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Research Article

Constant Rate Distributions on Partially Ordered Sets

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We consider probability distributions with constant rate on partially ordered sets, generalizing distributions in the usual reliability setting $([0,\infty),\leq)$ that have constant failure rate. In spite of the minimal algebraic structure, there is a surprisingly rich theory, including moment results and results concerning ladder variables and point processes. We concentrate mostly on discrete posets, particularly posets whose graphs are rooted trees. We pose some questions on the existence of constant rate distributions for general discrete posets.

1. Preliminaries

1.1. Introduction

The exponential distribution on $[0,\infty)$ and the geometric distribution on $\mathbb{N}$ are characterized by the *constant rate* property: the density function (with respect to Lebesgue measure in the first case and counting measure in the second) is a multiple of the upper (right-tail) distribution function.

The natural mathematical home for the constant rate property is a partially ordered set (poset) with a reference measure for the density functions. In this paper we explore these distributions. In spite of the minimal algebraic structure of a poset, there is a surprisingly rich theory, including moment results and results concerning ladder variables and point processes. In many respects, constant rate distributions lead to the most random way to put ordered points in the poset. We will be particularly interested in the existence question—when does a poset support constant rate distributions?

1.2. Standard Posets

Suppose that $(S, \preceq)$ is a poset. For $x \in S$, let

$$I(x) = \{t \in S : t \succ x\}, \qquad I[x] = \{t \in S : t \succeq x\}, \qquad D[x] = \{t \in S : t \preceq x\}. \tag{1.1}$$

For $n \in \mathbb{N}_+$ and $x \in S$, let

$$\begin{aligned} D_n &= \{(x_1, x_2, \ldots, x_n) \in S^n : x_1 \preceq x_2 \preceq \cdots \preceq x_n\}, \\ D_n[x] &= \{(x_1, x_2, \ldots, x_n) \in S^n : x_1 \preceq x_2 \preceq \cdots \preceq x_n \preceq x\}. \end{aligned} \tag{1.2}$$

For $x, y \in S$, y is said to cover x if y is a minimal element of $I(x)$. If S is countable, the *Hasse graph* or *covering graph* of $(S, \preceq)$ has vertex set S and (directed) edge set $\{(x, y) \in S^2 : y \text{covers } x\}$. We write $x \perp y$ if x and y are comparable, and we write $x \| y$ if x and y are noncomparable. The poset $(S, \preceq)$ is *connected* if, for every $x, y \in S$, there exists a finite sequence $(x_0, x_1, \ldots, x_n)$ such that $x_0 = x$, $x_n = y$, and $x_{i-1} \perp x_i$ for $i = 1, \ldots, n$.

Now suppose that $\mathcal{S}$ is a σ-algebra on S, and let $\mathcal{S}^n$ denote the corresponding product σ-algebra on S^n for $n \in \mathbb{N}_+$. The main assumption that we make to connect the algebraic structure of S to the measure structure is that the partial order $\preceq$ is itself measurable, in the sense that $D_2 \in \mathcal{S}^2$. It then follows that $I[x],\ D[x] \in \mathcal{S}$ for $x \in S$ since these sets are simply the cross-sections of D_2 (see [1, 2]). Note that $\{x\} = D[x] \cap I[x] \in \mathcal{S}$, so in fact all of the "intervals" $I[x] \cap D[y]$, $I(x) \cap D[y]$, and so forth are measurable for $x \prec y$. Also, $D_n,\ D_n[x] \in \mathcal{S}^n$ for $x \in S$ and $n \in \mathbb{N}_+$. If S is countable, $\mathcal{S}$ is the power set of S. When S is uncountable, $\mathcal{S}$ is usually the Borel σ-algebra associated with an underlying topology (see [3, 4]).

Finally, we fix a positive, σ-finite measure λ on $\mathcal{S}$ as a *reference measure*. We assume that $\lambda(I[x]) > 0$ and $\lambda(D[x]) < \infty$ for each $x \in S$. When S is countable, we take λ to be counting measure on S unless otherwise noted. In this case, $D[x]$ must be finite for each x, so that S is locally finite in the terminology of discrete posets [5].

Definition 1.1. The term *standard poset* will refer to a poset $(S, \preceq)$ together with a measure space $(\mathcal{S}, \lambda)$ that satisfies the algebraic and measure theoretic assumptions. When the measure space is understood (in particular for discrete posets with counting measure), we will often omit the reference to this space.

An important special case is the poset associated with a *positive semigroup*. A positive semigroup $(S, \cdot)$ is a semigroup that has an identity element e, satisfies the left-cancellation law, and has no nontrivial inverses. The partial order $\preceq$ associated with $(S, \cdot)$ is given by

$$x \preceq y \quad \text{iff } xt = y \text{ for some } t \in S. \tag{1.3}$$

If $x \preceq y$, then t satisfying $xt = y$ is unique and is denoted by $x^{-1}y$. The space S has a topology that makes the mapping $(x, y) \to xy$ continuous, and $\mathcal{S}$ is the ordinary Borel σ-algebra. The reference measure λ is *left invariant:*

$$\lambda(xA) = \lambda(A)\ , \quad x \in S,\ A \in \mathcal{S}. \tag{1.4}$$

Probability distributions (particularly exponential type distributions) on positive semigroups have been studied in [6–10]. The critical feature of a positive semigroup is that, for each x, $I[x] = xS$ looks like the entire space S from algebraic, topological, and measure theoretic points of view. Of course, there is no such self-similarity for general standard posets. Nonetheless, the definition of a standard poset is supposed to capture the minimal structure of a temporal (i.e., time-like) set, particularly in probability applications such as reliability and queuing. The following examples are intended to give some idea of the structures that we have in mind; some of these will be explored in more detail in this paper.

Example 1.2. Consider that $([0,\infty),\leq)$, where $\leq$ is ordinary order (and where λ is Lebesgue measure on the Borel σ-algebra $\mathcal{S}$) is the standard model of continuous time. Of course, this poset is associated with the positive semigroup $([0,\infty),+)$.

Example 1.3. Consider that $(\mathbb{N},\leq)$, where $\leq$ is ordinary order, is the standard model of discrete time. This poset is associated with the positive semigroup $(\mathbb{N},+)$.

Example 1.4. If $(S_i, \preceq_i, \mathcal{S}_i, \lambda_i)$ is a standard poset for $i = 1,\ldots,n$, then so is $(S,\preceq,\mathcal{S},\lambda)$, where S is the Cartesian product, $\preceq$ the product order, $\mathcal{S}$ the product σ-algebra, and λ the product measure. Product sets occur frequently in multivariate settings of reliability and other applications. In particular, the n-fold powers of the posets in Examples 1.2 and 1.3 are the standard multivariate models for continuous and discrete time, respectively.

