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Research Article

On Second-Order Cone Functions

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We consider the second-order cone function (SOCF) $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ defined by $f(x) = c^T x + d - \|Ax + b\|$, with parameters $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$. Every SOCF is concave. We give necessary and sufficient conditions for strict concavity of f. The parameters A and b are not uniquely determined. We show that every SOCF can be written in the form $f(x) = c^T x + d - \sqrt{\delta^2 + (x - x_*)^T M (x - x_*)}$. We give necessary and sufficient conditions for the parameters c, d, δ, $M = A^T A$, and x_* to be uniquely determined. We also give necessary and sufficient conditions for f to be bounded above.

1. Introduction

Second-order cone programming is an important convex optimization problem [1–4]. A second-order cone constraint has the following form: $\|Ax + b\| \leq c^T x + d$, where $\|\cdot\|$ is the Euclidean norm. This second-order cone constraint is equivalent to the inequality $f(x) \geq 0$, where f is what we call a second-order cone function. The solution set of the constraint is convex, and the function f is concave [1, 5].

In the following definition, we use $\mathbb{R}$ to denote the set of real numbers and $\mathbb{R}^{m\times n}$ to denote the set of $m \times n$ matrices with real entries. Of course, m and n are positive integers.

Definition 1. A second-order cone function (SOCF) is a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ that can be written as

$$f(x) = c^T x + d - \|Ax + b\|, \tag{1}$$

with parameters $c \in \mathbb{R}^n$, $d \in \mathbb{R}$, $A \in \mathbb{R}^{m\times n}$, and $b \in \mathbb{R}^m$.

In second-order cone programming, a linear function of x is minimized subject to one or more second-order cone constraints, along with the constraint $Fx = g$, where $F \in \mathbb{R}^{p\times n}$ and $g \in \mathbb{R}^p$. The solution set of $Fx = g$ is an affine subspace, and we will show that the restriction of a SOCF to an affine subspace is another SOCF. Thus, from a mathematical point of view, the constraint $Fx = g$ is not necessary, although in applications it can be convenient. In this paper, we do not consider the constraint $Fx = g$ but instead focus on understanding the family of SOCFs.

There are interior-point methods for solving second-order cone programming problems. These methods usually use SOCFs to impose the second-order cone constraints [2, 5–7]. Solvers for second-order cone programming problems include CVXOPT and MATLAB [8, 9]. The study of second-order cone programming and its applications has continued to generate interest for over three decades [3, 10–15].

The current research was started to get a deeper understanding of SOCFs to improve interior-point algorithms for finding the weighted analytic center of a system of second-order cone constraints [7, 16, 17]. The current work can lead to improved algorithms.

In this paper, we give a thorough description of the family of SOCFs. In equation (1), the parameters A and b are not uniquely determined, since $\|Ax + b\| = \|Q(Ax + b)\| = \|(QA)x + (Qb)\|$ for any orthogonal $m \times m$ matrix Q. We show that every SOCF can be written in the following form:

$$f(x) = c^T x + d - \sqrt{\delta^2 + (x - x_*)^T M (x - x_*)}, \quad (2)$$

with the parameters $\delta \geq 0$ and $x_* \in \mathbb{R}^n$, and the positive semidefinite $M = A^T A \in \mathbb{R}^{n\times n}$ replace the parameters A and b. We show that these new parameters are unique if and only if M is positive definite.

It is known that every SOCF f is concave [1, 5]. We show that f is strictly concave if and only if $\operatorname{rank}(A) = n$ and $b \notin \operatorname{col}(A)$, where $\operatorname{col}(A)$ denotes the column space of A. In terms of the new parameters, the SOCF is strictly concave if and only if M is positive definite and $\delta > 0$.

In the case where M is positive definite, we show that f is bounded above if and only if $c^T M^{-1} c \leq 1$. We show that the convex set $\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$ is bounded if and only if M is positive definite and $c^T M^{-1} c < 1$.

Our results have computational implications for convex optimization problems involving second-order constraints such as the problem of minimizing weighted barrier functions presented in [16, 17]. This is related to the problem of finding a weighted analytic center for second-order cone constraints given in [7]. There are also computational implications for the problem of computing the region of weighted analytic centers of a system of several second-order cone constraints. This is under investigation as part of our current research is an extension of the work given in [7].

In the problems presented in [7, 16, 17], the boundedness of the feasible region guarantees the existence of a minimizer, and the strict convexity of the barrier function guarantees the uniqueness of the minimizer. Also, the strict convexity of the barrier function affects how quickly we can find the minimizer using these algorithms. The determination of the strict concavity of f is related to the strict convexity of the barrier function. The boundedness of the feasible region of the SOC constraints system is also related to the boundedness of f. If a single f is bounded, then the feasible region of the SOC constraints system is also bounded.

Convex optimization algorithms perform well and more efficiently when the problem is known to be bounded and the objective function is strictly convex. If a second-order cone function is strictly concave, its gradient and Hessian matrix is defined, and the Hessian is invertible. The corresponding barrier function is similarly well-behaved, and Newton's method and Newton-based methods work well for the problem. However, many optimization problems are not bounded or have objective functions that are not strictly convex. Our results would allow one to recognize convex optimization problems involving second-order cone constraints (as in [7, 16, 17]) that can be solved efficiently, or to assist in reformulating those that are hard to solve.

2. Properties of Second-Order Cone Functions

The SOCFs of $\mathbb{R}$ (that is, $n = 1$) are the simplest to understand, and give insight into the general case.

Example 1. We consider $f: \mathbb{R} \longrightarrow \mathbb{R}$ defined by equation (1) with $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $b = \begin{bmatrix} -x_* \\ \delta \end{bmatrix}$. Thus, $Ax + b = \begin{bmatrix} x - x_* \\ \delta \end{bmatrix}$, and $\|Ax + b\| = \sqrt{\delta^2 + (x - x_*)^2}$, so $f(x) = \mathrm{c}x + d - \sqrt{\delta^2 + (x - x_*)^2}$. Figure 1 shows several graphs with various values of the real parameters δ, x_*, c, and d. If $\delta \neq 0$, then f is smooth and strictly concave, as shown by the solid graphs. If $\delta = 0$, then $f(x) = \mathrm{d}x + d - |x - x_*|$ is piecewise linear with a corner at $(x_*, \mathrm{c}x_* + d)$, as shown by the dashed graphs. Note that $f(x_*) = \mathrm{c}x_* + d - |\delta|$ for any value of δ, so the solid graphs in Figure 1 (with $\delta = 0.2$) pass a distance of 0.2 below the corner of the dashed graphs (with $\delta = 0$), as indicated by the double arrows.

One important property of SOCFs is that their restriction to an affine subspace is another SOCF. We will frequently restrict to a 1-dimensional affine subspace.

