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Research Article

Multifractal Analysis of Infinite Products of Stationary Jump Processes

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There has been a growing interest in constructing stationary measures with known multifractal properties. In an earlier paper, the authors introduced the *multifractal products of stochastic processes* (MPSP) and provided basic properties concerning convergence, nondegeneracy, and scaling of moments. This paper considers a subclass of MPSP which is determined by jump processes with i.i.d. exponentially distributed interjump times. Particularly, the information dimension and a multifractal spectrum of the MPSP are computed. As a side result it is shown that the random partitions imprinted naturally by a family of Poisson point processes are sufficient to determine the spectrum in this case.

1. Introduction

The measures resulting from the limits of random multiplicative martingales have attracted much attention in the mathematical community since the the work by Kahane on positive martingales [1–3]; these martingales are of the form $\int Q_n(t)d\sigma(t)$, where $Q_n(t)$ forms a positive martingale for each t. Related early ideas go back to de Wijs [4, 5], Kolmogorov [6], Novikov and Stewart [7], Yaglom [8], and Mandelbrot [9–12], and emerged mostly in the context of turbulence. Recently, Barral and Mandelbrot have published a series of papers [13, 14] completing Kahane's general theory of T-martingales.

Research on multiplicative cascades has been very active. Especially, Mandelbrot's martingale [9, 10], a simple tree-based construction with independent random multipliers, has been considered in a large number of publications; first by Kahane and Peyrière [15], and the story still continues (see e.g. [16–23]). Extensions such as relaxing the independence assumption of the multipliers or randomizing the number of offsprings have also been studied. To give a short list without intention of being complete, we refer to Molchan [19] and Waymire and Williams [24, 25] regarding dependent multipliers, and to Peyrière [26], Arbeiter [27], and Burd and Waymire [28] regarding random numbers of offsprings.

Classes of Kahane's martingales that move away from a tree-based structure include Gaussian chaos by Kahane [1], Lévy chaos by Fan [29], random Gibbs measures by Fan and Shieh [30], random coverings by Kahane and Fan [2, 3, 31, 32], multifractal product of cylindrical pulses by Barral and Mandelbrot [13, 14, 33], an infinitely divisible cascades by Bacry and Muzy [34, 35], Chainais et al. [36, 37] and Riedi and Gershman [38], as well as products of stochastic processes by the present authors [39] and subsequently by Anh et al. [40, 41] and Matsui and Shieh [42].

Processes defined through iterative multiplication such as the above exhibit rich scaling properties and ubiquitous, fine details. Multifractal analysis, in a nutshell, strives at capturing the essence of such complex scaling behaviour through geometric and probabilistic descriptors. The multifractal spectrum of a given realization, on one hand, is given as the Hausdorff dimension of all points where the local Hölder regularity has a given degree; it is local and random in nature. The multifractal envelope, on the other hand, is defined in terms of the scaling of moments and constitutes, thus, a global and deterministic descriptor. It is well known that the envelope provides an almost sure upper bound to the spectrum, a fact that can easily be established through arguments akin to large deviation principles. The goal of multifractal analysis lies in establishing almost sure equality of spectrum and envelope.

Motivated by natural, desirable properties of a process such as stationarity, this paper studies the multifractal spectrum of an infinite product of stationary jump processes, defined as the limit of a particular choice of martingales of Kahane:

$$A(t) = \lim_{n \to \infty} \int_0^t \prod_{i=1}^{n} \Lambda^{(i)}(s) \mathrm{d}s, \tag{1.1}$$

where the $\Lambda^{(i)}$ are independent rescaled versions of a mean one, stochastic mother process Λ. We refer to Anh et al. [40, 41] for a collection of such products of stochastic processes for which the multifractal envelope is explicitly computed. Recently, the $\mathcal{L}_2$ convergence of $A(t)$ with a long range-dependent Λ is shown by Matsui and Shieh [42]. In this paper, we study the case $\Lambda^{(i)}(\cdot) \overset{\text{dist}}{=} \Lambda(b^i \cdot)$, where Λ is a stationary jump process with i.i.d. exponential interjump times.

