## Supplementary Materials

For convenience of the further analysis we use the following denominations:

$$A_{ij}^{a} = \frac{k_0^2}{\varepsilon_0} \int_{V_a} d\mathbf{R}'_a G_{ij}(\mathbf{R}_a, \mathbf{R}'_a) \ , \tag{S.1}$$

$$B_{ij}^{a} = \frac{k_0^2}{\varepsilon_0} \int_{V_b} d\mathbf{R}'_b G_{ij}(\mathbf{R}_a, \mathbf{R}'_b) = \frac{b_{ij}^{a}}{d^3} , \tag{S.2}$$

$$A_{ij}^{b} = \frac{k_0^2}{\varepsilon_0} \int_{V_b} d\mathbf{R}'_b G_{ij}(\mathbf{R}_b, \mathbf{R}'_b) , \tag{S.3}$$

$$B_{ij}^{b} = \frac{k_0^2}{\varepsilon_0} \int_{V_a} d\mathbf{R}'_a G_{ij}(\mathbf{R}_b, \mathbf{R}'_a) = \frac{b_{ij}^{b}}{d^3} . \tag{S.4}$$

Here $A_{ij}^{a}$ and $A_{ij}^{b}$ describe the self-action and define the effective susceptibility of the spherical nanoparticle in the free space, hence they do not depend on the nanoparticles coordinates. $B_{ij}^{a}$ and $B_{ij}^{b}$ describe the field interaction between the nanoparticles located at the distance d from each other. Hence, using the denominations and taking into account the quasi-point nanoparticle approximation allowing considering the dipolar moment in the integral as a constant:

$$\frac{\partial F}{\partial P_i^a(\mathbf{R}_a)} = \left\{ \frac{1}{\varepsilon_0} \lambda_{ij}^{a} P_j^a(\mathbf{R}_a) + \frac{1}{\varepsilon_0} \beta_{ijrs}^{a} P_j^a(\mathbf{R}_a) P_r^a(\mathbf{R}_a) P_s^a(\mathbf{R}_a) \right.$$
$$\left. - E_i^{ex}(\mathbf{R}_\alpha) - \mathbf{A}_{ji}^{a} P_j^a(\mathbf{R}_a) - \frac{1}{2} \frac{k_0^2}{\varepsilon_0} \mathbf{B}_{ji}^{b} P_j^b(\mathbf{R}_b) \right\} = 0 \ , \tag{S.5}$$

$$\frac{\partial F}{\partial P_i^b(\mathbf{R}_b)} = \left\{ \frac{1}{\varepsilon_0} \lambda_{ij}^b P_j^b(\mathbf{R}_b) + \frac{1}{\varepsilon_0} \beta_{ijrs}^b P_j^b(\mathbf{R}_b) P_r^b(\mathbf{R}_b) P_s^a(\mathbf{R}_b) \right.$$
$$\left. - E_i^{ex}(\mathbf{R}_b) - \mathbf{A}_{ji}^b P_j^b(\mathbf{R}_b) - \frac{1}{2} B_{ji}^a P_j^a(\mathbf{R}_a) \right\} = 0 \ . \quad \text{(S.6)}$$

Zero approximation of the equation solution may be found by assuming that the nanoparticles have linear properties and do not interact one which other. Here we present transformations just for $P_i^a$, as expressions for $P_i^b$ are similar.

$$\frac{1}{\varepsilon_0} \lambda_{ij}^a P_j^{a(0)}(\mathbf{R}_a) - E_i^{ex}(\mathbf{R}_a) - A_{ij}^a P_j^{a(0)}(\mathbf{R}_a) = 0 \ .$$

$$\left[ \frac{1}{\varepsilon_0} \lambda_{ij}^a - \frac{k_0^2}{\varepsilon_0} \int_{V_a} d\mathbf{R}'_a G_{ij}(\mathbf{R}_a, \mathbf{R}'_a) \right] P_j^{a(0)}(\mathbf{R}_a) = E_i^{ex}(\mathbf{R}_a) \ .$$

As a consequence:

$$P_j^{a(0)}(\mathbf{R}_a) = \left[ \frac{1}{\varepsilon_0} \lambda_{ij}^a - \frac{k_0^2}{\varepsilon_0} \int_{V_a} d\mathbf{R}'_a G_{ij}(\mathbf{R}_a, \mathbf{R}'_a) \right]^{-1} E_i^{ex}(\mathbf{R}_a) = \mathrm{X}_{ji}^{(a)}(\mathbf{R}_a) E_i^{ex}(\mathbf{R}_a) \ , \quad \text{(S.7)}$$

$$P_j^{b(0)}(\mathbf{R}_b) = \left[ \frac{1}{\varepsilon_0} \lambda_{ij}^b - \frac{k_0^2}{\varepsilon_0} \int_{V_b} d\mathbf{R}'_b G_{ij}(\mathbf{R}_b, \mathbf{R}'_b) \right]^{-1} E_i^{ex}(\mathbf{R}_b) = \mathrm{X}_{ji}^{(b)}(\mathbf{R}_b) E_i^{ex}(\mathbf{R}_b) \ , \quad \text{(S.8)}$$

where

$$\mathrm{X}_{ij}^{(a,b)}(\mathbf{R}_{a,b}) = \left[ \frac{1}{\varepsilon_0} \lambda_{ij}^{a,b} - \frac{k_0^2}{\varepsilon_0} \int_{V_{a,b}} d\mathbf{R}'_{a,b} G_{ji}(\mathbf{R}_{a,b}, \mathbf{R}'_{a,b}) \right]^{-1} \quad \text{(S.9)}$$

is the linear response of the nanoparticles to the external local field[1] and is the property of the nanoparticles and the medium in which it is located. The next approximation may be obtained from

