667	5.333333×10^{-3}

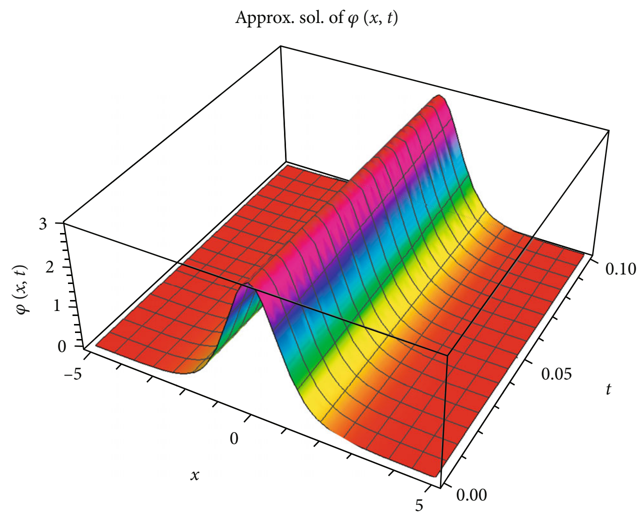

Figure 1: 2nd-order NIM solution of $\varphi(x, t)$ FDSW system.

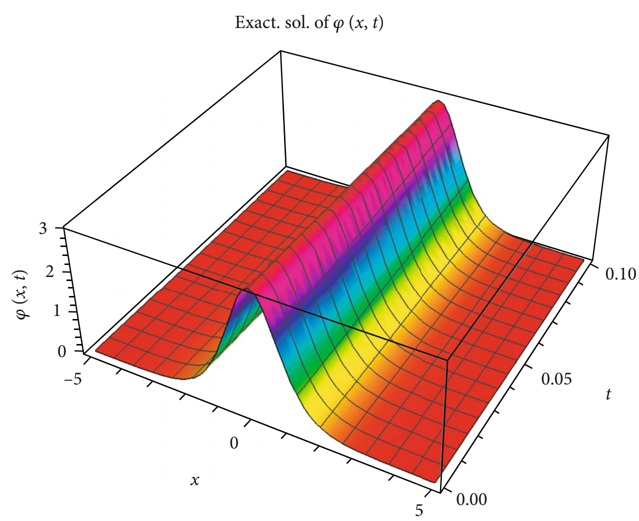

Figure 2: Exact solution of $\varphi(x, t)$ FDSW system.

Equations (9)–(11) are substituted in Equation (8) to obtain

$$\sum_{i=1}^{\infty} \psi_i = h + \sum_{m=0}^{\infty} L(\psi_m) + \aleph(\psi_0) + \sum_{m=1}^{\infty} \left\{ \aleph\left(\sum_{j=0}^{m} \psi_j \right) - \aleph\left(\sum_{j=0}^{m-1} \psi_j \right) \right\}. \tag{12}$$

We define the recursive relation as follows:

$$\begin{cases} \psi_0\left(\underline{\chi}, t\right) = h, \\ \psi_1\left(\underline{\chi}, t\right) = L(\psi_0) + \aleph(\psi_0), \\ \psi_2\left(\underline{\chi}, t\right) = L(\psi_1) + \aleph(\psi_0 + \psi_1) - \aleph(\psi_0), \\ \vdots \\ \psi_m\left(\underline{\chi}, t\right) = L(\psi_{m-1}) + \aleph(\psi_0 + \psi_1 + \cdots + \psi_{m-1}) - \aleph(\psi_0 + \psi_1 + \cdots + \psi_{m-2}), \\ \quad m = 1, 2, 3 \cdots, \end{cases} \tag{13}$$

and then,

$$\psi_m\left(\underline{\chi}, t\right) = L(\psi_0 + \psi_1 + \cdots + \psi_{m-1}) + \aleph(\psi_0 + \psi_1 + \cdots + \psi_{m-1}),$$
$$m = 1, 2, 3, \cdots,$$
$$\sum_{m=0}^{\infty} \psi_m\left(\underline{\chi}, t\right) = h\left(\underline{\chi}, t\right) + L\left(\sum_{m=0}^{\infty} \psi_m\left(\underline{\chi}, t\right) \right) + \aleph\left(\sum_{m=0}^{\infty} \psi_m\left(\underline{\chi}, t\right) \right). \tag{14}$$

The n-term NIM solution of Equations (8) and (9) is

$$\psi\left(\underline{\chi}, t\right) = \psi_0 + \psi_1 + \cdots + \psi_{n-1}. \tag{15}$$

3.1. NIM Convergence. In this section, the conditions for the convergence of NIM are given in the following theorems for the series $\sum_{m=0}^{\infty} \psi_m(\underline{\chi}, t)$.