Remark 2. Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be written in the form of equation (1). The restriction of f to the affine subspace $\{x_0 + By \mid y \in \mathbb{R}^k\}$, for some $x_0 \in \mathbb{R}^n$ and $B \in \mathbb{R}^{n\times k}$ is

$$f(x_0 + \mathrm{By}) = (c^T B) y + (c^T x_0 + d) - \|(\mathrm{AB}) y + (A x_0 + b)\|, \quad (3)$$

which is a SOCF on $\mathbb{R}^k$ with the variable y.

We recall that a function $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ is concave provided that $f((1-t)x_0 + \mathrm{t}x_1) \geq (1-t) f(x_0) + \mathrm{t}f(x_1)$ for all $x_0 \neq x_1 \in \mathbb{R}^n$, and all $t \in (0, 1)$. The function is strictly concave if the inequality is strict. A twice differentiable function $f: \mathbb{R} \longrightarrow \mathbb{R}$ is concave if $f''(x) \leq 0$ for all x, and strictly concave if the inequality is strict.

Lemma 3. *Let $f: \mathbb{R} \longrightarrow \mathbb{R}$ be the general SOCF of one variable, defined by $f(x) = cx + d - \|Ax + b\|$ with parameters $c, d \in \mathbb{R}$, and $A, b \in \mathbb{R}^m$. The function f is concave for all parameters, and f is strictly concave if and only if $A \neq 0$ and $b \notin col(\mathrm{A})$.*

Proof. If $A = 0$, then $f(x) = \mathrm{c}x + d - \|b\|$ is linear, and hence concave but not strictly concave.

Assume $A \neq 0$. Then, Ax_*, where $x_* = -(A^T b)/(A^T A)$ is the point in $\operatorname{col}(A) = \operatorname{span}(A)$ closest to $-b$. Let $\delta = \|Ax_* + b\|$ be the distance from Ax_* to $-b$. Thus, $\|Ax + b\|^2 = \delta^2 + \|A(x - x_*)\|^2$ by the Pythagorean theorem, and $f(x) = \mathrm{c}x + d - \sqrt{\delta^2 + \|A(x - x_*)\|^2} = \mathrm{c}x + d - \sqrt{\delta^2 + (A^T A)(x - x_*)^2}$. The constant $A^T A$ is a positive real number. The geometry is shown in Figure 2. Note that $\delta = 0$ if and only if $b \in \operatorname{col}(A)$. If $\delta = 0$, then $f(x) = \mathrm{c}x + d - \sqrt{A^T A}|x - x^*|$ is piecewise linear with a downward bend at x_*, and hence concave but not strictly concave.

So far, we have proved that f is concave but not strictly concave if $A = 0$ or $b \in \operatorname{col}(A)$.

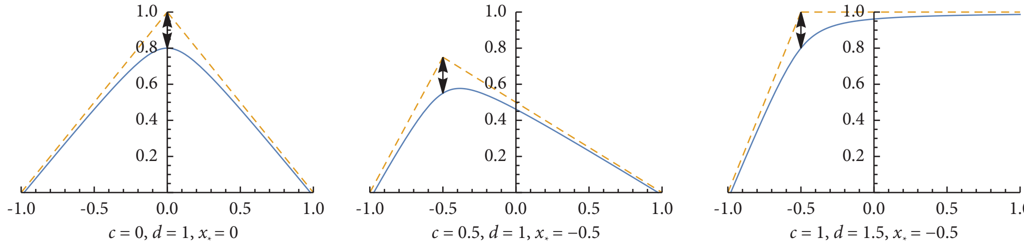

Figure 1: Graphs of second-order cone functions $f(x) = \mathrm{c}x + d - \sqrt{\delta^2 + (x - x_*)^2}$, as described in Example 1. In each of the three plots, the parameters c, d, and x_* are indicated. The dashed curve has $\delta = 0$, and the solid curve has $\delta = 0.2$.

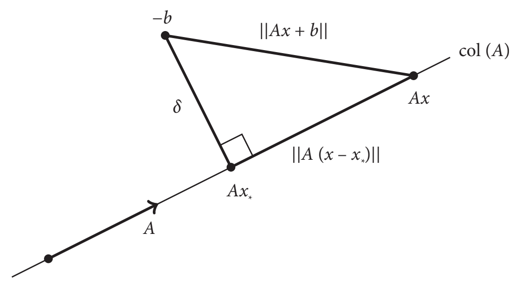

Figure 2: The geometry of a SOCF on $\mathbb{R}$. In this case, $A \in \mathbb{R}^2 = \mathbb{R}^{2\times 1}$. Note that Ax_* is the point in $\mathrm{col}(A)$ that is closest to $-b$ (see Lemma 3).

Assume $A \neq 0$ and $b \notin \mathrm{col}(A)$. Then, $\delta > 0$, and f is strictly concave, since

$$f''(x) = \frac{-\delta^2 A^T A}{\left(\delta^2 + A^T A (x - x_*)^2\right)^{3/2}}, \tag{4}$$

is defined and negative for all x. □

Theorem 4. *Every second-order cone function f is concave. Furthermore, f is strictly concave if and only if rank$(A) = n$ and $b \notin$ col(A), by using the parameters in Definition 1.*

Proof. Let $x_0 \neq x_1 \in \mathbb{R}^n$, and we define that $v = x_1 - x_0$. Let $g: \mathbb{R} \longrightarrow \mathbb{R}$ be defined by $g(t){:} = f((1-t)x_0 + tx_1) = f(x_0 + \mathrm{t}v) = c^T(x_0 + \mathrm{t}v) + d - \|A(x_0 + \mathrm{t}v) + b\|$. It follows directly from the definition that f is (strictly) concave if and only if g is (strictly) concave for all $x_0 \neq x_1$. Note that $Av \in \mathbb{R}^m$. If $Av = 0$, then g is linear. If $Av \neq 0$, then, we have

$$g(t) = \tilde{c}t + \tilde{d} - \sqrt{\tilde{\delta}^2 + \|Av\|^2 (t - t_*)^2}, \tag{5}$$

where $\tilde{c} = c^T v$, $\tilde{d} = c^T x_0 + d$, $\tilde{\delta} = \|A(x_0 + t_* v) + b\|$, and $t_* = -(Av)^T(Ax_0 + b)/(Av)^T Av$ are all real numbers. Thus, g is a second-order cone function of one variable. By Lemma 3, g is concave for all choices of x_0 and x_1, and hence f is concave.

Since $A \in \mathbb{R}^{m\times n}$, it follows that $\mathrm{rank}(A) \leq n$. If $\mathrm{rank}(A) < n$, then $A^T A$ is singular, and there exists $x_0 \neq x_1 = x_0 + v$ such that $Av = 0$ and hence g is linear. If $b \in \mathrm{col}(A)$, then there exists x_0 such that $Ax_0 + b = 0$. Thus, $t_* = 0$ and $\tilde{\delta} = 0$, and g is piecewise linear with a downward corner. Thus, if $\mathrm{rank}(A) < n$ or $b \in \mathrm{col}(A)$ (or both), we can find $x_0 \neq x_1$ such that g is concave but not strictly concave, and hence f is not strictly concave.