Overview and contributions: Section 2 reviews the multifractal analysis in the context of restricted coverings, recalls the less known semispectra needed in our approach, and establishes the corresponding multifractal formalism. The main purpose of this section is to provide solid grounds for the reader less familiar with the field of fractals and to clearly state results that are not found in common books. Section 3 develops the multifractal properties of the infinite product of jump processes and constitutes the heart of the paper. To compute the multifractal spectrum we establish the multifractal formalism, for the infinite product of jump processes. To our best of knowledge, these are the first processes that may exhibit *temporal dependence* over arbitrarily large lags and for which the formalism has been established. Notably, the absence of an a priori bound on the length of the temporal dependence precludes the use of common methods as found, for example, in [14]. To overcome this hurdle, we take advantage of a more general version of the multifractal formalism than is usually applied. More precisely, we base the multifractal computations on partitions created naturally by the Poisson processes underlying the processes at hand. While Poisson partitions can in general not be used to compute the ordinary Hausdorff dimension of sets, they will suffice here. Moreover, the Poisson partition yields the same multifractal envelope as the one obtained with the usual dyadic partitions in earlier work [39].

2. Multifractal Spectra and Formalism

The multifractal spectrum of a process is defined pathwise. Therefore, we start in sections 2.1–2.3 by providing the relevant notions for a given (deterministic) increasing function, or its related measure. We also develop tools which allow for computing the multifractal spectrum of a measure in ways which are adapted to the inherent structure of the processes we will study in this paper. In 2.4 we proceed to random measures, providing almost sure upper bounds on the pathwise multifractal spectrum. It will be the task of the remainder of the paper to establish conditions under which these upper bounds are actually tight.

In this paper, we consider scaling exponents and multifractal spectra which are defined via nonhomogenous partitions. The results have analogous counter-parts in the standard setting but it should be noted that these are not fully equivalent notions.

2.1. Hausdorff Dimension

Let us start by recalling notions related to Hausdorff dimension. Consider a set $F \subset \mathbb{R}^d$ and a class $\mathcal{C}$ of subsets of $\mathbb{R}^d$. Then, the s-dimensional *Hausdorff measure with respect to (w.r.t.)* $\mathcal{C}$ is given by

$$h_s^{\mathcal{C}}(F) \doteq \lim_{\delta \to 0} \inf\left\{ \sum_i |U_i|^s : \ 0 < |U_i| \leq \delta, \ U_i \in \mathcal{C} \ \forall i; \ F \subseteq \bigcup_i U_i \right\}, \tag{2.1}$$

where the diameter of set U is denoted by $|U| = \sup\{|x-y| : \ x, y \in U\}$. Then

$$\dim^{\mathcal{C}}(F) \doteq \sup_s\{s \geq 0 : \ h_s(F) = \infty\} = \inf_s\{s \geq 0 : \ h_s(F) = 0\} \tag{2.2}$$

is called the *Hausdorff dimension of set* F *w.r.t.* $\mathcal{C}$. By definition, when choosing the powerset $2^{\mathbb{R}^d}$, that is, the class of all subsets of $\mathbb{R}^d$ as the class $\mathcal{C}$ in (2.1), then these notions reduce to the usual s-dimensional Hausdorff measure h_s and dimension dim:

$$h_s(F) = h_s^{2^{\mathbb{R}^d}}(F), \qquad \dim(F) = \dim^{2^{\mathbb{R}^d}}(F). \tag{2.3}$$

Clearly, $\dim(F) \leq \dim^{\mathcal{C}}(F)$. The Hausdorff dimension w.r.t. $\mathcal{C}$ is σ-stable, that is, $\dim^{\mathcal{C}}(\cup_j F_j) = \sup_j \dim^{\mathcal{C}}(F_j)$ for any countable family of sets F_j. This can be established as in the classical case. See Rogers [43] for more properties of Hausdorff measures.