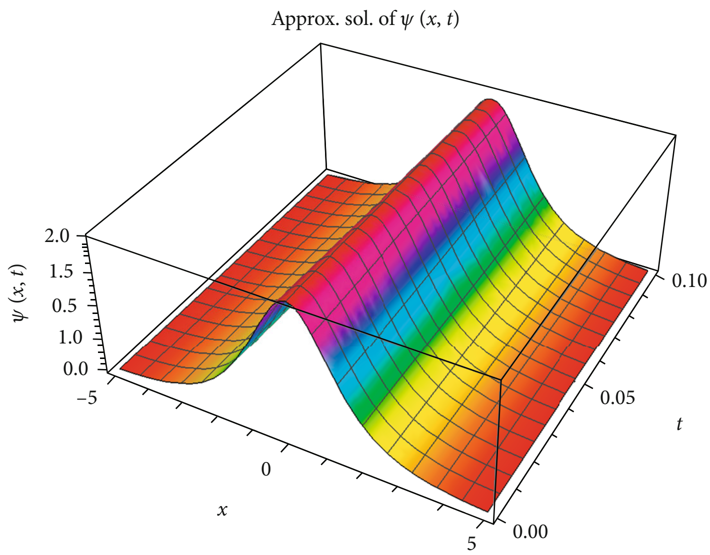

FIGURE 3: 2^{nd}-order NIM solution of $\psi(x,t)$ FDSW system at $c=2$.

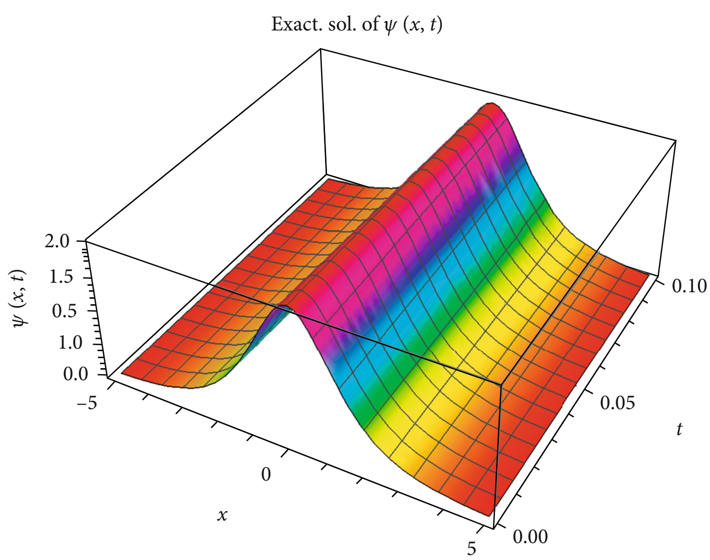

FIGURE 4: Exact solution of $\psi(x,t)$ FDSW system at $c=2$.

Theorem 4. *If $\aleph$ is $C^{(\infty)}$ in a neighborhood of ψ_0 and $\|\aleph^n(\psi_0)\| \le L$, for some real $L>0$ and any n $\|\psi_j\| \le M < 1/e, j=1,2,\cdots$, then the series $\sum_{n=0}^{\infty} G_n$ is convergent, and moreover, $\|G_n\| \le LM^n e^{n-1}(e-1), n=1,2,\cdots$.*

Theorem 5. *If $\aleph$ is C^{∞} and $\|\aleph^n(\psi_0)\| \le M \le e^{-1} \forall n$, then the series $\sum_{n=0}^{\infty} G_n$ is convergent. The detail of NIM convergence can be seen in article written by Bhalekar and Daftardar-Geiji in [25].*

4. Implementation of NIM

In this section, we implement NIM firstly to the fractional Drinfeld–Sokolov–Wilson coupled system and then to the fractional shallow water coupled system. The implementation is done by considering the fractional derivative in Caputo's sense, and the R-L integral is applied to the equations. The method is applied directly for obtaining an approximate solution.

4.1. Fractional Drinfeld–Sokolov–Wilson (FDSW) Coupled System. Consider the FDSW system of the form by rearranging Equation (1), and we write [18]

$$D_t^{\beta}\varphi(x,t) = -3\psi(x,t)\psi(x,t)_x, \tag{16}$$

$$D_t^{\beta}\psi(x,t) = -2\psi(x,t)_{xxx} - 2\varphi(x,t)\psi(x,t)_x - \varphi(x,t)_x\psi(x,t), \tag{17}$$

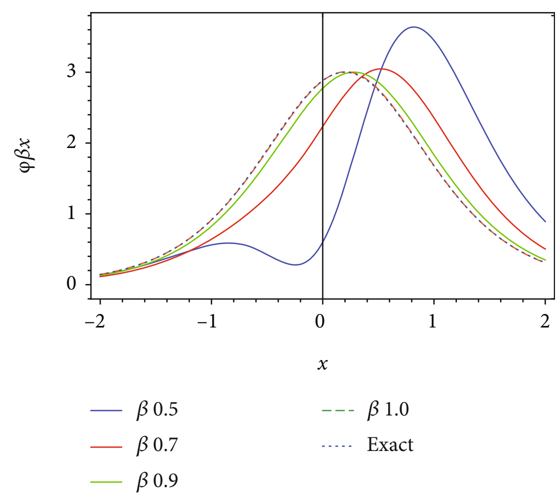

FIGURE 5: 2nd-order NIM and exact solution of $\varphi(x,t)$ at $t = 0.1$ of FDSW equation and for fractional values of β.