Now, assume $\mathrm{rank}(A) = n$ and $b \notin \mathrm{col}(A)$. It follows that $Av \neq 0$ and $\tilde{\delta} > 0$ for all $x_0 \neq x_1$. Lemma 3 implies that g is strictly concave for all $x_0 \neq x_1$, and it follows that f is strictly concave. □

Note that $A \in \mathbb{R}^{n\times n}$ cannot satisfy $\mathrm{rank}(A) = n$ and $b \notin \mathrm{col}(A)$. Therefore, any SOCF with $A \in \mathbb{R}^{n\times n}$ is concave but not strictly concave.

Example 2. We give four examples of SOCFs on $\mathbb{R}^2$, with different truth values of $\mathrm{rank}(A) = 2$ or $b \in \mathrm{col}(A)$. These SOCFs have $c = 0$ and $d = 0$, so $f(x) = -\|Ax + b\|$ (Figure 3).

(a) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}$ yields $f(x) = -\sqrt{0.09 + x_1^2 + x_2^2}$.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yields $f(x) = -\sqrt{x_1^2 + x_2^2}$.

(c) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \\ 0.3 \end{bmatrix}$ yields $f(x) = -\sqrt{0.09 + x_1^2}$.

(d) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $b = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ yields $f(x) = -|x_1|$.

Notice that, we have frequently rewritten $\|Ax + b\|$ in terms of a square root, as shown in Examples 1 and 2. We have also noted that $\|Ax + b\| = \|\mathrm{Q}Ax + \mathrm{Q}b\|$ for any

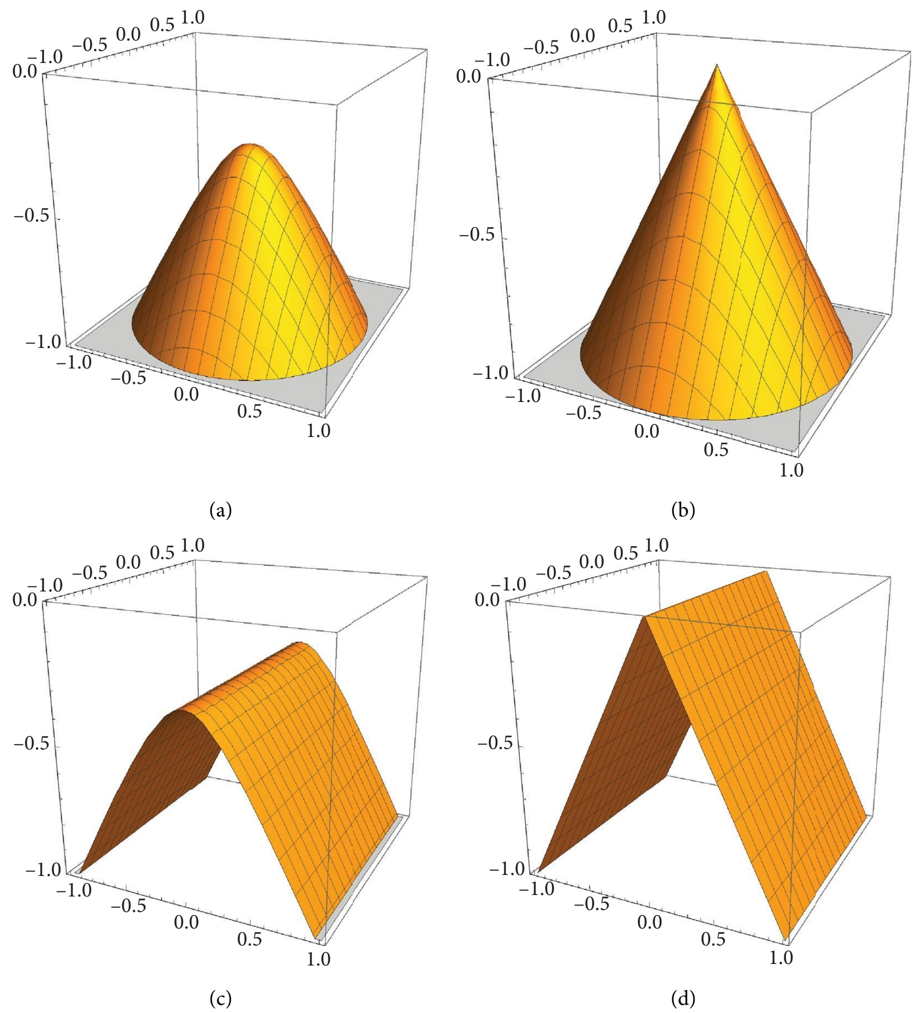

Figure 3: Graphs of four SOCFs on $\mathbb{R}^2$. Note that the graph of the SOCF (b) is indeed a cone. The top row shows functions with $\operatorname{rank}(A) = 2$ and the bottom row shows $\operatorname{rank}(A) = 1$. The left column shows $b \notin \operatorname{col}(A)$ and the right column shows $b \in \operatorname{col}(A)$. All the functions graphed are concave, but only the upper left function is strictly concave, which is in agreement with Theorem 4 (see Example 2).

orthogonal matrix Q, so many different choices of A and b define the same SOCF. The next theorem describes a useful way to write a SOCF.

This theorem uses the Moore–Penrose Inverse of a matrix, also called the pseudoinverse, which has many interesting properties found in [18]. For example, $x = A^{+}b$ is the least squares solution to $Ax = b$, where $A^{+} \in \mathbb{R}^{n\times m}$ is the pseudoinverse of $A \in \mathbb{R}^{m\times n}$.

The next theorem mentions the well-known fact that A^TA is a positive semidefinite matrix, which means that it is symmetric with non-negative eigenvalues. A positive definite matrix is a symmetric matrix with all positive eigenvalues. If $A \in \mathbb{R}^{m\times n}$, then A^TA is positive definite if and only if the rank of A is n.

Theorem 5. *Every SOCF of the form $f(x) = c^Tx + d - \|Ax + b\|$ is identically equal to*

$$f(x) = c^Tx + d - \sqrt{\delta^2 + (x - x_*)^T M (x - x_*)}, \tag{6}$$

where $M = A^TA$ is positive semidefinite, $x_* = -A^{+}b$, and $\delta = \|Ax_* + b\|$.