Numerous classes of sets $\mathcal{C}$ are known to result in the usual Hausdorff dimension (see e.g., [44–46]). For our purposes, it will be sufficient to consider certain nested classes $\mathcal{C}$ and to establish a simple sufficient condition to warrant that $\dim(F) = \dim^{\mathcal{C}}(F)$. The simplest example of such a family is the b-ary cubes of the form

$$[k_1 b^{-n}, (k_1+1)b^{-n}) \times \cdots \times [k_d b^{-n}, (k_d+1)b^{-n}) \tag{2.4}$$

where $b > 1$. To generalize this fact, recall that a collection is called *nested* if for all pairs of its sets A, B, we have $A \cap B = \emptyset A \subseteq B$, or $B \subseteq A$. Also, we use the term *cube* for any product of intervals of equal lengths.

Lemma 2.1. *Let $F \subset \mathbb{R}^d$ and $\mathcal{C}$ be a nested collection of arbitrary cubes. Assume that for all $x \in F$ there exists a sequence $\{C_n(x)\}_{\mathbb{N}} \in \mathcal{C}$ such that $x \in C_n(x) \neq \{x\}$ for all n,*

$$\lim_{n\to\infty} |C_n(x)| = 0, \qquad \limsup_{n\to\infty} \frac{\log|C_n(x)|}{\log|C_{n-1}(x)|} \le 1. \tag{2.5}$$

Then $\dim\,(F) = \dim^{\mathcal{C}}(F)$.

The use of nested covers with a condition akin to (2.5) is quite standard. In [45], for example, a Frostman-type result based on nested cubes is derived. In [46], the equality $\dim\,(F) = \dim^{\mathcal{C}}(F)$ is established under assumptions which are somewhat more restrictive than (2.5) and prevent the use of Poisson covers (compare lemma 3.6). As we do not need this result directly we leave the proof to the interested reader who will have no problems rewriting the arguments of [45] for a proof of the result given here. In doing so, exploit that the covers are nested to avoid the need of a lower bound on $|C_n(x)|$ as used in [45]. A proof is contained in the technical report [47].

A standard method to get a lower bound for the Hausdorff dimension of a set is based on a scaling law, which appears as *mass distribution principle* in [44]. For our purposes, we need the following stronger lemma which follows from [45]. Note that condition (ii) can be replaced by the less restrictive (2.5), following the classical argument of the Frostman lemma applied to the covers $\mathcal{C}$, combined with lemma 2.1.

Lemma 2.2 (see [45, Lemma 4.3.2.b]). *Let ν be a Borel measure on $\mathbb{R}^d$, F a Borel subset, and $\mathcal{C} = \{C_k^n : k = 1, \ldots, N_n, n \in \mathbb{N}\}$ a nested collection of cubes. Assume $\nu(F) > 0$,*

(i) *$\{C_k^n\}_{k=1}^{N_n}$ cover F for all n,*

(ii) *there exists $b > 1$ such that $(1/n)\log|C_k^n| \to -\log b$ as $C_k^n \to \{x\}$ for all $x \in F$,*

(iii) *$\log\nu(C_k^n)/\log|C_k^n| \to s$ as $C_k^n \to \{x\}$ for all $x \in F$.*

Then, $\dim(F) = \dim^{\mathcal{C}}(F) \ge s$.

2.2. Multifractal Spectrum

Let ν be a Borel measure in $\mathbb{R}^d$. Its Hölder exponent at x, or equivalently local dimension at x, is given by

$$\dim_{\mathrm{loc}}\nu(x) = \lim_{r\to 0} \frac{\log\nu(B(x,r))}{\log r} \tag{2.6}$$

if this limit exists. In multifractal analysis, we are usually interested in the properties like fractal dimension of the set $\{x : \dim_{\text{loc}}\nu(x) = \alpha\}$.

A more approachable framework to study scaling properties is to consider a sequence of finer and finer partitions $C^n \doteq \{C_k^n : k = 1, \ldots, N_n\}$ of a set containing the support of measure. The standard example in $\mathbb{R}$ is to partition the unit interval by the dyadic intervals $[(k-1)2^{-n}, k2^{-n})$, $k = 1, \ldots, 2^n$.