$$P_j^{a(1)}(\mathbf{R}_a) = -\alpha_{ji}^a \beta_{ijrs}^a P_j^{a(0)}(\mathbf{R}_a) P_r^{a(0)}(\mathbf{R}_a) P_s^{a(0)}(\mathbf{R}_a) + \\ + k_0^2 \int_{V_b} d\mathbf{R}'_b \alpha_{ji}^a G_{ij}(\mathbf{R}_a, \mathbf{R}'_b) P_j^{b(0)}(\mathbf{R}_b) \ , \tag{S.10}$$

and

$$P_j^{b(1)}(\mathbf{R}_b) = -\alpha_{ji}^b \beta_{ijrs}^b P_j^{b(0)}(\mathbf{R}_b) P_r^{b(0)}(\mathbf{R}_b) P_s^{b(0)}(\mathbf{R}_b) + \\ + k_0^2 \int_{V_a} d\mathbf{R}'_a \alpha_{ji}^a G_{ij}(\mathbf{R}_b, \mathbf{R}'_a) P_j^{b(0)}(\mathbf{R}_b) \ . \tag{S.11}$$

Hence, dependence of the nanoparticles dipolar moments on the external field is the following:

$$P_j^a(\mathbf{R}_a) = \varepsilon_0 \mathrm{X}_{jk}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) + \varepsilon_0 \Xi_{jk}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_b) \ + \\ + \varepsilon_0 \Omega_{jklm}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) E_l^{ex}(\mathbf{R}_a) E_m^{ex}(\mathbf{R}_a) \ , \tag{S.12}$$

where

$$\Omega_{jklm}^a(\mathbf{R}_a) = -\varepsilon_0{}^2 \alpha_{ji}^a \beta_{ijrs}^a \mathrm{X}_{jk}^b(\mathbf{R}_a) \mathrm{X}_{rl}^b(\mathbf{R}_a) \mathrm{X}_{sm}^b(\mathbf{R}_a) = -\varepsilon_0{}^2 \tilde{\beta}_{jklm}^a \ , \tag{S.13}$$

$$\Xi_{jk}^a(\mathbf{R}_a) = \alpha_{ji}^a B_{il}^a\left(\mathbf{R}_a, \mathbf{R}_b\right) \mathrm{X}_{lk}^b(\mathbf{R}_b) = \frac{\xi_{jk}^a}{\left|\mathbf{R}_a - \mathbf{R}_b\right|^3} \ , \tag{S.14}$$

In (S.12) the first term describes the influence of the linear effective susceptibility, the second one describes the influence of the nonlinear nanoparticle susceptibility (S.13), and the third one describes the influence of

[1] V. Lozovski, “The Effective Susceptibility Concept in the Electrodynamics of Nano-Systems,” *Journal of Computational and Theoretical Nanoscience*, vol. 7, no. 10, pp. 2077–2093, 2010.

the interaction between the nanoparticles (S.14). Similar for the second nanoparticle:

$$P_j^b(\mathbf{R}_b) = \varepsilon_0 \mathrm{X}_{jk}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) + \varepsilon_0 \Xi_{jk}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_a) + \\ + \varepsilon_0 \Omega_{jklm}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) E_l^{ex}(\mathbf{R}_b) E_m^{ex}(\mathbf{R}_b)\ , \tag{S.15}$$

$$\Xi_{jk}^b(\mathbf{R}_b) = \alpha_{ji}^b B_{il}^b(\mathbf{R}_b, \mathbf{R}_a) \mathrm{X}_{lk}^b(\mathbf{R}_a) = \frac{\xi_{jk}^b}{\left|\mathbf{R}_a - \mathbf{R}_b\right|^3}\ , \tag{S.16}$$

$$\Omega_{jklm}^b(\mathbf{R}_b) = -\varepsilon_0{}^2 \alpha_{ji}^a \beta_{ijrs}^a \mathrm{X}_{jk}^a(\mathbf{R}_b) \mathrm{X}_{rl}^a(\mathbf{R}_b) \mathrm{X}_{sm}^a(\mathbf{R}_b) = -\varepsilon_0{}^2 \tilde{\beta}_{jklm}^b\ . \tag{S.17}$$

Then the self-consistent local field is:

$$E_i^a(\mathbf{R}_a) = E_i^{ex}(\mathbf{R}_\alpha) + \frac{1}{\varepsilon_0} A_{ij}^a(\mathbf{R}_a) \Big\{ \varepsilon_0 \mathrm{X}_{jk}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) + \\ + \varepsilon_0 \Xi_{jk}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_b) + \varepsilon_0 \Omega_{jklm}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) E_l^{ex}(\mathbf{R}_a) E_m^{ex}(\mathbf{R}_a) \Big\} + \\ + \frac{1}{\varepsilon_0} B_{ij}^a(\mathbf{R}_a, \mathbf{R}_b) \Big\{ \varepsilon_0 \mathrm{X}_{jk}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) + \varepsilon_0 \Xi_{jk}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_a) + \\ + \varepsilon_0 \Omega_{jklm}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) E_l^{ex}(\mathbf{R}_b) E_m^{ex}(\mathbf{R}_b) \Big\}\ . \tag{S.18}$$

$$E_i^b(\mathbf{R}_b) = E_i^{ex}(\mathbf{R}_b) + \frac{1}{\varepsilon_0} A_{ij}^b(\mathbf{R}_b) \Big\{ \varepsilon_0 \mathrm{X}_{jk}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) + \\ + \varepsilon_0 \Xi_{jk}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_a) + \varepsilon_0 \Omega_{jklm}^b(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) E_l^{ex}(\mathbf{R}_b) E_m^{ex}(\mathbf{R}_b) \Big\} + \\ + \frac{1}{\varepsilon_0} B_{ij}^b(\mathbf{R}_b, \mathbf{R}_a) \Big\{ \varepsilon_0 \mathrm{X}_{jk}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) + \varepsilon_0 \Xi_{jk}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_b) + \\ + \varepsilon_0 \Omega_{jklm}^a(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) E_l^{ex}(\mathbf{R}_a) E_m^{ex}(\mathbf{R}_a) \Big\}\ . \tag{S.19}$$