Proof. It is well-known that the least squares solution to $Ax = -b$ is $x_* = -A^{+}b$, and that $Ax_* = -AA^{+}b$ is the orthogonal projection of $-b$ onto $\operatorname{col}(A)$. That is, Ax_* is the point in $\operatorname{col}(A)$ that is closest to $-b$. Thus, the distance squared from Ax to $-b$ is the distance squared from Ax to Ax_* plus the distance squared from Ax_* to $-b$. That is,

$$\begin{aligned} \|Ax + b\|^2 &= \|Ax - Ax_*\|^2 + \|Ax_* + b\|^2 \\ &= \|A(x - x_*)\|^2 + \|Ax_* + b\|^2 \\ &= (x - x_*)^T A^T A (x - x_*) + \|Ax_* + b\|^2 \\ &= (x - x_*)^T M (x - x_*) + \delta^2. \end{aligned} \tag{7}$$

The last equality uses the definitions of M and δ. The results are as follows. □

Remark 6. For $A \in \mathbb{R}^{m\times n}$, note that $\operatorname{rank}(A) = n$ if and only if $A^T A \in \mathbb{R}^{n\times n}$ is positive definite. The definition of δ in Theorem 5 makes it clear that $b \in \operatorname{col}(A)$ if and only if $\delta = 0$. Therefore, Theorem 4 implies that a SOCF written in the form of equation (6) is strictly concave if and only if M is positive definite and $\delta > 0$.

Example 3. The left half of Figure 4 shows the critical point and one contour of the SOCF $f(x) = -\|Ax + b\|$, with

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 1 \\ 0 & 2 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1 \\ -1 \end{bmatrix}. \tag{8}$$

The right part of the same figure shows the geometry behind Theorem 5, which describes how to write the function in the form $f(x) = -\sqrt{\delta^2 + (x - x_*)^T M (x - x_*)}$. The calculations show that

$$A^+ = \frac{1}{9}\begin{bmatrix} 5 & -4 & 2 \\ 1 & 1 & 4 \end{bmatrix}, \quad M = A^T A = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}, \quad x_* = \left(\frac{1}{9}, \frac{2}{9}\right), \text{ and } \quad \delta = \frac{5}{3}. \tag{9}$$

The image of the square in $\mathbb{R}^2$ under A is the light blue parallelogram in $\mathbb{R}^3$, shown on the right side of Figure 4. The vectors in $\mathbb{R}^3$ are the first (blue) and second (red) columns of A. These span the column space of A in $\mathbb{R}^3$. The dot in $\mathbb{R}^2$ is x_*, and the dot in the column space is $Ax_* = -\mathrm{AA}^+ b = (1/9, 1/9, 4/9)$, which is the orthogonal projection of $-b$ onto $\operatorname{col}(\mathrm{A})$. The other dot in $\mathbb{R}^3$ is $-b$. The distance from Ax_* to $-b$ is $\delta = 5/3$, so $f(x_*) = -5/3$. The ellipse on the left is the contour of f with height -2. The image of the ellipse under A is the circle on the right, which is the set of points in the column space that are at a distance of 2 from $-b$.

The proof Theorem 5, to follow, is subtle. While it is obvious that changing one parameter will change the function f, it is difficult to eliminate the possibility that more than one parameter can be changed while leaving the function unchanged. For example, with the form of equation (1), the function f is unchanged when $A \mapsto \mathrm{QA}$ and $b \mapsto \mathrm{Qb}$ for an orthogonal matrix Q. The strategy in the proof is to uniquely determine one parameter at a time in a specific order.

Theorem 7. *We assume a SOCF is written in the form of equation (6), and that the same SOCF is written with possibly different parameters satisfying the same requirements, so*

$$f(x) = c^T x + d - \sqrt{\delta^2 + (x - x_*)^T M (x - x^*)} = \tilde{c}^T x + \tilde{d} - \sqrt{\tilde{\delta}^2 + \left(x - \tilde{x}_*\right)^T \tilde{M}\left(x - \tilde{x}^*\right)}, \tag{10}$$

for all x.

(i) If $M = 0$ (the zero matrix), then $\tilde{c} = c$, $\tilde{M} = 0$, $\tilde{d} - \tilde{\delta} = d - \delta$, and $\tilde{x}_*$ is arbitrary.

(ii) If $M \neq 0$, then $\tilde{c} = c$, $\tilde{d} = d$, $\tilde{\delta} = \delta$, $\tilde{M} = M$, and $M\tilde{x}_* = \mathrm{Mx}_*$.

As a consequence, the parameterization of a SOCF in the form of equation (6) is unique if and only if M is positive definite.

Proof. We recall that M and, $\tilde{M}$ are positive semidefinite. It follows that $\mathrm{Mv} = 0$ if and only if $v^T \mathrm{Mv} = 0$. Also, we recall that δ and $\tilde{\delta}$ are non-negative real numbers. □

For nonzero $v \in \mathbb{R}^n$ and $t \in [0, \infty)$, we consider the function $f(\mathrm{vt})$ and its asymptotic behavior as $t \longrightarrow \infty$. If $v^T \mathrm{Mv} = 0$, then $f(\mathrm{vt}) = c^T v t + d - \sqrt{\delta^2 + x_*^T \mathrm{Mx}_*}$. If $v^T \mathrm{Mv} \neq 0$, then, we have

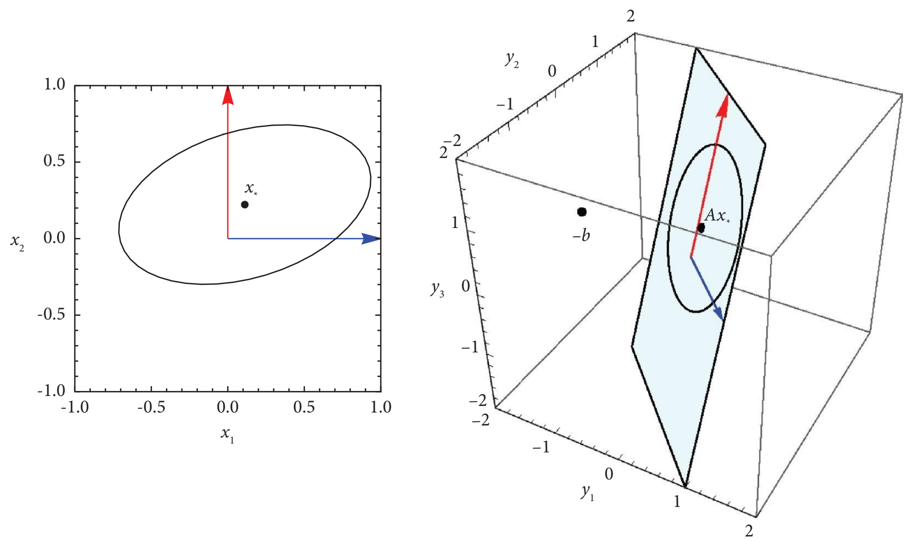

Figure 4: The geometry of the second-order cone function $f(x) = -\|Ax + b\|$, with $A \in \mathbb{R}^{3\times 2}$ and $b \in \mathbb{R}^3$, is defined in Example 3. The function can also be written as $f(x) = -\sqrt{\delta^2 + (x - x_*)^T M (x - x_*)}$, where $M = A^T A$. The maximum of f is at $x_* = -A^+ b$, and the maximum value is $f(x_*) = -\delta$. The orthogonal projection of $-b$ onto the column space of A is $Ax_* = -AA^+ b$. The distance from Ax_* to $-b$ is δ. One contour of f is shown. The image of this contour is a circle of points in $\text{col}(B)$ that are equidistant from $-b$.