Example 1.5. Consider that $(\mathbb{N}_+,\preceq)$, where $x \preceq y$ means that x divides y, is a standard poset and corresponds to the positive semigroup $(\mathbb{N}_+,\cdot)$. Exponential distributions for this semigroup were explored in [10].

Example 1.6. Suppose that $(R,\preceq_R)$ is a poset and that $(S_x,\preceq_x)$ is a poset for each $x \in R$. The *lexicographic sum* of $(S_x,\preceq_x)$ over $x \in R$ is the poset $(T,\preceq)$, where $T = \bigcup_{x\in R}\{x\} \times S_x$ and where $(u,v)\preceq(x,y)$ if and only if either $u\prec_R x$ or $u = x$ and $v\preceq_x y$. In the special case that $(S_x,\preceq_x) = (S,\preceq_S)$ for each $x \in R$, $(T,\preceq)$ is the lexicographic product of $(R,\preceq_R)$ and $(S,\preceq_S)$. In the special case that $\preceq_R$ is the equality relation, $(T,\preceq)$ is the *simple sum* of $(S_x,\preceq_x)$ over $x \in R$.

Given appropriate σ-algebras and reference measures, these become standard posets and have possible applications. For example, suppose that $(S_i,\preceq_i)$ is a poset in a reliability model, for each $i \in \{1,2,\ldots,n\}$. The simple sum could represent the appropriate space when n devices are run in parallel. On the other hand, sometimes the more exotic posets lead to additional insights into simple posets. For example, $([0,\infty),\leq)$ is isomorphic to the lexicographic product of $(\mathbb{N},\leq)$ with $([0,1),\leq)$ (just consider the integer part and remainder of $x \in [0,\infty)$).

Example 1.7. Suppose that $(S,\preceq,\mathcal{S},\lambda)$ is a standard poset and that $T \in \mathcal{S}$ with $\lambda(I[x] \cap T) > 0$ for each $x \in T$. Then $(T,\preceq_T,\mathcal{T},\lambda_T)$ is also a standard poset, where $\preceq_T$ is the restriction of $\preceq$ to T and λ_T is the restriction of λ to $\mathcal{T} = \{A \cap T : A \in \mathcal{S}\}$. An incredibly rich variety of new posets can be constructed from a given poset in this way. Moreover, such subposets can be used to impose restrictions in the reliability setting and other applications.

Example 1.8. A rooted tree is the Hasse graph of a standard poset, and such trees have myriad applications to various kinds of data structures. One common application is to reproducing objects (e.g., organisms in a colony or electrons in a multiplier). In this interpretation, that y covers x means that y is a child of x, and, more generally, $x\preceq y$ means that y is an ancestor of x.

Example 1.9. If S denotes the set of all finite subsets of $\mathbb{N}_+$, then $(S, \subseteq)$ is a standard discrete poset. Somewhat surprisingly, this poset corresponds to a positive semigroup. Exponential type distributions for this semigroup were explored in [9]. Random sets have numerous applications (see [11]); in particular, the problem of choosing a subset of $\mathbb{N}_+$ in the "most random" way is important in statistics.

Example 1.10. Various collections of finite graphs can be made into standard posets under the subgraph relation. Depending on the model of interest, the graphs could be labeled or unlabeled and might be restricted (e.g., to finite rooted trees). Of course, random graphs in general and random trees in particular are large and important areas of research (see [12, 13]).

For the remainder of this paper, unless otherwise noted, we assume that $(S, \preceq, \mathcal{S}, \lambda)$ is a standard poset.

1.3. Operators and Cumulative Functions

Let $\mathfrak{D}(S)$ denote the set of measurable functions from S into $\mathbb{R}$ that are bounded on $D[x]$ for each $x \in S$. Define the *lower operator* L on $\mathfrak{D}(S)$ by

$$Lf(x) = \int_{D[x]} f(t)d\lambda(t), \quad x \in S. \tag{1.5}$$

Next, let $\mathcal{L}(S)$ denote the usual Banach space of measurable functions $f : S \to \mathbb{R}$, with $\|f\| = \int_S |f(x)|d\lambda(x) < \infty$. Define the *upper operator* U on $\mathcal{L}(S)$ by

$$Uf(x) = \int_{I[x]} f(y)d\lambda(y), \quad x \in S. \tag{1.6}$$

A simple application of Fubini's theorem gives the following duality relationship between the linear operators L and U:

$$\int_S Lf(x)g(x)d\lambda(x) = \int_S f(x)Ug(x)d\lambda(x), \tag{1.7}$$

assuming, of course, the appropriate integrability conditions. Both operators can be written as integral operators with a kernel function. Define $\rho : S \times S \to \mathbb{R}$ by

$$\rho(x, y) = \begin{cases} 1 & \text{if } x \preceq y, \\ 0 & \text{otherwise.} \end{cases} \tag{1.8}$$

Then

$$Lf(x) = \int_S \rho(t,x) f(t) d\lambda(t), \quad x \in S,$$
$$Uf(x) = \int_S \rho(x,t) f(t) d\lambda(t), \quad x \in S. \tag{1.9}$$

In the discrete case, ρ is the *Riemann function* in the terminology of Möbius inversion [5], and its inverse m (also in the sense of this theory) is the *Möbius function*. The lower operator L is invertible, and if $g = Lf$, then

$$f(x) = \sum_{t \in S} g(t) m(t,x). \tag{1.10}$$

As we will see in Example 2.2, the upper operator U is not invertible in general, even in the discrete case.

Now let $\mathbf{1}$ denote the constant function 1 on S, and define $\lambda_n : S \to [0,\infty)$ by $\lambda_n = L^n \mathbf{1}$ for $n \in \mathbb{N}$. Equivalently, $\lambda_n(x) = \lambda^n(D_n[x])$ for $n \in \mathbb{N}_+$ and $x \in S$, where λ^n is n-fold product measure on $(S^n, \mathcal{S}^n)$. We will refer to λ_n as the *cumulative function of order* n; these functions play an important role in the study of probability distribution on $(S, \preceq)$. For the poset of Example 1.2, the cumulative functions are

$$\lambda_n(x) = \frac{x^n}{n!}, \quad x \in [0,\infty), \ n \in \mathbb{N}. \tag{1.11}$$

For the poset of Example 1.3, the cumulative functions are

$$\lambda_n(x) = \binom{n+x}{x}, \quad x, n \in \mathbb{N}. \tag{1.12}$$

For the poset of Example 1.5, $\lambda_n(x)$ is the number of $n+1$ factorings of $x \in \mathbb{N}_+$; these are important functions in number theory. For the poset of Example 1.9, the cumulative functions are

$$\lambda_n(x) = (n+1)^{\#(x)}, \quad x \in S, \ n \in \mathbb{N}. \tag{1.13}$$

The ordinary generating function of $n \mapsto \lambda_n(x)$ is the function Λ given by

$$\Lambda(x,t) = \sum_{n=0}^{\infty} \lambda_n(x) t^n, \tag{1.14}$$

for $x \in S$ and for $t \in \mathbb{R}$ for which the series converges absolutely. The generating function Λ arises in the study of the point process associated with a constant rate distribution. For the poset of Example 1.2, $\Lambda(x,t) = e^{tx}$. For the poset of Example 1.3,

$$\Lambda(x,t) = \frac{1}{(1-t)^{x+1}}, \quad x \in \mathbb{N},\ |t| < 1. \tag{1.15}$$

2. Probability Distributions

2.1. Distribution Functions

Suppose now that X is a random variable taking values in S. We assume that $\mathbb{P}(X \in A) > 0$ if and only if $\lambda(A) > 0$ for $A \in \mathcal{S}$. Thus, the distribution of X is absolutely continuous with respect to λ, and the support of X is the entire space S in a sense. Let f denote the probability density function (PDF) of X, with respect to λ.