Definition 2.3. Let F be a Borel subset of $\mathbb{R}^d$ and $C^n = \{C_k^n : k = 1, \ldots, N_n\}$ a partition of F for each $n = 1, 2, \ldots$. We call the C^n *nested partitions*, if any $C_k^n \in C^n$ $(n \geq 2)$, $k = 1, \ldots, N_n$, is a subset of some $C_{k'}^{n-1} \in C^{n-1}$, $k' \in \{1, \ldots, N_{n-1}\}$.

Let

$$\alpha_n^{\mathcal{C}}(x) \doteq \frac{\log \nu(C^n(x))}{\log|C^n(x)|}, \tag{2.7}$$

where $\mathcal{C} = \{C^n : n = 1, \ldots\}$ is a collection of nested partitions C^n of supp, ν and $C^n(x)$ is the unique $C_k^n \in C^n$ containing x. Then the local scaling exponent (w.r.t. $\mathcal{C}$) of ν at x is given by

$$\alpha^{\mathcal{C}}(x) \doteq \lim_{n \to \infty} \alpha_n^{\mathcal{C}}(x) \tag{2.8}$$

if this limit exists. One should think of $\alpha^{\mathcal{C}}(x)$ as giving approximately the degree of Hölder regularity of ν at x, that is, an approximation to $\dim_{\text{loc}}\nu(x)$. It is tempting to conjecture that these two notions are equal, except in a set of dimension zero, provided that the partitions satisfy the conditions of Lemma 2.2. While this conjecture is known to hold for certain (random) measures ν such as the binomial cascade, a general proof seems hard to come by.

We consider the sets

$$E_\alpha^{\mathcal{C}} \doteq \left\{x : \lim_n \alpha_n^{\mathcal{C}}(x) = \alpha\right\}, \qquad V_\alpha^{\mathcal{C}} \doteq \left\{x : \liminf_n \alpha_n^{\mathcal{C}}(x) \leq \alpha\right\} \tag{2.9}$$

characterizing the local scaling behaviour. The classical literature on multifractals studies typically the sets $E_\alpha^{\mathcal{C}}$, calling dim $E_\alpha^{\mathcal{C}}$ the *multifractal spectrum* of the measure ν. The sets $V_\alpha^{\mathcal{C}}$ are somewhat easier to study, yet provide spectral information in the sense of multifractal analysis, as we will point out in this section. Also the sets $\{x : \liminf_n \alpha_n^{\mathcal{C}}(x) = \alpha\}$ would be of interest since they form a partition of space. However, their dimension would lie between $\dim(E_\alpha^{\mathcal{C}})$ and $\dim(V_\alpha^{\mathcal{C}})$. Trivially, $E_\alpha^{\mathcal{C}} \subseteq V_\alpha^{\mathcal{C}}$ and $\dim(E_\alpha^{\mathcal{C}}) \leq \dim(V_\alpha^{\mathcal{C}})$.

2.3. Pathwise Partition Function and Coarse Grain Spectra

In order to make the presentation simpler, we consider only the 1-dimensional case assuming that supp $\nu \subseteq [0, 1]$, but all the definitions are easily extended to compactly supported measures on $\mathbb{R}^d$.

Consider nested partitions $J^n = \{J_k^n : k = 1, \ldots, N_n\}$ of the unit interval. Analogous to the previous section, denote $\mathcal{C} = \{J^n : n = 1, \ldots\}$ the nested collection of sets which is determined by the J^n. The partition sum of the measure ν is defined by

$$S_n^{\mathcal{C}}(q,\gamma) \doteq S_n(q,\gamma) = \sum_{k=1}^{N_n} \nu(J_k^n)^q \left|J_k^n\right|^{-\gamma}, \tag{2.10}$$

where we adapt the convention $0^q \doteq 0$ for all $q \in \mathbb{R}$. The only purpose of this convention is to provide a convenient way to ensure that the sets J_k^n with $\nu(J_k^n) = 0$ do not contribute to any of the $S_n(q)$. Define then the partition function as

$$\tau_{\mathcal{C}}(q) \doteq \tau(q) = \sup\left\{\gamma : \sum_n S_n^{\mathcal{C}}(q,\gamma) < \infty\right\}. \tag{2.11}$$

We drop the index $\mathcal{C}$ whenever the choice of the partitions is clear from the context.