$$\begin{aligned} f(\mathrm{vt}) &= c^T vt + d - \sqrt{\delta^2 + (\mathrm{vt} - x_*)^T M (\mathrm{vt} - x_*)} \\ &= c^T vt + d - \sqrt{v^T \mathrm{Mv}t^2 - 2v^T \mathrm{Mx}_* t + x_*^T \mathrm{Mx}_* + \delta^2} \\ &= c^T vt + d - \sqrt{v^T \mathrm{Mv}t}\sqrt{1 + \frac{-2v^T \mathrm{Mx}_* t + x_*^T \mathrm{Mx}_* + \delta^2}{v^T \mathrm{Mv}t^2}} \\ &= \left(c^T v - \sqrt{v^T \mathrm{Mv}}\right) t + d + \frac{v^T \mathrm{Mx}_*}{\sqrt{v^T \mathrm{Mv}}} + O\left(\frac{1}{t}\right) \text{ as } t \longrightarrow \infty. \end{aligned} \tag{11}$$

The third equation uses the fact that $t \geq 0$, and the fourth equation uses the Taylor series $\sqrt{1+\varepsilon} = 1 + \varepsilon/2 + O(\varepsilon^2)$ as $\varepsilon \longrightarrow 0$. The fourth equation describes the slant asymptote of the graph of $f(\mathrm{vt})$, and is crucial for the remainder of the proof.

For all $v \neq 0$, equation (11) implies that

$$\frac{f(\mathrm{vt}) - f(-\mathrm{vt})}{2} = \begin{cases} c^T v t, & \text{if } v^T \mathrm{Mv} = 0, \\ c^T v t + \dfrac{v^T \mathrm{Mx}_*}{v^T \mathrm{Mv}} + O\left(\dfrac{1}{t}\right), & \text{if } v^T \mathrm{Mv} \neq 0. \end{cases} \tag{12}$$

Which is a similar expression where c is replaced by $\tilde{c}$ holds. If $v^T \mathrm{Mv} = 0$, then $\tilde{c}^T v = c^T v$. If $v^T \mathrm{Mv} \neq 0$, then the slope of the slant asymptote is the same for both sets of parameters, so again $\tilde{c}^T v = c^T v$. This holds for all v, so $\tilde{c} = c$.

For all $v \neq 0$, equation (11) implies that

$$\frac{f(\mathrm{vt}) + f(-\mathrm{vt})}{2} = \begin{cases} d - \sqrt{\delta^2 + x_*^T \mathrm{Mx}_*}, & \text{if } v^T \mathrm{Mv} = 0, \\ d - \sqrt{v^T \mathrm{Mv}}\, t + O\left(\dfrac{1}{t}\right), & \text{if } v^T \mathrm{Mv} \neq 0, \end{cases} \tag{13}$$

along with a similar expression where d is replaced by $\tilde{d}$, etc. If $\tilde{M} \neq M$, then there is some vector v such that $v^T \tilde{M} v \neq v^T \mathrm{Mv}$. This leads to a contradiction since the slope of the slant asymptote in equation (13) would be different. Thus, $\tilde{M} = M$.

Assume $M = 0$. Then, $f(x) = c^T x + d - \delta = c^T x + \tilde{d} - \tilde{\delta}$, since $\tilde{c} = c$, and $\tilde{M} = M = 0$. Thus, $\tilde{d} - \tilde{\delta} = d - \delta$.

Assume $M \neq 0$. Then, there exists $v \in \mathbb{R}^n$ that satisfies $\mathrm{Mv} \neq 0$. Using equation (13) with $v^T \tilde{M} v = v^T \mathrm{Mv} \neq 0$, we find that $\tilde{d} = d$. At this point, we conclude that from the equality of the two expressions for f, that $\delta^2 + (x - x_*)^T M(x - x_*) = \tilde{\delta}^2 + (x - \tilde{x}_*)^T M(x - \tilde{x}_*)^T$ for all x. By expanding the quadratic term and canceling like terms, we find that $\delta^2 - 2x^T \mathrm{M} x_* = \tilde{\delta}^2 - 2x^T M\tilde{x}_*$ for all x. Thus, $\tilde{\delta} = \delta$ and $M\tilde{x}_* = \mathrm{M}x_*$.

Now, we show that the parameterization of f is unique if and only if M is positive definite. If M is not positive definite, there exists $x_* \neq \tilde{x}_*$ such that $M\tilde{x}_* = \mathrm{M}x_*$. If M is positive definite, then $M \neq 0$ and M is invertible, so $\tilde{x}_* = x_*$ and all of the parameters are unique.

Example 4. Let $f(x_1, x_2) = -\sqrt{4 + (x_1 - 1)^2}$ be the SOCF on $\mathbb{R}^2$ defined by $c = 0$, $d = 0$, $\delta = 2$, $M = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, and $x_* = (1, 0)$. Note that M is not positive definite. The null space of M is span$\{(0,1)\}$. The parameterization is not unique since any $x_* \in \{(1, a) \mid a \in \mathbb{R}\}$ yields the same SOCF.

While many choices of A and b in the form of equation (1) yield the same function, there is a canonical choice for A and b starting with the function in the form of equation (6). We recall that a positive semidefinite matrix M has a unique positive semidefinite square root, denoted by $M^{1/2}$.

Theorem 8. *Let $M \in \mathbb{R}^{n\times n}$ be positive semidefinite, $x_* \in \mathbb{R}^n$, and $\delta \in \mathbb{R}$. Then, $\delta^2 + (x - x_*)^T M(x - x_*) = \|Ax + b\|^2$ for*

$$A = \begin{bmatrix} M^{1/2} \\ 0 \end{bmatrix}, \text{and} \quad b = \begin{bmatrix} -\mathrm{M}^{1/2}\mathrm{x}_* \\ \delta \end{bmatrix}. \tag{14}$$

The last row of $A \in \mathbb{R}^{(n+1)\times n}$ is all 0s, and the last component of $b \in \mathbb{R}^{n+1}$ is δ.

Proof. Note that $M^{1/2}$ is symmetric, and

$$Ax + b = \begin{bmatrix} M^{1/2}x \\ 0 \end{bmatrix} + \begin{bmatrix} -M^{1/2}x_* \\ \delta \end{bmatrix} = \begin{bmatrix} M^{1/2}(x - x_*) \\ \delta \end{bmatrix}. \tag{15}$$

Thus, $\|Ax + b\|^2 = \delta^2 + (x - x_*)^T M^{1/2} M^{1/2} (x - x_*) = \delta^2 + (x - x_*)^T M(x - x^*)$. □

Remark 9. It follows from this theorem that any SOCF can be defined in the form of equation (1) with $A \in \mathbb{R}^{(n+1)\times n}$. While A is an $m \times n$ matrix with any m, using $m > n + 1$ is never needed.