Definition 2.1. The *upper probability function* (UPF) of X is the function $F : S \to (0,1]$ given by

$$F(x) = \mathbb{P}(X \succeq x) = \mathbb{P}(X \in I[x]), \quad x \in S. \tag{2.1}$$

If $(S, \preceq)$ is interpreted as a temporal space and X as the failure time of a device, then F is the *reliability function*; $F(x)$ is the probability that the failure of the device occurs at or after x (in the sense of the partial order). This is the usual meaning of the reliability function in the standard spaces of Examples 1.2 and 1.3 and of the multivariate reliability function in the setting of Example 1.4 (see [14]). The definition is relevant for more exotic posets as well, such as the lexicographic order discussed in Example 1.6. Note that

$$F(x) = Uf(x) = \int_{I[x]} f(t)\,d\lambda(t), \quad x \in S. \tag{2.2}$$

In general, the UPF of X does not uniquely determine the distribution of X.

Example 2.2. Let A be fixed set with k elements ($k \geq 2$), and let $(S, \preceq)$ denote the lexicographic sum of the antichains $(A_n, =)$ over $(\mathbb{N}, \leq)$, where $A_0 = \{e\}$ and $A_n = A$ for $n \in \mathbb{N}_+$. Let f be a PDF on S with UPF F. Define g by

$$g(n,x) = f(n,x) + \left(-\frac{1}{k-1}\right)^n c, \tag{2.3}$$

where c is a constant. It is straightforward to show that

$$\sum_{(m,y) \succeq (n,x)} g(m,y) = F(n,x), \quad (n,x) \in S. \tag{2.4}$$

In particular, $\sum_{(n,x)\in S} g(n,x) = 1$. Hence if we can choose a PDF f and a nonzero constant c so that $g(n,x) > 0$ for every $(n,x) \in S$, then g is a PDF different from f but with the same UPF. This can always be done when $k \geq 3$.

In the discrete case, a simple application of the inclusion-exclusion rule shows that the distribution of X is determined by the generalized UPF, defined for finite $A \subseteq S$ by

$$F(A) = \mathbb{P}\left(X \succeq x \forall x \in A\right). \tag{2.5}$$

An interesting problem is to give conditions on the poset $(S, \preceq)$ that ensure that a distribution on S is determined by its ordinary UPF. This holds for a discrete upper semilattice, since $F(A) = F(\sup(A))$ for A finite. It also holds for trees, as we will see in Section 5.

The following proposition gives a simple result that relates expected value to the lower operator L. For positive semigroups, this result was given in [10].

Proposition 2.3. *Suppose that X has UPF F. For $g \in \mathfrak{D}(S)$ and $n \in \mathbb{N}$,*

$$\int_S L^n g(x) F(x) d\lambda(x) = \mathbb{E}\left[L^{n+1} g(X)\right]. \tag{2.6}$$

Proof. Using Fubini's theorem,

$$\begin{aligned} \int_S L^n g(x) \mathbb{P}\left(X \succeq x\right) d\lambda(x) &= \int_S L^n g(x) \mathbb{E}\left[1\left(X \succeq x\right)\right] d\lambda(x) \\ &= \mathbb{E}\left(\int_{D[X]} L^n(g)(x) d\lambda(x)\right) \\ &= \mathbb{E}\left[L^{n+1} g(X)\right]. \end{aligned} \tag{2.7}$$

□

When X has a PDF f, (2.6) also follows from (1.7). In particular, letting $g = 1$ gives

$$\int_S \lambda_n(x) F(x) d\lambda(x) = \mathbb{E}[\lambda_{n+1}(X)], \quad n \in \mathbb{N}, \tag{2.8}$$

and when $n = 0$, (2.8) becomes

$$\int_S F(x) d\lambda(x) = \mathbb{E}(\lambda(D[X])). \tag{2.9}$$

For the poset of Example 1.2, (2.8) becomes

$$\int_0^\infty \frac{x^n}{n!} \mathbb{P}(X \geq x) = \mathbb{E}\left(\frac{X^{n+1}}{(n+1)!}\right) \tag{2.10}$$

and (2.9) reduces to the standard result $\int_0^\infty \mathbb{P}(X \geq x) dx = \mathbb{E}(X)$. For the poset of Example 1.3, (2.8) becomes

$$\sum_{x=0}^{\infty} \binom{n+x}{n} \mathbb{P}(X \geq x) = \mathbb{E}\binom{X+n+1}{n+1} \tag{2.11}$$

and (2.9) reduces to the standard result $\sum_{n=0}^{\infty} \mathbb{P}(X \geq n) = \mathbb{E}(X) + 1$. For the poset of Example 1.9, (2.8) becomes

$$\sum_{x \in S} (n+1)^{\#(x)} P(X \supseteq x) = \mathbb{E}\left((n+2)^{\#(X)}\right). \tag{2.12}$$

Definition 2.4. Suppose that X has UPF F and PDF f. The *rate function* of X is the function $r : S \to (0, \infty)$ defined by

$$r(x) = \frac{f(x)}{F(x)}, \quad x \in S. \tag{2.13}$$

Again, if $(S, \preceq)$ is interpreted as a temporal space and X as the failure time of a device, then r is the failure rate function: $r(x)$ gives the probability density of failure at x given that failure occurs at or after x. As mentioned before, this agrees with the usual meaning of failure rate in the standard spaces of Examples 1.2 and 1.3 and with the usual multivariate meaning for the product of these spaces (see [14]).

In general, X has *constant rate* if r is constant on S, *increasing rate* if r is increasing on S, and *decreasing rate* if r is decreasing on S. In the reliability setting, devices with increasing failure rate deteriorate over time, in a sense, while devices with decreasing failure rate improve. Devices with constant failure rate do not age, in a sense.