The energy of interaction is:

$$U(d) = -\frac{1}{2}\int_0^\infty \frac{d\omega}{2\pi}\left\langle \int_{V_a} d\mathbf{R}_a P_i^a(\mathbf{R}_a,\omega)E_i^a(\mathbf{R}_a,\omega) + \int_{V_b} d\mathbf{R}_b P_i^b(\mathbf{R}_b,\omega)E_i^b(\mathbf{R}_b,\omega)\right\rangle - \delta U, \quad \text{(S.20)}$$

where

$$\delta U = -\int_0^\infty \frac{d\omega}{2\pi}\left\langle \frac{1}{2}\int_{V_a} d\mathbf{R}_a P_i^a(\mathbf{R}_a,\omega)E_i^a(\mathbf{R}_a,\omega) + \frac{1}{2}\int_{V_b} d\mathbf{R}_b P_i^b(\mathbf{R}_b,\omega)E_i^b(\mathbf{R}_b,\omega)\right\rangle\Bigg|_{d\to\infty}, \quad \text{(S.21)}$$

is the impact of the internal energy of the nanoparticles, namely this is the energy of the system, when the particles are located at the infinite distance. Obviously, the interaction potential consists only the terms depending both on $\mathbf{R}_a$ and on $\mathbf{R}_b$. In the approximation of the quasi-point nanoparticles (S.21) may be simplified:

$$U(d) = -\frac{1}{2}\int_0^\infty \frac{d\omega}{2\pi}\left\langle P_i^a E_i^a + P_i^b E_i^b\right\rangle - \delta U .$$

In order to obtain the adsorption potential let us consider its parts:

$$\begin{aligned} P_i^a(\mathbf{R}_a,\omega)E_i^a(\mathbf{R}_a,\omega) = &\Big\{\varepsilon_0 \mathrm{X}_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a) + \varepsilon_0 \Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b) + \\ &+ \varepsilon_0 \Omega_{iklm}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)E_m^{ex}(\mathbf{R}_a)\Big\}\Big\{E_i^{ex}(\mathbf{R}_\alpha) \\ &+ \frac{1}{\varepsilon_0}A_{ij}^a(\mathbf{R}_a)\Big[\varepsilon_0 \mathrm{X}_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a) + \\ &+ \varepsilon_0 \Xi_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b) + \varepsilon_0 \Omega_{jklm}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)E_m^{ex}(\mathbf{R}_a)\Big] + \\ &+ \frac{1}{\varepsilon_0}B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\Big[\varepsilon_0 \mathrm{X}_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b) + \varepsilon_0 \Xi_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_a) + \\ &+ \varepsilon_0 \Omega_{jklm}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)E_m^{ex}(\mathbf{R}_b)\Big]\Big\} \end{aligned}$$

and

$$P_i^b(\mathbf{R}_b,\omega)E_i^b(\mathbf{R}_b,\omega)=\Big\{\varepsilon_0\mathrm{X}_{ik}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)+\varepsilon_0\Xi_{ik}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_a)\ +$$
$$+\varepsilon_0\Omega_{iklm}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)E_m^{ex}(\mathbf{R}_b)\ \Big\}\Big\{E_i^{ex}(\mathbf{R}_b)$$
$$+\frac{1}{\varepsilon_0}A_{ij}^b(\mathbf{R}_b)\Big[\varepsilon_0\mathrm{X}_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Xi_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_a)+\varepsilon_0\Omega_{jklm}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)E_m^{ex}(\mathbf{R}_b)\Big]+$$
$$+\frac{1}{\varepsilon_0}B_{ij}^b(\mathbf{R}_b,\mathbf{R}_a)\Big[\varepsilon_0\mathrm{X}_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)+\varepsilon_0\Xi_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Omega_{jklm}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)E_m^{ex}(\mathbf{R}_a)\Big]\Big\},$$

which may be rewritten as

$$P_i^a(\mathbf{R}_a,\omega)E_i^a(\mathbf{R}_a,\omega)=\varepsilon_0\mathrm{X}_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)A_{ij}^a(\mathbf{R}_a)\Xi_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\mathrm{X}_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\mathrm{X}_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\mathrm{X}_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\Xi_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_a)+$$
$$+\varepsilon_0\mathrm{X}_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\Omega_{jklm}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)E_m^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)E_i^{ex}(\mathbf{R}_a)+$$
$$+\varepsilon_0\Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)A_{ij}^a(\mathbf{R}_a)\mathrm{X}_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)+$$
$$+\varepsilon_0\Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)A_{ij}^a(\mathbf{R}_a)\Xi_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)A_{ij}^a(\mathbf{R}_a)\Omega_{jklm}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)E_m^{ex}(\mathbf{R}_a)+$$
$$+\varepsilon_0\Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\mathrm{X}_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\Xi_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_a)+$$
$$+\varepsilon_0\Xi_{ik}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\Omega_{jklm}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)E_m^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Omega_{iklm}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)E_m^{ex}(\mathbf{R}_a)A_{ij}^a(\mathbf{R}_a)\Xi_{jk}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Omega_{iklm}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)E_m^{ex}(\mathbf{R}_a)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\mathrm{X}_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)+$$
$$+\varepsilon_0\Omega_{iklm}^a(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)E_m^{ex}(\mathbf{R}_a)B_{ij}^a(\mathbf{R}_a,\mathbf{R}_b)\Xi_{jk}^b(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_a)$$