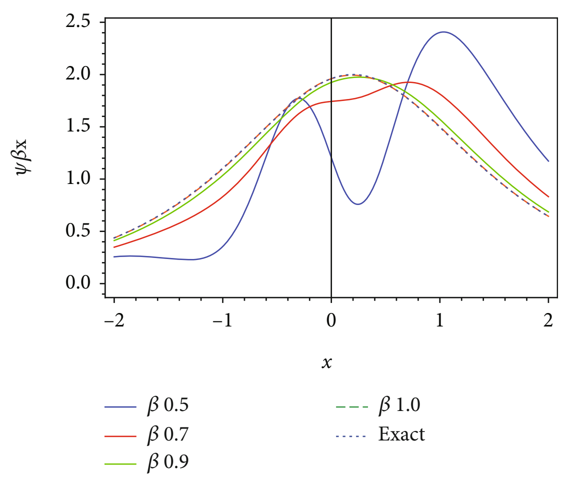

FIGURE 6: 2nd-order NIM and exact solution of $\psi(x,t)$ at $t = 0.1$ of FDSW equation and for fractional values of β.

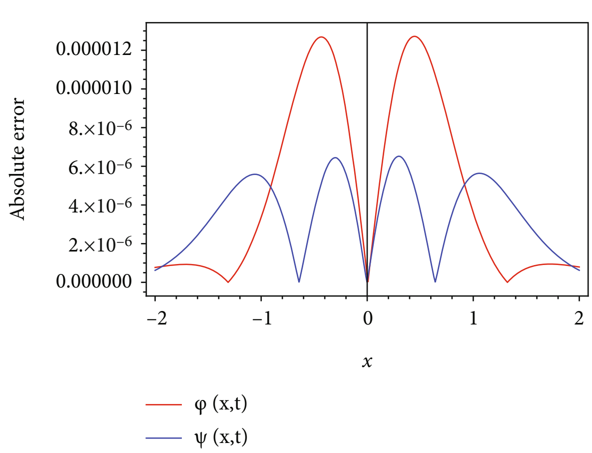

FIGURE 7: Comparison of the absolute error of the 2nd-order NIM solution for $\varphi(x,t)$ and $\psi(x,t)$ at $t = 0.01$ for the fractional DSW equations.

together with the initial condition

$$\begin{aligned} \varphi(x,0) &= \frac{1}{2}(3c)\,\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}x\right), \\ \psi(x,0) &= c\,\mathrm{sech}\left(\sqrt{\frac{c}{2}}x\right). \end{aligned} \tag{18}$$

Equation (16) has the exact solution for $\beta = 1$ as

$$\begin{aligned} \varphi(x,t) &= \frac{1}{2}(3c)\,\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}(x-ct)\right), \\ \psi(x,t) &= c\,\mathrm{sech}\left(\sqrt{\frac{c}{2}}(x-ct)\right). \end{aligned} \tag{19}$$

Applying I_t^β to Equation (16), we have

$$\begin{aligned} I_t^\beta D_t^\beta \varphi(x,t) &= \varphi(x,0) + I_t^\beta\left\{-3\psi(x,t)\psi(x,t)_x\right\}, \\ I_t^\beta D_t^\beta \psi(x,t) &= \psi(x,0) + I_t^\beta\left\{-2\psi(x,t)_{xxx} - 2\varphi(x,t)\psi(x,t)_x - \varphi(x,t)_x\psi(x,t)\right\}. \end{aligned} \tag{20}$$

By substituting the initial condition, we get

$$\begin{aligned} \varphi(x,t) &= \frac{1}{2}(3c)\,\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}(x-ct)\right) + I_t^\beta\left\{-3\psi(x,t)\psi(x,t)_x\right\}, \\ \psi(x,t) &= c\,\mathrm{sech}\left(\sqrt{\frac{c}{2}}(x-ct)\right) + I_t^\beta\left\{-2\psi(x,t)_{xxx} - 2\varphi(x,t)\psi(x,t)_x - \varphi(x,t)_x\psi(x,t)\right\}. \end{aligned} \tag{21}$$

By NIM algorithm, the zeroth-order component of $\varphi(x,t)$ and $\psi(x,t)$solution is as follows:

$$\left\{\begin{aligned} \varphi_0(x,t) &= \frac{1}{2}(3c)\,\mathrm{sech}^2\left(\sqrt{\frac{c}{2}}x\right) \\ \psi_0(x,t) &= c\,\mathrm{sech}\left(\sqrt{\frac{c}{2}}x\right) \end{aligned}\right\}. \tag{22}$$

The first-order component of solution is as follows:

$$\left\{\begin{aligned} \varphi_1(x,t) &= \frac{3c^{5/2}t^\beta \tanh\left(\sqrt{c}x/\sqrt{2}\right)\mathrm{sech}^2\left(\sqrt{c}x/\sqrt{2}\right)}{\sqrt{2}\Gamma(\beta+1)} \\ \psi_1(x,t) &= \frac{c^{5/2}t^\beta \tanh\left(\sqrt{c}x/\sqrt{2}\right)\mathrm{sech}\left(\sqrt{c}x/\sqrt{2}\right)}{\sqrt{2}\Gamma(\beta+1)} \end{aligned}\right\}. \tag{23}$$