The above functionals can be used to characterize multifractal properties through the Legendre transform

$$\tau^*(\alpha) \doteq \inf_q(\alpha q - \tau(q)). \tag{2.12}$$

Then $\tau^*(\alpha)$ denotes a pathwise large deviations spectrum. In the special case where the partitions of $\mathcal{C}$ consist of the b-ary intervals, the above definition of $\tau_{\mathcal{C}}$ reduces to the standard one (for an overview see [48]).

To provide upper bounds on the multifractal spectra, it is useful to introduce the so-called "coarse grain" spectra. To this end, the following collections of intervals are of interest:

$$N_n^+(\alpha) \doteq \left\{k : J_k^n \cap [0,1] \neq \emptyset, \frac{\log \nu(J_k^n)}{\log\left|J_k^n\right|} < \alpha\right\}. \tag{2.13}$$

The reference to large deviations relies on the fact that N_n^+ may be represented as a rare event or a "large deviation from the mean" if one considers $\log \nu(J_k^n)/\log|J_k^n|$ to be the random variable, where k is random but ν and J^n are fixed and $n \to \infty$. In the proper setting, the following spectrum $f_{\mathcal{C}}^+$ becomes a large deviation rate function:

$$f_{\mathcal{C}}^+(\alpha) \doteq f^+(\alpha) \doteq \inf\left\{\delta > 0 : \sum_{n \geq m} \sum_{k \in N_n^+(\alpha)} \left|J_k^n\right|^{\delta} \longrightarrow 0 \text{ as } m \longrightarrow \infty\right\}. \tag{2.14}$$

In the following, we collect some properties needed to order the different spectra. Note that these results specifically hold for spectra and partition functions defined via non-homogeneous partitions. The proofs are quite straightforward but they are shown for the sake of completeness.

Lemma 2.4. *For Borel measure ν, the following holds. If* $\max_k|J_k^n| \to 0$ *as* $n \to \infty$ *then*

$$\dim^{\mathcal{C}}\left(V_\alpha^{\mathcal{C}}\right) \le f_{\mathcal{C}}^+(\alpha+) \doteq \lim_{\varepsilon\downarrow 0} f_{\mathcal{C}}^+(\alpha+\varepsilon). \tag{2.15}$$

Proof. Fix α and take any γ such that $\gamma > f^+(\alpha+)$. Then there exists $\varepsilon > 0$ such that $f^+(\alpha+\varepsilon) < \gamma$. Let m be an arbitrary positive integer. Then, for any $t \in V_\alpha^{\mathcal{C}}$ there exists $n > m$ such that $\alpha_n(t) < \alpha+\varepsilon$ and, thus, there is $k \in N_n^+(\alpha+\varepsilon)$ such that $J_n(t) = J_k^n$. Since $t \in J_n(t)$, we get

$$V_\alpha^{\mathcal{C}} \subseteq \bigcup_{n\ge m}\ \bigcup_{k\in N_n^+(\alpha+\varepsilon)} J_k^n. \tag{2.16}$$

This means that for every m we have constructed a cover of $V_\alpha^{\mathcal{C}}$ with J_k^n such that

$$\max_{n\ge m, k\in N_n^+(\alpha+\varepsilon)} \left|J_k^n\right| \to 0, \qquad \sum_{n\ge m}\ \sum_{k\in N_n^+(\alpha+\varepsilon)} \left|J_k^n\right|^\gamma \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{2.17}$$

We conclude that $h_\gamma^{\mathcal{C}}(V_\alpha^{\mathcal{C}}) = 0$ and $\dim^{\mathcal{C}}(V_\alpha^{\mathcal{C}}) \le \gamma$. □

Lemma 2.5. *Let ν be a Borel measure and let the J^n be nested partitions. Then, for all real α, $f^+(\alpha) \le q\alpha - \tau(q)$ for $q \ge 0$, and $\tau(q) \le (f^+)^*(q)$ for $q > 0$.*