and similar formula for the nanoparticle ‘b’. Hence, energy of interaction between two nanoparticles in the medium describing with the Green function $G_{ij}(\mathbf{R},\mathbf{R}',\omega)$ is defined as:

$$
\begin{aligned}
U(d) = -\frac{1}{2}\int_0^{\infty}\frac{d\omega}{2\pi}\Bigg\{ \int_{V_a} d\mathbf{R}_a \Big[ & S_{ij}^{(1a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_b)\right\rangle + \\
& + S_{ij}^{(2a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)\right\rangle + \\
& + S_{ij}^{(3a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_a)\right\rangle + \\
& + Q_{ijkl}^{(1a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle + \\
& + Q_{ijkl}^{(2a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle + \\
& + Q_{ijkl}^{(3a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle + \\
& + Q_{ijkl}^{(4a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle \Big] + \\
+ \int_{V_b} d\mathbf{R}_b \Big[ & S_{ij}^{(1b)}(\mathbf{R}_b,\mathbf{R}_a,\omega)\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_a)\right\rangle + \\
& + S_{ij}^{(2b)}(\mathbf{R}_b,\mathbf{R}_a,\omega)\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_a)\right\rangle + \\
& + S_{ij}^{(3b)}(\mathbf{R}_b,\mathbf{R}_a,\omega)\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)\right\rangle + \\
& + Q_{ijkl}^{(1b)}(\mathbf{R}_b,\mathbf{R}_a,\omega)\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle + \\
& + Q_{ijkl}^{(2b)}(\mathbf{R}_b,\mathbf{R}_a,\omega)\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle + \\
& + Q_{ijkl}^{(3b)}(\mathbf{R}_b,\mathbf{R}_a,\omega)\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle + \\
& + Q_{ijkl}^{(4b)}(\mathbf{R}_b,\mathbf{R}_a,\omega)\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle \Big]\Bigg\} ,
\end{aligned}
\tag{S.22}
$$

where

$$
\begin{aligned}
S_{ij}^{(1a)}(\mathbf{R}_a,\mathbf{R}_b,\omega) = {} & \mathrm{X}_{ki}^{a}(\mathbf{R}_a)A_{kl}^{a}(\mathbf{R}_a)\Xi_{lj}^{a}(\mathbf{R}_a) + \mathrm{X}_{ki}^{a}(\mathbf{R}_a)B_{kl}^{a}(\mathbf{R}_a,\mathbf{R}_b)\mathrm{X}_{lj}^{b}(\mathbf{R}_b) + \\
& + \Xi_{ij}^{a}(\mathbf{R}_a) + \Xi_{kj}^{a}(\mathbf{R}_a)A_{kl}^{a}(\mathbf{R}_a)\mathrm{X}_{li}^{a}(\mathbf{R}_a) + \Xi_{kj}^{a}(\mathbf{R}_a)B_{kl}^{a}(\mathbf{R}_a,\mathbf{R}_b)\Xi_{li}^{b}(\mathbf{R}_b) \\
& = \frac{s_{ij}^{(1a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^3} + \frac{t_{ij}^{(1a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^9},
\end{aligned}
\tag{S.23}
$$

$$S_{ij}^{(2a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\Xi_{ki}^{a}(\mathbf{R}_a)A_{kl}^{a}(\mathbf{R}_a)\Xi_{lj}^{a}(\mathbf{R}_a)+\Xi_{ki}^{a}(\mathbf{R}_a)B_{kl}^{a}(\mathbf{R}_a,\mathbf{R}_b)\mathrm{X}_{lj}^{b}(\mathbf{R}_b)$$
$$=\frac{s_{ij}^{(2a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}+\frac{t_{ij}^{(2a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}\ , \qquad (S.24)$$

$$S_{ij}^{(3a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\mathrm{X}_{ki}^{a}(\mathbf{R}_a)B_{kl}^{a}(\mathbf{R}_a,\mathbf{R}_b)\Xi_{lj}^{b}(\mathbf{R}_b)=\frac{s_{ij}^{(3a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}, \qquad (S.25)$$

$$Q_{ijkl}^{(1a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\mathrm{X}_{mi}^{a}(\mathbf{R}_a)B_{mn}^{a}(\mathbf{R}_a,\mathbf{R}_b)\Omega_{njkl}^{b}(\mathbf{R}_b)=-\varepsilon_0^2\frac{u_{ijkl}^{(1a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^3}, \qquad (S.26)$$

$$Q_{ijkl}^{(2a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\Xi_{mi}^{a}(\mathbf{R}_a)A_{mn}^{a}(\mathbf{R}_a)\Omega_{njkl}^{a}(\mathbf{R}_a)+$$
$$+\Omega_{mljk}^{a}(\mathbf{R}_a)A_{mn}^{a}(\mathbf{R}_a)\Xi_{ni}^{a}(\mathbf{R}_a)+\Omega_{mljk}^{a}(\mathbf{R}_a)B_{mn}^{a}(\mathbf{R}_a,\mathbf{R}_b)\mathrm{X}_{ni}^{b}(\mathbf{R}_b)= \qquad (S.27)$$
$$=-\varepsilon_0^2\frac{u_{ijkl}^{(2a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^3}\ ,$$

$$Q_{ijkl}^{(3a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\Omega_{mijk}^{a}(\mathbf{R}_a)B_{mn}^{a}(\mathbf{R}_a,\mathbf{R}_b)\Xi_{nl}^{b}(\mathbf{R}_b)=-\varepsilon_0^2\frac{u_{ijkl}^{(3a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}, \qquad (S.28)$$