We recall that any nonconstant SOCF is not bounded below, since it is concave. We give necessary and sufficient conditions for a SOCF to be bounded above with the two theorems. The next theorem assumes that M is positive definite, and the case where M is positive semidefinite is handled in Theorem 13.

Theorem 10. *The SOCF $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ can be written in the form of equation (6).*

$$f(x) = c^T x + d - \sqrt{\delta^2 + (x - x_*)^T M(x - x_*)}, \tag{16}$$

with M positive definite is bounded above if and only if $c^T M^{-1} c \leq 1$.

(1) If $c^T M^{-1} c < 1$ and $\delta = 0$, then x_* is the unique critical point of f, and $f(x_*) = c^T x_* + d$ is the global maximum value of f.

(2) If $c^T M^{-1} c = 1$ and $\delta = 0$, then every point in the ray $\{x_* + tM^{-1}c \mid t \geq 0\}$ is a critical point of f, on which f attains its maximum value of $f(x_*) = c^T x_* + d$.

(3) If $c^T M^{-1} c > 1$ and $\delta = 0$, then x_* is the unique critical point of f, but f is not bounded above.

(4) If $c^T M^{-1} c < 1$ and $\delta > 0$, then $x_{cp} := x_* + \delta M^{-1} c/\sqrt{1 - c^T M^{-1} c}$ is the unique critical point of f, and $f(x_{cp}) = c^T x_{cp} + d - \delta\sqrt{1 - c^T M^{-1} c}$ is the global maximum value of f.

(5) If $c^T M^{-1} c = 1$ and $\delta > 0$, then f has no critical points and f does not have a global maximum value, but f is bounded above by $c^T x_* + d$.

(6) If $c^T M^{-1} c > 1$ and $\delta > 0$, then f has no critical points and f is not bounded above.

Proof. To simplify the proof, we will analyze the SOCF $\tilde{f}(x) := c^T x - \sqrt{\delta^2 + x^T \mathrm{M}x}$. Note that $f(x + x_*) = \tilde{f}(x) + (d + c^T x_*)$, and $\tilde{f}(x - x_*) = f(x) - d + c^T x_*$, so we can easily relate the critical points, and the upper bounds, of f and $\tilde{f}$. □

Case 1. $\delta = 0$. In this case, $\tilde{f}(x) = c^T x - \sqrt{x^T \mathrm{M}x}$. Let v be any nonzero vector in $\mathbb{R}^n$. Since $v^T \mathrm{M}v > 0$, the function $t \mapsto \tilde{f}(vt) = (c^T v)t - \sqrt{v^T \mathrm{M}v}\,|t|$ is not differentiable at $t = 0$. Thus, $0 \in \mathbb{R}^n$ is a critical point of $\tilde{f}$ at which $\tilde{f}$ is not differentiable, and f has a critical point at x_*. To determine if $\tilde{f}$ has a global maximum at 0, we define $g_v: [0, \infty) \longrightarrow \mathbb{R}$ by $g_v(t) = \tilde{f}(\mathrm{v}t) = (c^T v)t - t\sqrt{v^T \mathrm{M}v}$. Note that g_v is a linear function giving the value of $\tilde{f}$ along a ray starting at $0 \in \mathbb{R}^n$ with the direction vector v. The function $\tilde{f}$ is bounded above if and only if the slope of g_v is nonpositive for all directions v.

Let $\mathscr{E} = \{x \in \mathbb{R}^n \mid x^T \mathrm{M}x = 1\}$. Note that $\mathscr{E}$ is an ellipsoid centered at 0, since M is positive definite. Furthermore, $g_v(0) = 0$, so $\tilde{f}$ is bounded above if and only if the maximum value of $\tilde{f}$, restricted to $\mathscr{E}$, is nonpositive. We compute this maximum value by using the method of Lagrange multipliers. The extreme values of $\tilde{f}$ restricted to $\mathscr{E}$ occur at places where $\nabla(c^T x) = \lambda\nabla\sqrt{x^T \mathrm{M}x}$. This is equivalent to $c = \lambda \mathrm{M}x/\sqrt{x^T \mathrm{M}x}$ or $\lambda \mathrm{M}x = c$ since $x^T \mathrm{M}x = 1$ on $\mathscr{E}$. Thus, the extrema of $\tilde{f}$ are at $x = 1/\lambda M^{-1} c$, where λ is determined by $x^T \mathrm{M}x = 1$. Thus, $1/\lambda^2 c^T M^{-1} M M c = 1$, so $\lambda^2 = c^T M^{-1} c$. There are two antipodal points on $\mathscr{E}$, $x_\pm = \pm 1/\sqrt{c^T M^{-1} c} M^{-1} c$, with extreme values of $\tilde{f}$ restricted to $\mathscr{E}$. We see that $\tilde{f}(x_{\pm}) = c^T x_{\pm} - 1 = \pm\sqrt{c^T M^{-1} c} - 1$. The maximum value of $\tilde{f}$ restricted to $\mathscr{E}$ is $\sqrt{c^T M^{-1} c} - 1$, which occurs at x_+. Thus, the maximum slope of g_v occurs when v is a positive scalar multiple of $M^{-1}c$, and that the maximum slope has the same sign as $\sqrt{c^T M^{-1} c} - 1$. Thus, $\tilde{f}$ is bounded above if and only if $c^T M^{-1} c \le 1$.

If $c^T M^{-1} c < 1$, then 0 is the unique critical point of $\tilde{f}$, and $\tilde{f}(0) = 0$ is the global maximum value of $\tilde{f}$. Thus, x_* is the unique critical point of f, and $f(x_*) = c^T x_* + d$ is the global maximum value of f. This proves part 1 in the theorem. If $c^T M^{-1} c = 1$, then the linear function g_v has a slope 0 when $v = M^{-1}c$, and $\tilde{f}$ achieves its maximum value of 0 at each point on the ray from 0 through $M^{-1}c$. Every point in this ray, $\mathscr{C} = \{tM^{-1}c \mid t \ge 0\}$, is a critical point. Translating this result to the original f proves part 2. If $c^T M^{-1} c > 1$, then the slope of g_v is positive for some v. Thus, $\tilde{f}$ has an isolated critical point at 0, but $\tilde{f}$ is not bounded above. This proves part 3 of the theorem.