The second-order component of solution is as follows:

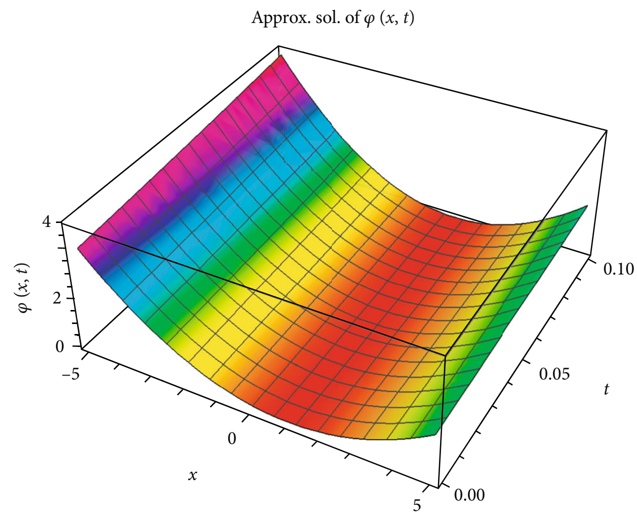

FIGURE 8: 2^{nd}-order NIM solution of $\varphi(x, t)$ FSW system.

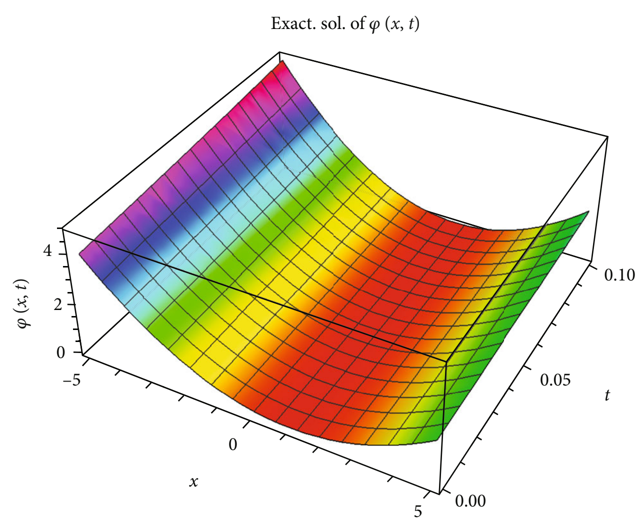

FIGURE 9: Exact solution of $\varphi(x, t)$ DSW system.

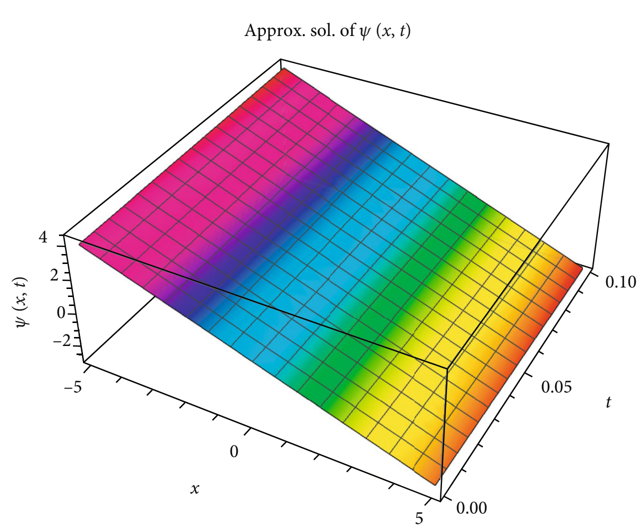

FIGURE 10: 2^{nd}-order NIM solution of $\varphi(x, t)$ DSW system.

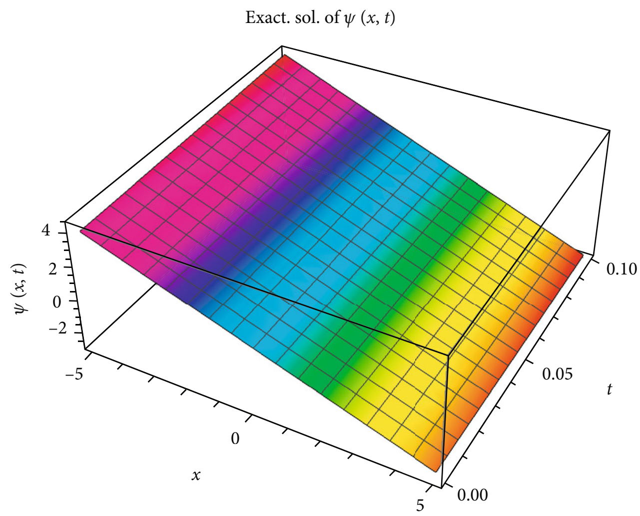

Figure 11: Exact solution of $\psi(x,t)$ DSW system.