Proposition 2.5. *Suppose that $(S, \preceq)$ is a standard discrete poset and that X has rate function r. Then $r(x) \leq 1$ for $x \in S$, and $r(x) = 1$ if and only if x is maximal.*

Proof. Let f denote the PDF of X and F the UPF. For $x \in S$,

$$F(x) = f(x) + \mathbb{P}(X \succ x) = r(x)F(x) + \mathbb{P}(X \succ x). \tag{2.14}$$

Hence

$$\mathbb{P}(X \succ x) = [1 - r(x)]F(x). \tag{2.15}$$

By (2.15), $r(x) \leq 1$ for all $x \in S$. If x is maximal then $\mathbb{P}(X \succ x) = \mathbb{P}(\emptyset) = 0$ so $r(x) = 1$. Conversely, if x is not maximal, then $\mathbb{P}(X \succ x) > 0$ (since X has support S) and hence $r(x) < 1$. □

2.2. Ladder Variables and Partial Products

Let $\mathbf{X} = (X_1, X_2, \ldots)$ be a sequence of independent, identically distributed random variables, taking values in S, with common UPF F and PDF f. We define the sequence of *ladder variables* $\mathbf{Y} = (Y_1, Y_2, \ldots)$ associated with X as follows. First let

$$N_1 = 1, \qquad Y_1 = X_1, \tag{2.16}$$

and then recursively define

$$N_{n+1} = \min\{k > N_n : X_k \succeq Y_n\}, \quad Y_{n+1} = X_{N_{n+1}}. \tag{2.17}$$

Proposition 2.6. *The sequence* $\mathbf{Y}$ *is a homogeneous Markov chain with transition density* g *given by*

$$g(y,z) = \frac{f(z)}{F(y)}, \quad (y,z) \in D_2. \tag{2.18}$$

Proof. Let $(y_1,\ldots,y_{n-1},y,z) \in D_{n+1}$. The conditional distribution of Y_{n+1} given $\{Y_1 = y_1,\ldots,Y_{n-1} = y_{n-1}, Y_n = y\}$ corresponds to observing independent copies of X until a variable occurs with a value greater than y (in the partial order). The distribution of this last variable is the same as the conditional distribution of X given $X \succeq y$, which has density $z \mapsto f(z)/F(y)$ on $I[y]$. □

Since $Y_1 = X_1$ has PDF f, it follows immediately from Proposition 2.6 that $(Y_1, Y_2, \ldots, Y_n)$ has PDF g_n (with respect to λ^n) given by

$$g_n(y_1,y_2,\ldots,y_n) = f(y_1)\frac{f(y_2)}{F(y_1)}\cdots\frac{f(y_n)}{F(y_{n-1})}, \quad (y_1,y_2,\ldots,y_n) \in D_n. \tag{2.19}$$

This PDF has a simple representation in terms of the rate function r:

$$g_n(y_1,y_2,\ldots,y_n) = r(y_1)r(y_2)\cdots r(y_{n-1})f(y_n), \quad (y_1,y_2,\ldots,y_n) \in D_n. \tag{2.20}$$

Suppose now that $(S,\cdot)$ is a positive semigroup and that $\mathbf{X} = (X_1, X_2, \ldots)$ is an IID sequence in S with PDF f. Let $Z_n = X_1 \cdots X_n$ for $n \in \mathbb{N}_+$, so that $\mathbf{Z} = (Z_1, Z_2, \ldots)$ is the partial product sequence associated with $\mathbf{Z}$.

Proposition 2.7. *The sequence* $\mathbf{Z}$ *is a homogeneous Markov chain with transition probability density* h *given by*

$$h(y,z) = f\left(y^{-1}z\right), \quad (y,z) \in D_2. \tag{2.21}$$

Since $Z_1 = X_1$ has PDF f, it follows immediately from Proposition 2.7 that $(Z_1, Z_2, \ldots, Z_n)$ has PDF h_n (with respect to λ^n) given by

$$h_n(z_1,z_2,\ldots,z_n) = f(z_1)f\left(z_1^{-1}z_2\right)\cdots f\left(z_{n-1}^{-1}z_n\right), \quad (z_1,z_2,\ldots,z_n) \in D_n. \tag{2.22}$$

So in the case of a positive semigroup, there are two natural processes associated with an IID sequence $\mathbf{X}$: the sequence of ladder variables $\mathbf{Y}$ and the partial product sequence $\mathbf{Z}$. In general, these sequences are not equivalent but, as we will see in Section 3, *are* equivalent when the underlying distribution of $\mathbf{X}$ has constant rate. Moreover, this equivalence characterizes constant rate distributions.

Return now to the setting of a standard poset. If $\mathbf{W} = (W_1, W_2, \ldots)$ is an increasing sequence of random variables in S (such as a sequence of ladder variables or, in the special case of a positive semigroup, a partial product sequence), we can construct a point process in the usual way. For $x \in S$, let

$$N_x = \#\{n \in \mathbb{N}_+ : W_n \preceq x\}, \tag{2.23}$$

so that N_x is the number of random points in $D[x]$. We have the usual inverse relation between the processes $\mathbf{W}$ and $\mathbf{N} = (N_x : x \in S)$, namely, $W_n \preceq x$ if and only if $N_x \geq n$ for $n \in \mathbb{N}_+$ and $x \in S$. For $n \in \mathbb{N}_+$, let G_n denote the *lower* probability function of W_n, so that $G_n(x) = \mathbb{P}(W_n \preceq x)$ for $x \in S$. Then $\mathbb{P}(N_x \geq n) = \mathbb{P}(W_n \preceq x) = G_n(x)$ for $n \in \mathbb{N}_+$. Of course, $\mathbb{P}(N_x \geq 0) = 1$. If we define $G_0(x) = 1$ for all $x \in S$, then for fixed x, $n \mapsto G_n(x)$ is the *upper* probability function of N_x.

The sequence of random points $\mathbf{W}$ can be *thinned* in the usual way. Specifically, suppose that each point is accepted with probability $p \in (0,1)$ and rejected with probability $1 - p$, independently from point to point. Then the first accepted point is W_M where M is independent of $\mathbf{W}$ and has the geometric distribution on $\mathbb{N}_+$ with parameter p.

3. Distributions with Constant Rate

3.1. Characterizations and Properties

Suppose that X is a random variable taking values in S with UPF F. Recall that X has *constant rate* $\alpha > 0$ if $f = \alpha F$ is a PDF of X.

If $(S, \preceq)$ is associated with a positive semigroup $(S, \cdot)$, then X has an *exponential* distribution if

$$\mathbb{P}(X \in xA) = F(x)\mathbb{P}(X \in A), \quad x \in S,\ A \in S. \tag{3.1}$$

Equivalently, the conditional distribution of $x^{-1}X$ given by $X \succeq x$ is the same as the distribution of X. Exponential distributions on positive semigroups are studied in [6–10]. In particular, it is shown in [7] that a distribution is exponential if and only if it has constant rate and

$$F(xy) = F(x)F(y), \quad x, y \in S. \tag{3.2}$$

However there are constant rate distributions that are not exponential. Moreover, of course, the exponential property makes no sense for a general poset.