Case 2. $\delta > 0$. The gradient of $\tilde{f}$ at x is

$$\nabla\tilde{f}(x) = c - \frac{\mathrm{Mx}}{\sqrt{\delta^2 + x^T \mathrm{Mx}}}. \tag{17}$$

In this case, $\tilde{f}$ is smooth, and the critical points of $\tilde{f}$ are solutions to $\nabla\tilde{f}(x) = 0$. Since M is positive definite, $\tilde{f}$ is strictly concave by Theorem 4, and $\tilde{f}$ has at most one critical point. If $\tilde{f}$ has a critical point then it must be a global maximum and hence $\tilde{f}$ is bounded above. We denote the critical point of $\tilde{f}$ as x_{cp}, if it exists, which satisfies $\mathrm{Mx}_{\mathrm{cp}} = c\sqrt{\delta^2 + x_{\mathrm{cp}}{}^T \mathrm{Mx}_{\mathrm{cp}}}$. It follows that the critical point is a scalar multiple of $M^{-1}c$. Let $x_{cp} = \alpha M^{-1}c$. The scalar α satisfies $\alpha = \sqrt{\delta^2 + \alpha^2 (M^{-1}c)^T M (M^{-1}c)} = \sqrt{\delta^2 + \alpha^2 c^T M^{-1} c}$. If $c^T M^{-1} c < 1$, then the unique solution is $\alpha_s = \delta/\sqrt{1 - c^T M^{-1} c}$, and if $c^T M^{-1} c \ge 1$, then there are no solutions for α. Thus, if $c^T M^{-1} c < 1$, the function $\tilde{f}$ has the critical point $\alpha_s M^{-1} c$, and the critical point of f is $x_{\mathrm{cp}} = x_* + \alpha_s M^{-1} c$, and a calculation of $f(x_{\mathrm{cp}})$ completes the proof of part 4.

We have already seen that $\tilde{f}$, and therefore f, has no critical points when $c^T M^{-1} c \ge 1$. The results about boundedness and upper bounds need the following asymptotic analysis. When $\|x\|$ is large, then $x^T \mathrm{Mx}$ is large of order $O(\|x\|^2)$ because M is positive definite, and $\sqrt{\delta^2 + x^T \mathrm{Mx}} = \sqrt{x^T \mathrm{Mx}}\sqrt{1 + \delta^2/(x^T \mathrm{Mx})} > \sqrt{x^T \mathrm{Mx}}$. The Taylor expansion $\sqrt{1+\varepsilon} = 1 + \varepsilon/2 + O(\varepsilon^2)$ shows that $\sqrt{\delta^2 + x^T \mathrm{Mx}} = \sqrt{x^T \mathrm{Mx}}(1 + \delta^2/(2x^T \mathrm{Mx}) + O(\|x\|^{-4}))$. Thus, a SOCF with $\delta > 0$ is always less than the corresponding SOCF with $\delta = 0$, and the difference approaches 0 as $\|x\| \longrightarrow \infty$. We now present parts 5 and 6 of the theorem which are given as follows.

Example 5. Figure 5 shows the contour diagrams of 6 SOCFs of the form $f(x) = c^T x - \sqrt{\delta^2 + x^T \mathrm{Mx}}$, with $M = \begin{bmatrix} 2 & -1 \\ -1 & 5 \end{bmatrix}$. The eigenvalues of M are $(7 \pm \sqrt{13})/2$, so M is positive definite and Theorem 10 applies with the parameters $d = 0$ and $x_* = (0, 0)$. A calculation shows that $M^{-1} = (1/9)\begin{bmatrix} 5 & 1 \\ 1 & 2 \end{bmatrix}$. The other parameters are $\delta = 0$ in the top row, $\delta = 1$ in the bottom row, $c = (0.7, 0.7)$ in the left column, $c = (1, 1)$ in the middle column, and $c = (1.3, 1.3)$ in the right column. These values of c give $c^T M^{-1} c = 0.7^2, 1$, and 1.3^2, respectively. In the top row, $(0, 0)$ is always the critical point and $f(0, 0) = 0$. In the top middle figure, the contour with height 0 is the ray in the direction $M^{-1}c = (2/3, 1/3)$. In the bottom left figure, we find that $x_{cp} = (1.4/3, 0.7/3)/\sqrt{1 - .7^2} \approx (0.65, 0.33)$ and $f(x_{\mathrm{cp}}) = -\sqrt{1 - .7^2} \approx -.71$. In the bottom middle figure, f is bounded above by 0.

Theorem 11. *The SOCF $f\colon \mathbb{R}^n \longrightarrow \mathbb{R}$ written in the form of equation (6)*

$$f(x) = c^T x + d - \sqrt{\delta^2 + (x - x_*)^T M (x - x_*)}, \tag{18}$$

with M positive semidefinite is bounded above if and only if $c \in col(\mathrm{M})$ and $c^T M^+ c \le 1$.

Proof. Since M is symmetric, the fundamental theorem of linear algebra states that the null space of M is the orthogonal complement of the column space of M. We can write any $x \in \mathbb{R}^n$ as $x = x_n + x_r$ for unique $x_n \in N(M)$ and $x_r \in \mathrm{col}(M)$. Similarly, we split $c = c_n + c_r$. For a fixed x_*, we write $x \in \mathbb{R}^n$ as $x = x_* + x_n + x_r$. Thus, $M(x - x*) = M(x_n + x_r) = \mathrm{Mx}_r$, and the general second-order cone function is

$$f(x_* + x_n + x_r) = c_n^T x_n + c_r^T x_r + \left(c^T x_* + d\right) - \sqrt{\delta^2 + x_r{}^T \mathrm{Mx}_r}. \tag{19}$$

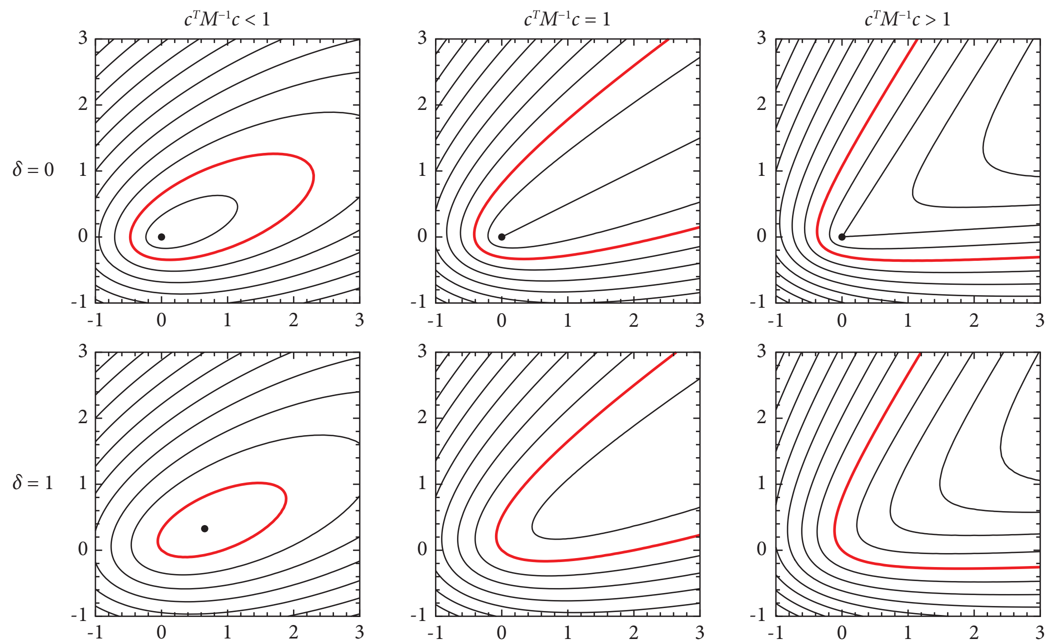

FIGURE 5: Contour plots of six different second-order cone functions defined in Example 5. All functions have the same positive definite matrix M. The parameters c and δ are chosen to illustrate Theorem 10, which says that f is bounded above if and only if $c^T M^{-1} c \leq 1$. The six parts of Theorem 10 correspond to the six contour plots. The contour with $f(x) = -1$ is a thick red curve, and the spacing between contours is $\Delta f = 0.5$.