$$Q_{ijkl}^{(4a)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\Xi_{mi}^{a}(\mathbf{R}_a)B_{mn}^{a}(\mathbf{R}_a,\mathbf{R}_b)\Omega_{njkl}^{b}(\mathbf{R}_b)=-\varepsilon_0^2\frac{u_{ijkl}^{(4a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}, \qquad (S.29)$$

$$S_{ij}^{(1b)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\mathrm{X}_{ki}^{b}(\mathbf{R}_b)A_{kl}^{b}(\mathbf{R}_b)\Xi_{lj}^{b}(\mathbf{R}_b)+\mathrm{X}_{ki}^{b}(\mathbf{R}_b)B_{kl}^{b}(\mathbf{R}_b,\mathbf{R}_a)\mathrm{X}_{lj}^{a}(\mathbf{R}_a)+$$
$$+\Xi_{ij}^{b}(\mathbf{R}_b)+\Xi_{kj}^{b}(\mathbf{R}_b)A_{kl}^{b}(\mathbf{R}_b)\mathrm{X}_{li}^{b}(\mathbf{R}_b)+\Xi_{kj}^{b}(\mathbf{R}_b)B_{kl}^{b}(\mathbf{R}_b,\mathbf{R}_a)\Xi_{li}^{a}(\mathbf{R}_a)= \qquad (S.30)$$
$$=\frac{s_{ij}^{(1b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^3}+\frac{t_{ij}^{(1b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^9}\ ,$$

$$S_{ij}^{(2b)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\frac{s_{ij}^{(2b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}+\frac{t_{ij}^{(2b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}\ , \qquad (S.31)$$

$$S_{ij}^{(3b)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\mathrm{X}_{ki}^{b}(\mathbf{R}_b)B_{kl}^{b}(\mathbf{R}_b,\mathbf{R}_a)\Xi_{lj}^{a}(\mathbf{R}_a)=\frac{s_{ij}^{(3b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}, \qquad (S.32)$$

$$Q_{ijkl}^{(1b)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\mathrm{X}_{mi}^{b}(\mathbf{R}_b)B_{mn}^{b}(\mathbf{R}_b,\mathbf{R}_a)\Omega_{njkl}^{a}(\mathbf{R}_a)=-\varepsilon_0^2\frac{u_{ijkl}^{(1b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^3}, \qquad (S.33)$$

$$Q_{ijkl}^{(2b)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\Xi_{mi}^{b}(\mathbf{R}_b)A_{mn}^{b}(\mathbf{R}_b)\Omega_{njkl}^{b}(\mathbf{R}_b)+$$

$$+\Omega_{mljk}^{b}(\mathbf{R}_b)A_{mn}^{b}(\mathbf{R}_b)\Xi_{ni}^{b}(\mathbf{R}_b)+\Omega_{mljk}^{b}(\mathbf{R}_b)B_{mn}^{b}(\mathbf{R}_b,\mathbf{R}_a)\mathrm{X}_{ni}^{a}(\mathbf{R}_a)= \quad \text{(S.34)}$$

$$=-\varepsilon_0^2\frac{u_{ijkl}^{(2a)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^3}\ ,$$

$$Q_{ijkl}^{(3b)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\Omega_{mijk}^{b}(\mathbf{R}_b)B_{mn}^{b}(\mathbf{R}_b,\mathbf{R}_a)\Xi_{nl}^{a}(\mathbf{R}_a)=-\varepsilon_0^2\frac{u_{ijkl}^{(3b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}, \quad \text{(S.35)}$$

$$Q_{ijkl}^{(4b)}(\mathbf{R}_a,\mathbf{R}_b,\omega)=\Xi_{mi}^{b}(\mathbf{R}_b)B_{mn}^{b}(\mathbf{R}_b,\mathbf{R}_a)\Omega_{njkl}^{a}(\mathbf{R}_a)=-\varepsilon_0^2\frac{u_{ijkl}^{(4b)}}{\left|\mathbf{R}_a-\mathbf{R}_b\right|^6}. \quad \text{(S.36)}$$

In the formulae (S.23)-(S.36) the dependence on the distance between the nanoparticles was revealed and the following denominations were used:

$$s_{ij}^{(1a)}=\mathrm{X}_{ki}^{a}(\mathbf{R}_a)A_{kl}^{a}(\mathbf{R}_a)\xi_{lj}^{a}+\xi_{ij}^{a}, \quad \text{(S.37)}$$

$$t_{ij}^{(1a)}=\xi_{li}^{b}b_{kl}^{(a)}\xi_{kj}^{a}, \quad \text{(S.38)}$$

$$s_{ij}^{(2a)}=\xi_{ki}^{a}A_{kl}^{a}(\mathbf{R}_a)\xi_{lj}^{a}, \quad \text{(S.39)}$$

$$t_{ij}^{(2a)}=\xi_{ki}^{a}b_{kl}^{(a)}\mathrm{X}_{lj}^{b}(\mathbf{R}_b), \quad \text{(S.40)}$$

$$s_{ij}^{(3a)}=\mathrm{X}_{ki}^{a}(\mathbf{R}_a)b_{kl}^{a}\xi_{lj}^{b}, \quad \text{(S.41)}$$

$$u_{ijkl}^{(1a)}=\mathrm{X}_{mi}^{a}(\mathbf{R}_a)b_{mn}^{a}\tilde{\beta}_{njkl}^{b}, \quad \text{(S.42)}$$

$$u_{ijkl}^{(2a)}=\xi_{mi}^{a}A_{mn}^{a}(\mathbf{R}_a)\tilde{\beta}_{njkl}^{a}+\tilde{\beta}_{mljk}^{a}A_{mn}^{a}(\mathbf{R}_a)\xi_{ni}^{a}+\tilde{\beta}_{mljk}^{a}b_{mn}^{a}\mathrm{X}_{ni}^{b}(\mathbf{R}_b), \quad \text{(S.43)}$$