Constant rate distributions can be characterized in terms of the upper operator U or the lower operator L. In the first case, the characterization is an eigenvalue condition and in the second case a moment condition.

Proposition 3.1. *The poset $(S, \preceq, \lambda)$ supports a distribution with constant rate α if and only if there exists a strictly positive $g \in \mathcal{L}(S)$ with*

$$U(g) = \frac{1}{\alpha} g. \tag{3.3}$$

Proof. If F is the UPF of a distribution with constant rate α, then trivially $f = \alpha F$ satisfies the conditions of the proposition since f is a PDF with UPF F. Conversely, if $g \in \mathcal{L}(S)$ is strictly positive and satisfies (3.3), then $f := g/\|g\|$ is a PDF and $U(f) = (1/\alpha)f$, so the distribution with PDF f has constant rate α. □

Proposition 3.2. *Random variable X has constant rate α if and only if*

$$\mathbb{E}\left[Lg(X)\right] = \frac{1}{\alpha}\mathbb{E}\left[g(X)\right], \tag{3.4}$$

for every $g \in \mathfrak{D}(S)$.

Proof. Suppose that X has constant rate α and UPF F, so that $f = \alpha F$ is a PDF of X. Let $g \in \mathfrak{D}(S)$. From Proposition 2.3,

$$\mathbb{E}\left[Lg(X)\right] = \int_S g(x)F(x)d\lambda(x) = \int_S g(x)\frac{1}{\alpha}f(x)d\lambda(x) = \frac{1}{\alpha}\mathbb{E}\left[g(X)\right]. \tag{3.5}$$

Conversely, suppose that (3.4) holds for every $g \in \mathfrak{D}(S)$. Again let F denote the UPF of X. By Proposition 2.3, condition (3.4) is equivalent to

$$\int_S \alpha g(x)F(x)d\lambda(x) = \mathbb{E}\left[g(X)\right]. \tag{3.6}$$

It follows that $f = \alpha F$ is a PDF of X. □

If X has constant rate α, then iterating (3.4) gives

$$\mathbb{E}\left[L^n(g)(X)\right] = \frac{1}{\alpha^n}\mathbb{E}\left[g(X)\right], \quad n \in \mathbb{N}. \tag{3.7}$$

In particular, if $g = 1$, then

$$\mathbb{E}[\lambda_n(X)] = \frac{1}{\alpha^n}, \quad n \in \mathbb{N}, \tag{3.8}$$

and if $n = 1$, $\mathbb{E}[\lambda_1(X)] = \mathbb{E}(\lambda(D[X])) = 1/\alpha$.

Suppose now that $(S, \preceq)$ is a discrete standard poset and that X has constant rate α on S. From Proposition 2.5, $\alpha \leq 1$, and if $\alpha = 1$, all elements of S are maximal, so that $(S, \preceq)$ is an antichain. Conversely, if $(S, \preceq)$ is an antichain, then any distribution on S has constant rate 1. On the other hand, if $(S, \preceq)$ is not an antichain, then $\alpha < 1$ and S has no maximal elements.

Consider again a discrete standard poset $(S, \preceq)$. For $x, y \in S$, we say that x and y are *upper equivalent* if $I(x) = I(y)$. Upper equivalence is the equivalence relation associated with the function $s \mapsto I(s)$ from S to $\mathcal{D}(S)$. Suppose that X has constant rate α on S and UPF F. If $x,\ y$ are upper equivalent, then $\mathbb{P}(X \succ\ x) = \mathbb{P}(X \succ\ y)$; so from (2.15),

$$F(x) = \frac{\mathbb{P}(X \succ x)}{1-\alpha} = \frac{\mathbb{P}(X \succ y)}{1-\alpha} = F(y). \tag{3.9}$$

Thus, the UPF (and hence also the PDF) is constant on the equivalence classes.

For the poset in Example 1.2, of course, the constant rate distributions are the ordinary exponential distributions (see [15] for myriad characterizations). For the poset in Example 1.3, the constant rate distributions are the ordinary geometric distributions. For the poset $([0,\infty)^n, \leq)$ as described in Example 1.4, the constant rate distributions are mixtures of distributions that correspond to independent, exponentially distributed coordinates [16].

3.2. Ladder Variables and Partial Products

Assume that $(S, \preceq)$ is a standard poset. Suppose that $\mathbf{X} = (X_1, X_2, \ldots)$ is an IID sequence with common UPF F, and let $\mathbf{Y} = (Y_1, Y_2, \ldots)$ be the corresponding sequence of ladder variables.

Proposition 3.3. *If the distribution has constant rate α, then*

(1) $\mathbf{Y}$ *is a homogeneous Markov chain on S with transition probability density g given by*

$$g(y,z) = \alpha \frac{F(z)}{F(y)}, \quad (y,z) \in D_2; \tag{3.10}$$

(2) $(Y_1, Y_2, \ldots, Y_n)$ *has PDF g_n given by*

$$g_n(y_1, y_2, \ldots, y_n) = \alpha^n F(y_n), \quad (y_1, y_2, \ldots, y_n) \in D_n; \tag{3.11}$$

(3) Y_n *has PDF f_n given by*

$$f_n(y) = \alpha^n \lambda_{n-1}(y) F(y), \quad y \in S; \tag{3.12}$$

(4) *the conditional distribution of $(Y_1, Y_2, \ldots, Y_{n-1})$ given $Y_n = y$ is uniform on $D_{n-1}[y]$;*

Proof. Parts 1 and 2 follow immediately from Proposition 2.6. Part 3 follows from Part 2 and Part 4 from Parts 2 and 3. □

Part 4 shows that the sequence of ladder variables $\mathbf{Y}$ of an IID constant rate sequence $\mathbf{X}$ is the most random way to put a sequence of ordered points in S. In the context of Example 1.2, of course, $\mathbf{Y}$ is the sequence of arrival times of an ordinary Poisson process and the distribution in Part 3 is the ordinary gamma distribution of order n and rate α. In the context of Example 1.3, the distribution in Part 3 is the negative binomial distribution of order n and parameter α. Part 4 almost characterizes constant rate distributions.