$$\left\{\begin{array}{l} \varphi_2(x,t)=\dfrac{3c^4\operatorname{sech}^5\left(\sqrt{c}x/\sqrt{2}\right)}{16\Gamma(\beta+1)^2}\left(\dfrac{\sqrt{2}c^{3/2}\Gamma(2\beta+1)t^{3\beta}\left(\begin{array}{c}\sinh\left(3\sqrt{c}x/\sqrt{2}\right)\\ -7\sinh\left(\sqrt{c}x/\sqrt{2}\right)\end{array}\right)}{\Gamma(3\beta+1)}+\dfrac{4\Gamma(\beta+1)^2t^{2\beta}\left(\cosh\left(3\sqrt{c}x/\sqrt{2}\right)-3\cosh\left(\sqrt{c}x/\sqrt{2}\right)\right)}{\Gamma(2\beta+1)}\right) \\ \psi_2(x,t)=\dfrac{c^4\operatorname{sech}^3\left(\sqrt{c}x/\sqrt{2}\right)}{4\Gamma(\beta+1)^2}\left(\dfrac{3\sqrt{2}c^{3/2}\Gamma(2\beta+1)t^{3\beta}\left(\begin{array}{c}\sinh\left(3\sqrt{c}x/\sqrt{2}\right)\\ -6\sinh\left(\sqrt{c}x/\sqrt{2}\right)\end{array}\right)\operatorname{sech}^3\left(\sqrt{c}x/\sqrt{2}\right)}{\Gamma(3\beta+1)}+\dfrac{\Gamma(\beta+1)^2t^{2\beta}\left(\cosh\left(\sqrt{2}\sqrt{c}x\right)-3\right)}{\Gamma(2\beta+1)}\right)\end{array}\right\}. \quad (24)$$

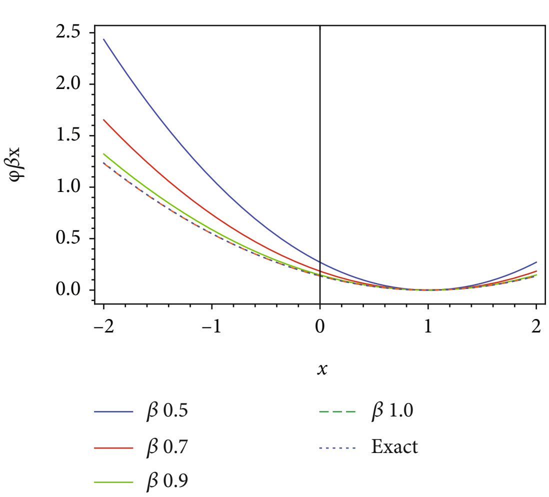

Figure 12: 2nd-order NIM solution and exact solution of $\varphi(x,t)$ at $t=0.1$ of FSW equation and for fractional values of β.

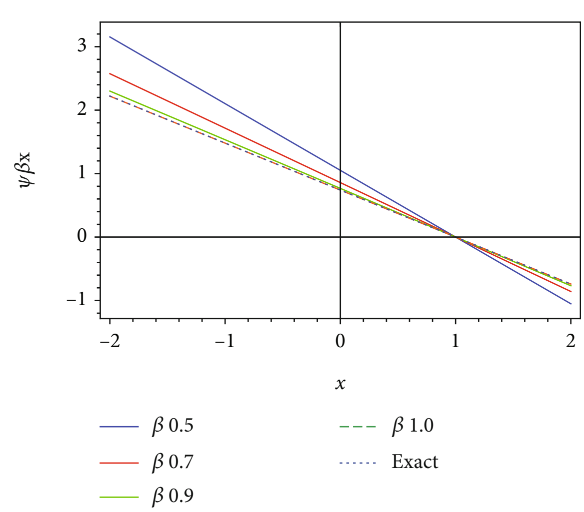

Figure 13: 2nd-order NIM solution and exact solution of $\psi(x,t)$ at $t=0.1$ of FSW equation and for fractional values of β.

$$\left\{\begin{array}{l}\varphi(x,t)=\varphi_0+\varphi_1+\varphi_2=\left\{\begin{array}{c}\dfrac{3}{2}c\,\mathrm{sech}^2\left(\dfrac{\sqrt{cx}}{\sqrt{2}}\right)+\dfrac{3c^{5/2}t^{\beta}\tanh\left(\sqrt{cx}/\sqrt{2}\right)\mathrm{sech}^2\left(\sqrt{cx}/\sqrt{2}\right)}{\sqrt{2}\Gamma(\beta+1)}\\ +\dfrac{3c^4\,\mathrm{sech}^5\left(\sqrt{cx}/\sqrt{2}\right)}{16\Gamma(\beta+1)^2}\left(\dfrac{\sqrt{2}c^{3/2}\Gamma(2\beta+1)t^{3\beta}\left(\begin{array}{c}\sinh\left(3\sqrt{cx}/\sqrt{2}\right)\\ -7\sinh\left(\sqrt{cx}/\sqrt{2}\right)\end{array}\right)}{\Gamma(3\beta+1)}+\dfrac{4\Gamma(\beta+1)^2t^{2\beta}\left(\cosh\left(3\sqrt{cx}/\sqrt{2}\right)-3\cosh\left(\sqrt{cx}/\sqrt{2}\right)\right)}{\Gamma(2\beta+1)}\right)\end{array}\right\}\\ \psi(x,t)=\psi_0+\psi_1+\psi_2=\left\{\begin{array}{c}c\,\mathrm{sech}\left(\dfrac{\sqrt{cx}}{\sqrt{2}}\right)+\dfrac{c^{5/2}t^{\beta}\tanh\left(\sqrt{cx}/\sqrt{2}\right)\mathrm{sech}\left(\sqrt{cx}/\sqrt{2}\right)}{\sqrt{2}\Gamma(\beta+1)}\\ +\dfrac{c^4\,\mathrm{sech}^3\left(\sqrt{cx}/\sqrt{2}\right)}{4\Gamma(\beta+1)^2}\left(\dfrac{3\sqrt{2}c^{3/2}\Gamma(2\beta+1)t^{3\beta}\left(\begin{array}{c}\sinh\left(3\sqrt{cx}/\sqrt{2}\right)\\ -6\sinh\left(\sqrt{cx}/\sqrt{2}\right)\end{array}\right)\mathrm{sech}^3\left(\sqrt{cx}/\sqrt{2}\right)}{\Gamma(3\beta+1)}+\dfrac{\Gamma(\beta+1)^2t^{2\beta}\left(\cosh\left(\sqrt{2}\sqrt{cx}\right)-3\right)}{\Gamma(2\beta+1)}\right)\end{array}\right\}\end{array}\right\}. \tag{25}$$