$$u_{ijkl}^{(3a)}=\tilde{\beta}_{mijk}^{a}b_{mn}^{a}\xi_{nl}^{b}, \quad \text{(S.44)}$$

$$u_{ijkl}^{(4a)}=\xi_{mi}^{a}b_{mn}^{a}\tilde{\beta}_{njkl}^{b}, \quad \text{(S.45)}$$

$$s_{ij}^{(1b)}=\mathrm{X}_{ki}^{b}(\mathbf{R}_b)A_{kl}^{b}(\mathbf{R}_b)\xi_{lj}^{b}+\xi_{ij}^{b}, \quad \text{(S.46)}$$

$$t_{ij}^{(1b)}=\xi_{li}^{a}b_{kl}^{(b)}\xi_{kj}^{b}, \quad \text{(S.47)}$$

$$s_{ij}^{(2b)}=\xi_{ki}^{b}A_{kl}^{b}(\mathbf{R}_b)\xi_{lj}^{b}, \quad \text{(S.48)}$$

$$t_{ij}^{(2b)} = \xi_{ki}^{b} b_{kl}^{(b)} \mathrm{X}_{lj}^{a}(\mathbf{R}_a)\,, \tag{S.49}$$

$$s_{ij}^{(3b)} = \mathrm{X}_{ki}^{b}(\mathbf{R}_b) b_{kl}^{b} \xi_{lj}^{a}\,, \tag{S.50}$$

$$u_{ijkl}^{(1b)} = \mathrm{X}_{mi}^{b}(\mathbf{R}_b) b_{mn}^{b} \tilde{\beta}_{njkl}^{a}\,, \tag{S.51}$$

$$u_{ijkl}^{(2b)} = \xi_{mi}^{b} A_{mn}^{b}(\mathbf{R}_b) \tilde{\beta}_{njkl}^{b} + \tilde{\beta}_{mljk}^{b} A_{mn}^{b}(\mathbf{R}_b) \xi_{ni}^{b} + \tilde{\beta}_{mljk}^{b} b_{mn}^{b} \mathrm{X}_{ni}^{a}(\mathbf{R}_a)\,, \tag{S.52}$$

$$u_{ijkl}^{(3b)} = \tilde{\beta}_{mijk}^{b} b_{mn}^{b} \xi_{nl}^{a}\,, \tag{S.53}$$

$$u_{ijkl}^{(4b)} = \xi_{mi}^{b} b_{mn}^{b} \tilde{\beta}_{njkl}^{a}\,. \tag{S.54}$$

Consequently, we have the following expression for the interaction potential:

$$
\begin{aligned}
U(d) = -\frac{1}{2}\int_0^\infty \frac{d\omega}{2\pi}\Bigg\{ \int_{V_a} d\mathbf{R}_a \Bigg[ & \varepsilon_0 \left( \frac{s_{ij}^{(1a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3} + \frac{t_{ij}^{(1a)}}{|\mathbf{R}_a-\mathbf{R}_b|^9} \right) \left\langle E_i^{ex}(\mathbf{R}_a) E_j^{ex}(\mathbf{R}_b) \right\rangle + \\
& + \varepsilon_0 \left( \frac{s_{ij}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} + \frac{t_{ij}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \right) \left\langle E_i^{ex}(\mathbf{R}_b) E_j^{ex}(\mathbf{R}_b) \right\rangle + \varepsilon_0 \frac{s_{ij}^{(3a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \left\langle E_i^{ex}(\mathbf{R}_a) E_j^{ex}(\mathbf{R}_a) \right\rangle + \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(1a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3} \left\langle E_i^{ex}(\mathbf{R}_a) E_j^{ex}(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) E_l^{ex}(\mathbf{R}_b) \right\rangle + \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3} \left\langle E_i^{ex}(\mathbf{R}_b) E_j^{ex}(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) E_l^{ex}(\mathbf{R}_a) \right\rangle + \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(3a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \left\langle E_i^{ex}(\mathbf{R}_a) E_j^{ex}(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) E_l^{ex}(\mathbf{R}_a) \right\rangle + \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(4a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \left\langle E_i^{ex}(\mathbf{R}_b) E_j^{ex}(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) E_l^{ex}(\mathbf{R}_b) \right\rangle \Bigg] + \\
+ \int_{V_b} d\mathbf{R}_b \Bigg[ & \varepsilon_0 \left( \frac{s_{ij}^{(1b)}}{|\mathbf{R}_a-\mathbf{R}_b|^3} + \frac{t_{ij}^{(1b)}}{|\mathbf{R}_a-\mathbf{R}_b|^9} \right) \left\langle E_i^{ex}(\mathbf{R}_b) E_j^{ex}(\mathbf{R}_a) \right\rangle + \\
& + \varepsilon_0 \left( \frac{s_{ij}^{(2b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} + \frac{t_{ij}^{(2b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \right) \left\langle E_i^{ex}(\mathbf{R}_a) E_j^{ex}(\mathbf{R}_a) \right\rangle + \varepsilon_0 \frac{s_{ij}^{(3b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \left\langle E_i^{ex}(\mathbf{R}_b) E_j^{ex}(\mathbf{R}_b) \right\rangle - \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(1b)}}{|\mathbf{R}_a-\mathbf{R}_b|^3} \left\langle E_i^{ex}(\mathbf{R}_b) E_j^{ex}(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) E_l^{ex}(\mathbf{R}_a) \right\rangle - \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3} \left\langle E_i^{ex}(\mathbf{R}_a) E_j^{ex}(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) E_l^{ex}(\mathbf{R}_b) \right\rangle - \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(3b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \left\langle E_i^{ex}(\mathbf{R}_b) E_j^{ex}(\mathbf{R}_b) E_k^{ex}(\mathbf{R}_b) E_l^{ex}(\mathbf{R}_b) \right\rangle + \\
& - \varepsilon_0^2 \frac{u_{ijkl}^{(4b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6} \left\langle E_i^{ex}(\mathbf{R}_a) E_j^{ex}(\mathbf{R}_a) E_k^{ex}(\mathbf{R}_a) E_l^{ex}(\mathbf{R}_a) \right\rangle \Bigg]\Bigg\} .
\end{aligned}
$$