Assume $c \notin \mathrm{col}(M)$. Then, $c_n \neq 0$ and f is not bounded since $f(x_* + x_n) = c_n^T x_n + (c^T x_* + d - \delta)$ is an unbounded linear function.

Assume $c \in \mathrm{col}(M)$, so $c_n = 0$. We define $g\colon \mathrm{col}(M) \longrightarrow \mathbb{R}$ by

$$g(x_r) \coloneqq f(x_* + x_n + x_r) = c_r^T x_r + \left(c^T x_* + d\right) - \sqrt{\delta^2 + x_r{}^T \mathrm{M} x_r}. \tag{20}$$

Note that M, restricted to $\mathrm{col}(M)$ is a nonsingular map, so we can apply Theorem 10 to g as follows. The pseudoinverse of M satisfies the following equation:

$$\begin{aligned} M^+(x_n + x_r) &= M^+ x_r, \\ \mathrm{MM}^+(x_n + x_r) &= M^+ M(x_n + x_r) = x_r. \end{aligned} \tag{21}$$

Thus, the restriction of M^+ to $\mathrm{col}(M)$ is the inverse of the restriction of M to $\mathrm{col}(M)$. Theorem 10 says that g is bounded above if and only if $c_r^T M^+ c_r \leq 1$. Note that $c^T M^+ c = c_r^T M^+ c_r$ for any $c \in \mathbb{R}^n$.

We have shown that f is not bounded above if $c \notin \mathrm{col}(M)$. We have also shown that if $c \in \mathrm{col}(M)$, then f is bounded above if and only if $c^T M^+ c \leq 1$. These two statements can be combined into one: f is bounded above if and only if $c \in \mathrm{col}(M)$ and $c^T M^+ c \leq 1$. □

Remark 12. If M is posivite definite, then $c \in \mathrm{col}(M) = \mathbb{R}^n$ and $M^+ = M^{-1}$. Thus, Theorem 13, in the case where M is posivite definite, implies that f is bounded above if and only if $c^T M^{-1} c \leq 1$, which is the first part of Theorem 10.

Example 6. We consider the SOCF on $\mathbb{R}^2$ with $M = \begin{bmatrix} 4 & 0 \\ 0 & 0 \end{bmatrix}$, $d = 0$, and $x_* = (0, 0)$ as

$$f(x) = c_1 x_1 + c_2 x_2 - \sqrt{\delta^2 + 4x_1^2}. \tag{22}$$

Note that $f(0, x_2) = c_2 x_2 - \delta$ is not bounded above if $c_2 \neq 0$. If $c_2 = 0$, then $f(x) = c_1 x_1 - \sqrt{\delta^2 + 4x_1^2} \leq c_1 x_1 - 2|x_1|$ and $f(x) \longrightarrow c_1 x_1 - 2|x_1|$ as $x_1 \longrightarrow \pm\infty$. Thus, f is bounded above if and only if $c_2 = 0$ and $c_1^2 \leq 4$.

This observation is predicted by Theorem 11. The column space of M is $\mathrm{col}(\mathrm{M}) = \{(\mathrm{a}, 0) \mid \mathrm{a} \in \mathbb{R}\}$, so $c \in \mathrm{col}(\mathrm{M})$ is equivalent to $c_2 = 0$. The pseudoinverse of M is $M^+ = 1/4 \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$, so $c^T M^+ c = c_1^2/4$, and $c^T M^{-1} c \leq 1$ is equivalent to $c_1^2 \leq 4$.

One of the main uses of SOCFs is to define convex sets for optimization problems. Optimization over a bounded set is very different from optimization over an unbounded set, so we finish this paper with a simple characterization.

Theorem 13. *Let $f: \mathbb{R}^n \longrightarrow \mathbb{R}$ be defined by $f(x) = c^T x + d - \sqrt{\delta^2 + (x - x_*)^T M (x - x^*)}$, where M is positive semidefinite. Then the set $\mathscr{R}: = \{x \in \mathbb{R}^n \mid f(x) \geq 0\}$ is closed and convex. Assuming $\mathscr{R}$ is not the empty set, $\mathscr{R}$ is bounded if and only if M is positive definite and $c^T M^{-1} c < 1$.*

Proof. The set $\mathscr{R}$ is convex since f is concave, and it is closed since f is continuous. If M is not positive definite, then Theorem 11 implies that $\mathscr{R}$ is not bounded, since f is unbounded if $c \in \operatorname{col}(M)$, and f satisfies $f(x + x_r) = f(x)$ for all $x_r \in N(M)$ if $c \in \operatorname{col}(M)$. If M is a positive definite, then Theorem 10 implies that $\mathscr{R}$ is bounded if and only if $c^T M^{-1} c < 1$. □

Remark 14. In the case where M is positive definite and $c^T M^{-1} c < 1$, the compact $\mathscr{R}$ might be trivial. Let $\tilde{d}: = d - \delta\sqrt{1 - c^T M^{-1} c}$. The set $\mathscr{R}$ is the empty set if $\tilde{d} < 0$, $\mathscr{R}$ is the singleton set $\{x_*\}$ if $\tilde{d} = 0$, and $\mathscr{R}$ has a nonempty interior if $\tilde{d} > 0$.

3. Conclusion

The second-order cone function has important applications in optimization problems. Our work gives necessary and sufficient conditions for strict concavity of a second-order cone function. We show that every SOCF can be written in the following form: $f(x) = c^T x + d - \sqrt{\delta^2 + (x - x_*)^T M (x - x_*)}$, which has unique parameters in many cases. This alternative parameterization gives a deep understanding of the family of SOCFs. This alternative description leads to new results on SOCFs. We characterize the critical points and global maxima of f, depending on the parameters. We give necessary and sufficient conditions for f to be bounded above, and for the set $\{x \in \mathbb{R}^n \mid f(x) \geq 0\}$ to be bounded. Our results can lead to improved algorithms for optimization problems involving second-order cone constraints.

Data Availability

The data used to support the findings of this study are included within the article.

Disclosure

The Cornell University arXiv posted a preprint of this article [19]. This research was performed as part of the employment of the authors at Northern Arizona University.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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