Proposition 3.4. *Suppose that $(S, \preceq)$ is a standard, connected poset and that $\mathbf{Y}$ is the sequence of ladder variables associated with an IID sequence $\mathbf{X}$. If the conditional distribution of Y_1 given $Y_2 = y$ is uniform on $D[y]$, then the common distribution of $\mathbf{X}$ has constant rate.*

Proof. From (2.20), the conditional PDF of Y_1 given $Y_2 = y \in S$ is

$$h_1(x \mid y) = \frac{1}{C(y)} r(x), \qquad x \in D[y], \tag{3.13}$$

where $C(y)$ is the normalizing constant. But this is constant in $x \in D[y]$ by assumption, and hence r is constant on $D[y]$ for each $y \in S$. Thus, it follows that $r(x) = r(y)$ whenever $x \perp y$. Since S is connected, r is constant on S. □

If the poset is not connected, it is easy to construct a counterexample to Proposition 3.4. Consider the simple sum of two copies of $(\mathbb{N}, \leq)$. Put proportion p of a geometric distribution with rate α on the first copy and $1 - p$ of a geometric distribution with rate β on the second copy, where $p, \alpha, \beta \in (0, 1)$ and $\alpha \neq \beta$. The resulting distribution has rate α on the first copy and rate β on the second copy. If $\mathbf{X}$ is an IID sequence with the distribution so constructed and $\mathbf{Y}$ the corresponding sequence of ladder variables, then the conditional distribution of $(Y_1, \ldots, Y_{n-1})$ given $Y_n = y$ is uniform for each $y \in S$.

Suppose now that $\mathbf{X}$ is an IID sequence from a distribution with constant rate α and UPF F. Let $\mathbf{Y}$ denote the corresponding sequence of ladder variables, and suppose that the sequence $\mathbf{Y}$ is thinned with probability $p \in (0, 1)$, as described in Section 2.2. Let Y_M denote the first accepted point.

Proposition 3.5. *The PDF g of Y_M is given by*

$$g(x) = p\alpha\Lambda[x, \alpha(1 - r)]F(x), \qquad x \in S, \tag{3.14}$$

where Λ is the generating function associated with $(\lambda_n : n \in \mathbb{N})$.

Proof. For $x \in S$,

$$\begin{aligned} g(x) = \mathbb{E}[f_M(x)] &= \sum_{n=1}^{\infty} p(1-p)^{n-1} f_n(x) \\ &= \sum_{n=1}^{\infty} p(1-p)^{n-1} \alpha^n \lambda_{n-1}(x) F(x) \\ &= \alpha p F(x) \sum_{n=1}^{\infty} [\alpha(1-p)]^{n-1} \lambda_{n-1}(x) \\ &= \alpha p F(x) \Lambda[x, \alpha(1-p)]. \end{aligned} \tag{3.15}$$

□

In general, Y_M does not have constant rate but, as we will see in Section 5, does have constant rate when $(S, \preceq)$ is a tree.

Suppose now that $(S, \cdot)$ is a positive semigroup and that $\mathbf{X} = (X_1, X_2, \ldots)$ is an IID sequence. Let $\mathbf{Y}$ denote the sequence of ladder variables and $\mathbf{Z}$ the partial product sequence. If the underlying distribution of $\mathbf{X}$ is exponential, then the distribution has constant rate, so

Proposition 3.3 applies. But by Proposition 2.7, **Z** is also a homogeneous Markov chain with transition probability

$$h(y,z) = f\left(y^{-1}z\right) = \alpha F\left(y^{-1}z\right) = \alpha\frac{F(y)}{F(z)}, \quad y \in S,\ z \in I[y], \tag{3.16}$$

where α is the rate constant, f the PDF, and F the UPF. Thus Z also satisfies the results in Proposition 3.3, and, in particular, **Y** and **Z** are equivalent. The converse is also true.

Proposition 3.6. *If* **Y** *and* **Z** *are equivalent, then the underlying distribution of* **X** *is exponential.*

Proof. Let f denote the common PDF of **X** and F the corresponding UPF. Since $Y_1 = Z_1 = X_1$, the equivalence of **Y** and **Z** means that the two Markov chains have the same transition probability density, almost surely with respect to λ. Thus we may assume that

$$\frac{f(z)}{F(y)} = f\left(y^{-1}z\right), \quad (y,z) \in D_2. \tag{3.17}$$

Equivalently,

$$f(xu) = F(x)f(u), \quad x,u \in S. \tag{3.18}$$

Letting $u = e$, we have $f(x) = f(e)F(x)$, so the distribution has constant rate $\alpha = f(e)$. But then we also have $\alpha F(xu) = F(x)\alpha F(u)$, so $F(xu) = F(x)F(u)$, and hence the distribution is exponential. □

4. Relative Aging

In most cases, the reference measure λ of a standard poset $(S, \preceq, \mathcal{S}, \lambda)$ is natural in some sense-counting measure for discrete posets, Lebesgue measure for Euclidean posets, for example. However, another possibility is to use a given probability distribution on S to construct a reference measure and then to study the rate functions of other probability distributions relative to this reference measure. In this way, we can study the "relative aging" of one probability distribution with respect to another. (This was done for positive semigroups in [6].)

Thus, suppose that $(S, \preceq)$ is a poset and that $\mathcal{S}$ is a σ-algebra of subsets of S satisfying the assumptions in Section 1.2. Suppose that X is a random variable taking values in S, with UPF F assumed to be strictly positive. Let μ denote the distribution of X, so that $\mu(A) = \mathbb{P}(X \in A)$ for $A \in \mathcal{S}$. If we take μ itself to be the reference measure, then, trivially of course, the PDF of X is the constant function 1 (so that X is "uniformly distributed" on S with respect to μ). Thus the rate function is $1/F$, so that, curiously, X has increasing rate with respect to its own distribution.

Of course, this is not quite what we want. We would like to construct a measure on S that gives X constant rate 1 and then study the rate of other distributions relative to this measure. Define the measure ν on S by

$$\nu(A) = \int_A \frac{1}{F(x)} d\mu(x) = \mathbb{E}\left(\frac{1}{F(X)}; X \in A\right). \tag{4.1}$$

Thus, $d\mu(x) = F(x)d\nu(x)$. So X has PDF F, and hence has constant rate 1, with respect to ν. Now if Y is another random variable with values in S, then the definitions and results of Sections 2 and 3 apply, relative to ν. It is easy to see that Y has rate function q with respect to X, then X has rate function $1/q$ with respect to Y. In particular, Y has increasing (decreasing) (constant) rate with respect to X if and only if X has decreasing (increasing) (constant) rate with respect to Y, respectively.

Finally, suppose that S does in fact have a natural reference measure λ, so that $(S, \preceq, \mathcal{S}, \lambda)$ is a standard poset. If X and Y are random variables with rate functions r and s (with respect to λ), respectively, then the rate function of Y with respect to X is s/r.

5. Trees

In this section we consider a standard discrete poset $(S, \preceq)$ whose covering graph is a rooted tree. Aside from the many applications of rooted trees, this is one of the few classes of posets for which explicit computations are possible.