By combining the zeroth, first, and second-order components of $\varphi(x,t)$ and $\psi(x,t)$ the solution, we obtain the 2^{nd}-order NIM solution as

4.2. Fractional Shallow Water (FSW) Coupled System. Consider the nonlinear FSW coupled system by rearranging Equation (3), and we have [19]

$$D_t^{\beta}\varphi(x,t)=-\psi(x,t)\varphi(x,t)_x-\varphi(x,t)\psi(x,t)_x, \tag{26}$$

$$D_t^{\beta}\psi(x,t)=-\psi(x,t)\psi(x,t)_x-\varphi(x,t)_x, \tag{27}$$

together with the initial condition

$$\varphi(x,0)=\frac{1}{9}\left(x^2-2x+1\right),\quad \psi(x,0)=\frac{2(1-x)}{3}, \tag{28}$$

where c is the wave front's velocity. The exact solution of Equation (26) is given as

$$\varphi(x,t)=\frac{(x-1)^2}{9(t-1)^2},\quad \psi(x,t)=\frac{2(x-1)}{3(t-1)}. \tag{29}$$

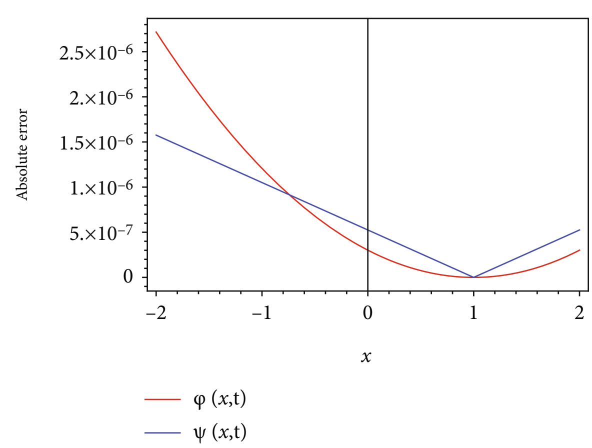

Figure 14: Comparison of the absolute error of the 2^{nd}-order NIM solution for $\varphi(x,t)$ and $\psi(x,t)$ at $t=0.01$ for the fractional SW equations.

Applying I_t^{β} to Equation (30), we have

$$I_t^{\beta}D_t^{\beta}\varphi(x,t)=\varphi(x,0)+I_t^{\beta}\left\{-\psi(x,t)\varphi(x,t)_x-\varphi(x,t)\psi(x,t)_x\right\}, \tag{30}$$

$$I_t^{\beta}D_t^{\beta}\psi(x,t)=\psi(x,0)+I_t^{\beta}\left\{-\psi(x,t)\psi(x,t)_x-\varphi(x,t)_x\right\}. \tag{31}$$

By substituting the initial condition, we get

$$\varphi(x,t)=\frac{1}{9}\left(x^2-2x+1\right)+I_t^{\beta}\left\{-\psi(x,t)\varphi(x,t)_x-\varphi(x,t)\psi(x,t)_x\right\},$$
$$\psi(x,t)=\frac{2(1-x)}{3}+I_t^{\beta}\left\{-\psi(x,t)\psi(x,t)_x-\varphi(x,t)_x\right\}. \tag{32}$$

Using the procedure of NIM, we have the zeroth-order component of the solution as

$$\left\{\begin{array}{c}\varphi_0(x,t)=\dfrac{1}{9}\left(x^2-2x+1\right)\\ \psi_0(x,t)=\dfrac{2(1-x)}{3}\end{array}\right\}. \tag{33}$$

The first-order component of solution is as follows:

$$\left\{\begin{array}{c}\varphi_1(x,t)=\dfrac{2(x-1)^2t^{\beta}}{9\Gamma(\beta+1)}\\ \psi_1(x,t)=-\dfrac{2(x-1)t^{\beta}}{3\Gamma(\beta+1)}\end{array}\right\}. \tag{34}$$