The following steps are concerned with the calculation of the field commutators for the electric field. As we consider the interaction due to vacuum field fluctuations, we may use the following expression for field commutators via the electrodynamic Green function of the medium[2]:

[2] S. Y. Buhmann, "*Dispersion Forces I, Macroscopic Quantum Electrodynamics and Ground-State Casimir, Casimir–Polder and van der Waals Forces*," Springer-Verlag Berlin Heidelberg, Berlin, Germany, 2012.

$$\left\langle E_i^{(0)}(\mathbf{R}_a,\omega)E_j^{(0)}(\mathbf{R}_b,\omega)\right\rangle_\omega = -\frac{\omega^2}{c^2}\hbar\,\mathrm{Im}\,G_{ij}^{(0)}(\mathbf{R}_a-\mathbf{R}_b,\omega)\,\mathrm{sign}\,\omega,$$

which gives us

$$\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_a)\right\rangle = \left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)\right\rangle = \frac{2}{3}\hbar\left(\frac{\omega}{c}\right)^3\delta_{ij}\,\mathrm{sgn}(\omega),$$

$$\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_b)\right\rangle = \left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_a)\right\rangle = -\hbar\frac{3n_in_j-\delta_{ij}}{\left|\mathbf{R}_b-\mathbf{R}_a\right|^3}\ ,\quad n_i=\frac{\left(\mathbf{R}_a-\mathbf{R}_b\right)_i}{\left|\mathbf{R}_a-\mathbf{R}_b\right|}.$$

The fourth order commutators are expressed vie the second order ones as in[3]:

$$\begin{aligned}\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle &= \left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_b)\right\rangle\left\langle E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle + \\ &+\left\langle E_i^{ex}(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_b)\right\rangle\left\langle E_j^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle + \\ &+\left\langle E_i^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_b)\right\rangle\left\langle E_k^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)\right\rangle = \\ &= -\frac{F_{ijkl}(\omega)}{\left|\mathbf{R}_b-\mathbf{R}_a\right|^3},\end{aligned}$$

[3] Yu. S. Barash, V. L. Ginzburg, "Some problems in the theory of van der Waals forces," Soviet Physics Uspekhi, vol. 27, pp. 467–491, 1984.

$$\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle = \left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_a)\right\rangle\left\langle E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle + \\ + \left\langle E_i^{ex}(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_a)\right\rangle\left\langle E_j^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle + \\ + \left\langle E_i^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_a)\right\rangle\left\langle E_j^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle = \\ = -\frac{F_{ijkl}(\omega)}{\left|\mathbf{R}_b - \mathbf{R}_a\right|^3} \ ,$$

$$\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_a)E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle = 3\left\langle E_i^{ex}(\mathbf{R}_a)E_j^{ex}(\mathbf{R}_a)\right\rangle\left\langle E_k^{ex}(\mathbf{R}_a)E_l^{ex}(\mathbf{R}_a)\right\rangle = \\ = 2\hbar\left(\frac{\omega}{c}\right)^6 \delta_{ij}\delta_{kl}\delta_{jk}\,\mathrm{sgn}^2(\omega) \ ,$$

$$\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle = 3\left\langle E_i^{ex}(\mathbf{R}_b)E_j^{ex}(\mathbf{R}_b)\right\rangle\left\langle E_k^{ex}(\mathbf{R}_b)E_l^{ex}(\mathbf{R}_b)\right\rangle = \\ = 2\hbar\left(\frac{\omega}{c}\right)^6 \delta_{ij}\delta_{kl}\delta_{jk}\,\mathrm{sgn}^2(\omega) \ ,$$

where:

$$F_{ijkl}(\omega) = \frac{2}{3}\mathrm{sgn}(\omega)\frac{\hbar^2\omega^3}{c^3}\left[\delta_{ij}(3n_k n_l - \delta_{kl}) + \delta_{ik}(3n_j n_l - \delta_{jl}) + \delta_{il}(3n_k n_j - \delta_{kj})\right].$$

Using the expressions for the local field commutators we obtain:

$$U(d)=-\frac{1}{2}\int_0^\infty\frac{d\omega}{2\pi}\left\{\int_{V_a}d\mathbf{R}_a\left[-\hbar\varepsilon_0\left(\frac{s_{ij}^{(1a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3}+\frac{t_{ij}^{(1a)}}{|\mathbf{R}_a-\mathbf{R}_b|^9}\right)\frac{3n_in_j-\delta_{ij}}{|\mathbf{R}_b-\mathbf{R}_a|^3}+\right.\right.$$

$$+\varepsilon_0\left(\frac{s_{ij}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}+\frac{t_{ij}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}\right)\frac{2}{3}\hbar\left(\frac{\omega}{c}\right)^3\delta_{ij}\,\mathrm{sgn}(\omega)+$$

$$+\varepsilon_0\frac{s_{ij}^{(3a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}\frac{2}{3}\hbar\left(\frac{\omega}{c}\right)^3\delta_{ij}\,\mathrm{sgn}(\omega)+\varepsilon_0^2\frac{u_{ijkl}^{(1a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3}\frac{F_{ijkl}(\omega)}{|\mathbf{R}_b-\mathbf{R}_a|^3},+$$