The root e is the minimum element. When $x \preceq y$, there is a unique path from x to y and we let $d(x, y)$ denote the distance from x to y. We abbreviate $d(e, x)$ by $d(x)$. Let $A(x)$ denote the children of x, and more generally let

$$A_n(x) = \{y \in S : x \preceq y,\ d(x, y) = n\}, \tag{5.1}$$

for $x \in S$ and $n \in \mathbb{N}$. Thus, $A_0(x) = \{x\}$ and $\{A_n(x) : n \in \mathbb{N}\}$ partitions $I[x]$. When $x = e$, we write A_n instead of $A_n(e)$.

The only trees that correspond to positive semigroups are those for which $A(x)$ has the same cardinality for every $x \in S$. In this case, the poset is isomorphic to the free semigroup on a countable alphabet, with concatenation as the semigroup operation [7].

Since there is a unique path from e to x, it follows from (1.12) that the cumulative function of order $n \in \mathbb{N}$ is

$$\lambda_n(x) = \binom{n + d(x)}{d(x)} = \binom{n + d(x)}{n}, \quad x \in S. \tag{5.2}$$

By (1.15), the corresponding generating function is

$$\Lambda(x, t) = \frac{1}{(1 - t)^{d(x)+1}}, \quad x \in S,\ |t| < 1. \tag{5.3}$$

5.1. Upper Probability Functions

Let X be a random variable with values in S having PDF f and UPF F. Then

$$F(x) = \mathbb{P}(X = x) + \sum_{y \in A(x)} \mathbb{P}(Y \succeq y) = f(x) + \sum_{y \in A(x)} F(y). \tag{5.4}$$

In particular, F uniquely determines f. Moreover, we can characterize upper probability functions.

Proposition 5.1. *Suppose that $F : S \to (0,1]$. Then F is the UPF of a distribution on S if and only if*

(1) $F(e) = 1$,

(2) $F(x) > \sum_{y \in A(x)} F(y)$ *for every* $x \in S$,

(3) $\sum_{x \in A_n} F(x) \to 0$ *as* $n \to \infty$.

Proof. Suppose first that F is the UPF of a random variable X taking values in S. Then trivially $F(e) = 1$, and, by (5.4),

$$F(x) - \sum_{y \in A(x)} F(y) = \mathbb{P}(X = x) > 0. \tag{5.5}$$

Next, $d(X) \geq n$ if and only if $X \succeq x$ for some $x \in A_n$. Moreover the events $\{X \succeq x\}$ are disjoint over $x \in A_n$. Thus

$$\mathbb{P}[d(X) \geq n] = \sum_{x \in A_n} F(x). \tag{5.6}$$

But by local finiteness, the random variable $d(X)$ (taking values in $\mathbb{N}$) has a proper (nondefective) distribution, so $\mathbb{P}[d(X) \geq n] \to 0$ as $n \to \infty$).

Conversely, suppose that $F : S \to (0,1]$ satisfies conditions (1.7)–(1.15). Define f on S by

$$f(x) = F(x) - \sum_{y \in A(x)} F(y), \quad x \in S. \tag{5.7}$$

Then $f(x) > 0$ for $x \in S$ by (1.12). Suppose that $x \in S$, and let $m = d(x)$. Then

$$\begin{aligned}
\sum_{k=0}^{n-1} \sum_{y \in A_k(x)} f(y) &= \sum_{k=0}^{n-1} \sum_{y \in A_k(x)} \left[F(y) - \sum_{z \in A(y)} F(z) \right] \\
&= \sum_{k=0}^{n-1} \left[\sum_{y \in A_k(x)} F(y) - \sum_{y \in A_k(x)} \sum_{z \in A(y)} F(z) \right] \\
&= \sum_{k=0}^{n-1} \left[\sum_{y \in A_k(x)} F(y) - \sum_{y \in A_{k+1}(x)} F(y) \right] \\
&= F(x) - \sum_{y \in A_n(x)} F(y)
\end{aligned} \tag{5.8}$$

since $A_0(x) = \{x\}$ and since the sum collapses. But

$$0 \le \sum_{y \in A_n(x)} F(y) \le \sum_{y \in A_{m+n}} F(y) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \tag{5.9}$$

Thus letting $n \to \infty$ in (5.8), we have

$$\sum_{y \in I[x]} f(y) = F(x), \quad x \in S. \tag{5.10}$$

Letting $x = e$ in (5.10) gives $\sum_{y \in S} f(y) = 1$ so f is a PDF on S. Another application of (5.10) then shows that F is the UPF of f. □

Note that $(S, \preceq)$ is a lower semilattice. Hence if X and Y are independent random variables with values in S, with UPFs F and G, respectively, then $X \wedge Y$ has UPF FG.

Proposition 5.2. *Suppose that $F : S \to (0,1]$ satisfies $F(e) = 1$ and*

$$F(x) \ge \sum_{y \in A(x)} F(y), \quad x \in S. \tag{5.11}$$

For $p \in (0,1)$, define $F_p : S \to (0,1]$ by $F_p(x) = p^{d(x)} F(x)$. Then F_p is an UPF on S.

Proof. First, $F_p(e) = p^0 F(e) = 1$. Next, for $x \in S$,

$$\begin{aligned}
\sum_{y \in A(x)} F_p(y) &= \sum_{y \in A(x)} p^{d(y)} F(y) = p^{d(x)+1} \sum_{y \in A(x)} F(x) \\
&\le p^{d(x)+1} F(x) < p^{d(x)} F(x) = F_p(x).
\end{aligned} \tag{5.12}$$

A simple induction using (5.11) shows that $\sum_{x\in A_n} F(x) \le F(e) = 1$ for $n \in \mathbb{N}$, so

$$\sum_{x\in A_n} F_p(x) = \sum_{x\in A_n} p^{d(x)} F(x) = p^n \sum_{x\in A_n} F(x) \le p^n \longrightarrow 0 \quad \text{as} n \longrightarrow \infty, \tag{5.13}$$

so it follows from Proposition 5.1 that F_p is an UPF. □

Note that $x \mapsto p^{d(x)}$ is not itself an UPF, unless the tree is a path, since properties (2) and (3) in Proposition 5.1 will fail in general. Thus, even when F is an UPF, we cannot view F_p simply as the product of two UPFs in general. However, we can give a probabilistic interpretation of the construction in Proposition 5.2 in this case. Thus, suppose that X is a random variable taking values in S with UPF F and PDF f. Moreover, suppose that each edge in the tree $(S, \preceq)$, independently of the other edges, is either *working* with probability p or failed with probability $1-p$. Define U by

$$U = \max\{u \preceq X : \text{the path from } e \text{to } u \text{is working}\}. \tag{5.14}$$

Corollary 5.3. *Random variable U has UPF F_p given in Proposition 5.2.*

Proof. If $X = x$ and $u \preceq x$, then $U \succeq u$ if and only if the path from e to u is working. Hence $\mathbb{P}(U \succeq u \mid X = x) = p^{d(u)}$ for $x \in S$ and $u \preceq x$. So conditioning on X gives

$$\mathbb{P}(U \succeq u) = \sum_{x \succeq u} p^{d(u)} f(x) = p^{d(u)} F(u). \tag{5.15}$$