The second-order component of solution is as follows:

$$\left\{\begin{array}{c}\varphi_2(x,t)=\dfrac{2(x-1)^2\left(3\Gamma(\beta+1)^2t^{2\beta}/\Gamma(2\beta+1)+2\Gamma(2\beta+1)t^{3\beta}/\Gamma(3\beta+1)\right)}{9\Gamma(\beta+1)^2}\\ \psi_2(x,t)=-\dfrac{4(x-1)\left(3\Gamma(\beta+1)^2t^{2\beta}/\Gamma(2\beta+1)+\Gamma(2\beta+1)t^{3\beta}/\Gamma(3\beta+1)\right)}{9\Gamma(\beta+1)^2}\end{array}\right\}. \tag{35}$$

By NIM algorithm, the zeroth-order component of $\varphi(x,t)$ and $\psi(x,t)$ solution is as follows:

$$\left\{\begin{array}{c}\varphi(x,t)=\varphi_0+\varphi_1+\varphi_2=\left\{\dfrac{1}{9}\left(x^2-2x+1\right)+\dfrac{2(x-1)^2t^{\beta}}{9\Gamma(\beta+1)}+\dfrac{2(x-1)^2\left(3\Gamma(\beta+1)^2t^{2\beta}/\Gamma(2\beta+1)+2\Gamma(2\beta+1)t^{3\beta}/\Gamma(3\beta+1)\right)}{9\Gamma(\beta+1)^2}\right\}\\ \psi(x,t)=\psi_0+\psi_1+\psi_2=\left\{\dfrac{2(1-x)}{3}-\dfrac{2(x-1)t^{\beta}}{3\Gamma(\beta+1)}-\dfrac{4(x-1)\left(3\Gamma(\beta+1)^2t^{2\beta}/\Gamma(2\beta+1)+\Gamma(2\beta+1)t^{3\beta}/\Gamma(3\beta+1)\right)}{9\Gamma(\beta+1)^2}\right\}\end{array}\right\}. \tag{36}$$

5. Numerical Results and Discussion

The fractional DSW and fractional SW coupled systems of PDEs have been solved by NIM. We calculated the approximate solution up to 2$^{\text{nd}}$ order and observed the convergence of the method. The results have been plotted with the help of 2D and 3D graphs and also shown in the tables through a numerical comparison for different values. The following discussion shows the detail of figures and tables.

Figures 1 and 2 show the 2$^{\text{nd}}$-order NIM and the exact solution, respectively, in 3D plots $\varphi(x,t)$ while Figures 3 and 4 show the 2$^{\text{nd}}$-order NIM and the exact solution, respectively, $\psi(x,t)$ for the fractional DSW equations. In Figures 5 and 6, the different fractional values of β of the 2$^{\text{nd}}$-order NIM solution are compared for $\varphi(x,t)$ and $\psi(x,t)$, respectively. In Figure 7, the 2D plot shows the absolute error for the 2$^{\text{nd}}$-order NIM solution for both $\varphi(x,t)$ and $\psi(x,t)$ of the fractional DSW equation. Similarly, the fractional SW coupled system of equations is discussed in Figures 8–14. The 3D plots of Figures 8 and 9 represent the 2$^{\text{nd}}$-order NIM solution and the exact solution, respectively, $\varphi(x,t)$. Figures 10 and 11 show the 2$^{\text{nd}}$-order approximate solution and exact solution for $\psi(x,t)$ of the fractional SW equations. The 2D plots in Figures 12 and 13 compare the different fractional values of β for the 2$^{\text{nd}}$-order approximate solution of $\varphi(x,t)$ and $\psi(x,t)$, respectively. In Figure 14, the absolute errors are compared for the 2$^{\text{nd}}$-order NIM solution of the fractional SW equations. In all these figures, we noted that as the fractional order of differential equation tends to 1, the approximate solution converges to exact solution and for $\beta = 1$, the approximate solution overlaps the exact solution which verifies the accuracy of our proposed method.

6. Conclusion

We implemented new iterative method (NIM) for the solution of the fractional Drinfeld–Sokolov–Wilson equations and fractional-order shallow water equations. The numerical comparison is made with the q-homotopy analysis transform method. The results show that NIM is conveniently convergent and provides an accurate approximate solution. The tables and figures show that as the value of β approaches the classical value (1 for these systems) of the differential equation, the approximate solution converges to the exact solution. Comparisons in tables and graphs verify that NIM converges more rapidly and is widely useful for obtaining the approximate solution of differential equations.

Data Availability

Data will be available on reasonable statement.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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References

[1] A. H. Nayfeh, *Introduction to perturbation techniques*, John Wiley & Sons, 2011.

[2] P. A. Lagerstrom and R. G. Casten, "Basic concepts underlying singular perturbation techniques," *SIAM Review*, vol. 14, no. 1, pp. 63–120, 1972.

[3] K. J. Bathe, *Finite element method*, Wiley Encyclopedia of Computer Science and Engineering, 2007.