$$+\varepsilon_0^2\frac{u_{ijkl}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3}\frac{F_{ijkl}(\omega)}{|\mathbf{R}_b-\mathbf{R}_a|^3}-\varepsilon_0^2\frac{u_{ijkl}^{(3a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}2\hbar\left(\frac{\omega}{c}\right)^6\delta_{ij}\delta_{kl}\delta_{jk}\,\mathrm{sgn}^2(\omega)+$$

$$\left.-\varepsilon_0^2\frac{u_{ijkl}^{(4a)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}2\hbar\left(\frac{\omega}{c}\right)^6\delta_{ij}\delta_{kl}\delta_{jk}\,\mathrm{sgn}^2(\omega)\right]+$$

$$+\int_{V_b}d\mathbf{R}_b\left[-\hbar\varepsilon_0\left(\frac{s_{ij}^{(1b)}}{|\mathbf{R}_a-\mathbf{R}_b|^3}+\frac{t_{ij}^{(1b)}}{|\mathbf{R}_a-\mathbf{R}_b|^9}\right)\frac{3n_in_j-\delta_{ij}}{|\mathbf{R}_b-\mathbf{R}_a|^3}+\right.$$

$$+\varepsilon_0\left(\frac{s_{ij}^{(2b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}+\frac{t_{ij}^{(2b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}\right)\frac{2}{3}\hbar\left(\frac{\omega}{c}\right)^3\delta_{ij}\,\mathrm{sgn}(\omega)+$$

$$+\varepsilon_0\frac{s_{ij}^{(3b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}\frac{2}{3}\hbar\left(\frac{\omega}{c}\right)^3\delta_{ij}\,\mathrm{sgn}(\omega)+\varepsilon_0^2\frac{u_{ijkl}^{(1b)}}{|\mathbf{R}_a-\mathbf{R}_b|^3}\frac{F_{ijkl}(\omega)}{|\mathbf{R}_b-\mathbf{R}_a|^3}+$$

$$+\varepsilon_0^2\frac{u_{ijkl}^{(2a)}}{|\mathbf{R}_a-\mathbf{R}_b|^3}\frac{F_{ijkl}(\omega)}{|\mathbf{R}_b-\mathbf{R}_a|^3}-\varepsilon_0^2\frac{u_{ijkl}^{(3b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}2\hbar\left(\frac{\omega}{c}\right)^6\delta_{ij}\delta_{kl}\delta_{jk}\,\mathrm{sgn}^2(\omega)-$$

$$\left.\left.-\varepsilon_0^2\frac{u_{ijkl}^{(4b)}}{|\mathbf{R}_a-\mathbf{R}_b|^6}2\hbar\left(\frac{\omega}{c}\right)^6\delta_{ij}\delta_{kl}\delta_{jk}\,\mathrm{sgn}^2(\omega)\right]\right\}.$$

Determining of the explicit dependence of the distance between nanoparticles $|\mathbf{R}_a-\mathbf{R}_b|=d$ gives:

$$U(d)=\frac{M}{d^{12}}-\frac{N}{d^6}\ ,\quad d=|\mathbf{R}_a-\mathbf{R}_b|, \tag{S.55}$$

where

$$M=\frac{\hbar\varepsilon_0}{2}\left(3n_in_j-\delta_{ij}\right)\int_0^\infty\frac{d\omega}{2\pi}\left[\int_{V_a}d\mathbf{R}_a t_{ij}^{(1a)}+\int_{V_b}d\mathbf{R}_b t_{ij}^{(1b)}\right], \tag{S.56}$$

$$N=\frac{\hbar\varepsilon_0}{2}\int_0^\infty\frac{d\omega}{2\pi}\cdot$$

$$\cdot\Bigg\{\int_{V_a}d\mathbf{R}_a\left[-s_{ij}^{(1a)}\left(3n_in_j-\delta_{ij}\right)+\left(s_{ij}^{(2a)}+t_{ij}^{(2a)}+s_{ij}^{(3a)}\right)\frac{2}{3}\left(\frac{\omega}{c}\right)^3\delta_{ij}\,\mathrm{sgn}(\omega)\right]+$$

$$+\varepsilon_0\int_{V_a}d\mathbf{R}_a\left[\left(u_{ijkl}^{(1a)}F_{ijkl}(\omega)+u_{ijkl}^{(2a)}F_{ijkl}(\omega)\right)-\left(u_{ijkl}^{(3a)}+u_{ijkl}^{(4a)}\right)2\left(\frac{\omega}{c}\right)^6\delta_{ij}\delta_{kl}\delta_{jk}\right]+ \quad \text{(S.57)}$$

$$+\int_{V_b}d\mathbf{R}_b\left[-s_{ij}^{(1b)}\left(3n_in_j-\delta_{ij}\right)+\left(s_{ij}^{(2b)}+t_{ij}^{(2b)}+s_{ij}^{(3b)}\right)\frac{2}{3}\left(\frac{\omega}{c}\right)^3\delta_{ij}\,\mathrm{sgn}(\omega)\right]+$$

$$+\varepsilon_0\int_{V_b}d\mathbf{R}_b\left[\left(u_{ijkl}^{(1b)}F_{ijkl}(\omega)+u_{ijkl}^{(2a)}F_{ijkl}(\omega)\right)-\left(u_{ijkl}^{(3b)}+u_{ijkl}^{(4b)}\right)2\left(\frac{\omega}{c}\right)^6\delta_{ij}\delta_{kl}\delta_{jk}\right]\Bigg\}\ .$$