□

5.2. Rate Functions

Next we are interested in characterizing rate functions of distributions that have support S. If r is such a function, then, from Proposition 2.5, $0 < r(x) \le 1$ and $r(x) = 1$ if and only if x is a leaf. Moreover, if F is the UPF, then $F(e) = 1$ and

$$\sum_{y\in A(x)} F(y) = [1 - r(x)]F(x). \tag{5.16}$$

Conversely, these conditions give a recursive procedure for constructing an UPF corresponding to a given rate function. Specifically, suppose that $r : S \to (0,1]$ and that $r(x) = 1$ for every leaf $x \in S$. First, we define $F(e) = 1$. Then if $F(x)$ has been defined for some $x \in S$ and x is not a leaf, then we define $F(y)$ for $y \in A(x)$ arbitrarily, subject only to the requirement that $F(y) > 0$ and that (5.16) holds. Note that F satisfies the first two conditions in Proposition 5.1. Hence if F satisfies the third condition, then F is the UPF of a distribution with support S and with the given rate function r. The following proposition gives a simple sufficient condition.

Proposition 5.4. *Suppose that $r : S \to (0,1]$ and that $r(x) = 1$ for each leaf $x \in S$. If there exists $\alpha > 0$ such that $r(x) \ge \alpha$ for all $x \in S$, then r is the rate function of a distribution with support S.*

Proof. Let $F : S \to (0,1]$ be any function constructed according to the recursive procedure above. Then as noted above, F satisfies the first two conditions in Proposition 5.1. A simple induction on n shows that

$$\sum_{x \in A_n} F(x) \leq (1-\alpha)^n, \quad n \in \mathbb{N}, \tag{5.17}$$

so the third condition in Proposition 5.1 holds as well. □

Condition (5.17) means that the distribution of $d(X)$ is stochastically smaller than the geometric distribution on $\mathbb{N}$ with rate constant α. If $(S, \preceq)$ is not a path, then the rate function does not uniquely determine the distribution. Indeed, if x has two or more children, then there are infinitely many ways to satisfy (5.16) given $F(x)$.

5.3. Constant Rate Distributions

Recall that if $(S, \preceq)$ has maximal elements (leaves), then there are no constant rate distribution with support S, except in the trivial case that $(S, \preceq)$ is an antichain.

Corollary 5.5. *Suppose that $(S, \preceq)$ is a rooted tree without leaves. Then $F : S \to (0,1]$ is the UPF of a distribution with constant rate α if and only if $F(e) = 1$ and*

$$\sum_{y \in A(x)} F(y) = (1-\alpha)F(x), \quad x \in S. \tag{5.18}$$

Proof. This follows immediately from Proposition 5.4. □

Corollary 5.6. *Suppose that X has constant rate α on $(S, \preceq)$. Then $d(X)$ has the geometric distribution on $\mathbb{N}$ with rate α.*

Proof. For $n \in \mathbb{N}$,

$$\mathbb{P}[d(X) \geq n] = \sum_{x \in A_n} \mathbb{P}(X \succeq x) = \sum_{x \in A_n} F(x) = (1-\alpha)^n. \tag{5.19}$$

□

As a special case of the comments in Section 5.2, we can construct the UPFs of constant rate distributions on $(S, \preceq)$ recursively: start with $F(e) = 1$. If $F(x)$ is defined for a given $x \in S$, then define $F(y)$ for $y \in A(x)$ arbitrarily, subject only to the conditions $F(y) > 0$ and that (5.18) holds.

5.4. Ladder Variables and the Point Process

Let F be the UPF of a distribution with constant rate α, so that F satisfies the conditions in Corollary 5.5. Let $\mathbf{X} = (X_1, X_2, \ldots)$ be an IID sequence with common distribution F, and let

$\mathbf{Y} = (Y_1, Y_2, \ldots)$ be the corresponding sequence of ladder variables. By Proposition 3.3, the distribution of Y_n has PDF

$$f_n(x) = \alpha^n \binom{n + d(x) - 1}{d(x)} F(x), \quad x \in S. \tag{5.20}$$

Consider now the thinned point process associated with $\mathbf{Y}$, where a point is accepted with probability p and rejected with probability $1 - p$, independently from point to point.

Proposition 5.7. *The distribution of the first accepted point has constant rate $p\alpha/(1 - \alpha + p\alpha)$.*

Proof. By Proposition 3.5, the PDF of the first accepted point is

$$\begin{aligned} g(x) &= p\alpha\Lambda(x, (1-p)\alpha)F(x) = p\alpha \frac{1}{[1 - (1-p)\alpha]^{d(x)+1}} F(x) \\ &= \frac{r\alpha}{1 - \alpha + p\alpha} \frac{F(x)}{(1 - \alpha + p\alpha)^{d(x)}}, \quad x \in S. \end{aligned} \tag{5.21}$$

Consider the function $G : S \to (0, 1]$ given by

$$G(x) = \frac{F(x)}{(1 - \alpha + p\alpha)^{d(x)}}, \quad x \in S. \tag{5.22}$$

Note that $G(e) = 1$ and for $x \in S$

$$\begin{aligned} \sum_{y \in A(x)} G(y) &= \sum_{y \in A(x)} \frac{F(y)}{(1 - \alpha + p\alpha)^{d(y)}} \\ &= \frac{1}{(1 - \alpha + p\alpha)^{d(x)+1}} \sum_{y \in A(x)} F(y) \\ &= \frac{1 - \alpha}{(1 - \alpha + p\alpha)^{d(x)+1}} F(x) \\ &= \frac{1 - \alpha}{1 - \alpha + p\alpha} G(x). \end{aligned} \tag{5.23}$$

□

In Proposition 5.7, the UPF F is related to the UPF G by the construction in Corollary 5.3. That is, suppose that Y denotes the first accepted point in the thinned process. Then the basic random variable X that we started with can be constructed as

$$X = \max\{x \preceq Y : \text{there is a working path from } e \text{ to } x\}, \tag{5.24}$$

where each edge is working, independently, with probability $1 - \alpha + p\alpha$.

6. The Existence Question

The existence of constant rate distributions for standard posets is an interesting mathematical question. Recall that the poset of finite subsets of $\mathbb{N}_+$ under set inclusion, considered in Example 1.9, is associated with a positive semigroup. This positive semigroup does not support exponential distributions [9], but we have been unable to determine if the poset supports constant rate distributions. We can show that if there is a random subset X with constant rate α, then the number of elements in X must have a Poisson distribution with parameter $-\ln(\alpha)$. A more general question is whether every standard discrete poset without maximal elements supports a constant rate distribution.

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