[4] R. J. LeVeque, "Finite difference methods for differential equations," *Draft Version for Use in a Math*, vol. 585, no. 6, p. 112, 1998.

[5] A. M. Wazwaz, "A study on linear and nonlinear Schrodinger equations by the variational iteration method," *Chaos, Solitons & Fractals*, vol. 37, no. 4, pp. 1136–1142, 2008.

[6] R. Lyons, A. S. Vatsala, and R. A. Chiquet, "Picard's iterative method for Caputo fractional differential equations with numerical results," *Mathematics*, vol. 5, no. 4, p. 65, 2017.

[7] "New integral transform: Shehu transform a generalization of Sumudu and Laplace transform for solving differential equations," https://arxiv.org/abs/1904.11370.

[8] J. S. Chen, C. W. Liu, and C. M. Liao, "Two-dimensional Laplace-transformed power series solution for solute transport in a radially convergent flow field," *Advances in Water Resources*, vol. 26, no. 10, pp. 1113–1124, 2003.

[9] R. Nawaz and L. Zada, "Solving time fractional Sharma-Tasso-Olever equation by optimal homotopy asymptotic method," *AIP Conference Proceedings*, vol. 1978, no. 1, article 310002, 2018.

[10] L. Zada and R. Nawaz, "Solution of time-fractional order RLW equation using optimal homotpy asymptotic method," *AIP Conference Proceedings*, vol. 2116, no. 1, article 300005, 2019.

[11] S. Liao, "Comparison between the homotopy analysis method and homotopy perturbation method," *Applied Mathematics and Computation*, vol. 169, no. 2, pp. 1186–1194, 2005.

[12] S. Bhalekar and V. Daftardar-Gejji, "New iterative method: application to partial differential equations," *Applied Mathematics and Computation*, vol. 203, no. 2, pp. 778–783, 2008.

[13] S. Bhalekar and V. Daftardar-Gejji, "Solving evolution equations using a new iterative method," *Numerical Methods for Partial Differential Equations: An International Journal*, vol. 26, no. 4, pp. 906–916, 2010.

[14] D. J. Evans and K. R. Raslan, "The Adomian decomposition method for solving delay differential equation," *International Journal of Computer Mathematics*, vol. 82, no. 1, pp. 49–54, 2005.

[15] L. Zada, R. Nawaz, S. Ahsan, K. S. Nisar, and D. Baleanu, "New iterative approach for the solutions of fractional order inhomogeneous partial differential equations," *AIMS Mathematics*, vol. 6, no. 2, pp. 1348–1365, 2021.

[16] R. Nawaz, N. Ali, L. Zada, Z. Shah, A. Tassaddiq, and N. A. Alreshidi, "Comparative analysis of natural transform decomposition method and new iterative method for fractional foam drainage problem and fractional order modified regularized long-wave equation," *Fractals*, vol. 28, no. 7, article 2050124, 2020.

[17] M. J. Khan, R. Nawaz, S. Farid, and J. Iqbal, "New iterative method for the solution of fractional damped burger and fractional Sharma-Tasso-Olver equations," *Complexity*, vol. 2018, Article ID 3249720, 7 pages, 2018.

[18] W. Gao, P. Veeresha, D. G. Prakasha, H. M. Baskonus, and G. Yel, "A powerful approach for fractional Drinfeld–Sokolov–Wilson equation with Mittag-Leffler law," *Alexandria Engineering Journal*, vol. 58, no. 4, pp. 1301–1311, 2019.

[19] S. Kumar, A. Kumar, Z. Odibat, M. Aldhaifallah, and K. S. Nisar, "A comparison study of two modified analytical approach for the solution of nonlinear fractional shallow water equations in fluid flow," *AIMS Mathematics*, vol. 5, no. 4, pp. 3035–3055, 2020.

[20] S. Bhatter, A. Mathur, D. Kumar, and J. Singh, "A new analysis of fractional Drinfeld–Sokolov–Wilson model with exponential memory," *Physica A: Statistical Mechanics and its Applications*, vol. 537, article 122578, 2020.

[21] J. Singh, D. Kumar, D. Baleanu, and S. Rathore, "An efficient numerical algorithm for the fractional Drinfeld–Sokolov–Wilson equation," *Applied Mathematics and Computation*, vol. 335, pp. 12–24, 2018.

[22] S. Kumar, "A numerical study for the solution of time fractional nonlinear shallow water equation in oceans," *Zeitschrift für Naturforschung A*, vol. 68, no. 8-9, pp. 547–553, 2013.

[23] D. H. Kim, Y. S. Cho, and W. G. Kim, "Weighted average flux-type scheme for shallow-water equations with fractional step method," *Journal of Engineering Mechanics*, vol. 130, no. 2, pp. 152–160, 2004.

[24] R. Gorenflo and F. Mainardi, “Fractional Calculus,” in *Fractals and Fractional Calculus in Continuum Mechanics*, Springer, Vienna, 1997.

[25] S. Bhalekar and V. Daftardar-Gejji, “Convergence of the new iterative method,” *International Journal of Differential Equations*, vol. 2011, Article ID 989065, 10 pages, 2011.