Proof. Let α, q, and γ be arbitrary real numbers for the moment. By definition, $\nu(J_k^n) > |J_k^n|^\alpha$ for all k in $N_n^+(\alpha)$. For *nonnegative* q we may, thus, estimate

$$S_n(q,\gamma) \ge \sum_{N_n^+(\alpha)} \nu(J_k^n)^q \left|J_k^n\right|^{-\gamma} \ge \sum_{N_n^+(\alpha)} \left|J_k^n\right|^{q\alpha-\gamma}. \tag{2.18}$$

Fix a real α and $q \ge 0$. Consider $\gamma < \tau(q)$. By definition of τ, we find

$$\sum_{n\ge m}\ \sum_{N_n^+(\alpha)} \left|J_k^n\right|^{q\alpha-\gamma} \longrightarrow 0 \quad \text{as } m \longrightarrow \infty. \tag{2.19}$$

Consequently, $q\alpha - \gamma \ge f^+(\alpha)$. Since γ can be chosen arbitrarily close to $\tau(q)$, this actually implies that $q\alpha - \tau(q) \ge f^+(\alpha)$ for all non-negative q, as claimed. □

The convexity of $\tau(q)$ can be established for $q > 0$ under mild conditions.

Lemma 2.6. *Let J^n form nested partitions such that $\sum_n \max_k|J_k^n| < \infty$. Let ν be a bounded Borel measure. Then $\tau(q) = (f^+)^*(q)$ and τ is convex for $q > 0$.*

Proof. Replacing ν by $c\nu$ for a sufficiently small c, we may assume that $\alpha_n(t) \ge 0$. Fix $q > 0$. Let $\gamma < \inf_\alpha(q\alpha - f^+(\alpha))$, let $\varepsilon > 0$ be such that $\gamma < q\alpha - f^+(\alpha) - \varepsilon q$ for every α, and let U_j

denote the interval $[(j-1)\varepsilon, j\varepsilon]$. Let M be sufficiently large such that $qM-\gamma>2$. Let p be large enough so that $p\varepsilon \geq M$. Then

$$\begin{aligned} S_n(q,\gamma) &\leq \left(\sum_{j=1}^{p}\ \sum_{k:\log\nu(J_k^n)/\log|J_k^n|\in U_j} + \sum_{k:\ \log\nu(J_k^n)/\log|J_k^n|>M}\right)\nu(J_k^n)^q\left|J_k^n\right|^{-\gamma} \\ &\leq \sum_{j=1}^{p}\ \sum_{k\in N_n^+(j\varepsilon)}\left|J_k^n\right|^{q((j-1)\varepsilon)}\left|J_k^n\right|^{-\gamma} + \sum_{k:\ J_k^n\cap[0,1]\neq\emptyset}\left|J_k^n\right|^{qM}\left|J_k^n\right|^{-\gamma}. \end{aligned} \tag{2.20}$$

By choice of M, and since the J_k^n are disjoint subsets of $[-1,2]$ for n large enough,

$$\sum_{J_k^n\cap[0,1]\neq\emptyset}|J_k^n|^{qM}|J_k^n|^{-\gamma} \leq \sum_{J_k^n\cap[0,1]\neq\emptyset}\left|J_k^n\right|^2 \leq \max_k\left|J_k^n\right|\sum_{J_k^n\cap[0,1]\neq\emptyset}\left|J_k^n\right| \leq 3\max_k\left|J_k^n\right|. \tag{2.21}$$

By choice of γ, it follows that $\tau(q) \geq \gamma$, thus $\tau(q) \geq (f^+)^*(q)$. The equality follows from Lemma 2.5 and a Legendre transformation is always convex. □

2.4. Deterministic Envelope for Random Measures

Let us now consider a random measure μ defined on $[0,1]$ and a random nested collection $\mathcal{C}$. The asymptotics of ensemble moments are given by

$$T_{\mathcal{C}}(\mathrm{q}) \doteq T(q) = \sup\left\{\gamma:\ \sum_{n=1}^{\infty}\mathbf{E}\left[S_n^{\mathcal{C}}(q,\gamma)\right] < \infty\right\}. \tag{2.22}$$

Then $T^*(\alpha)$ denotes the deterministic large deviation spectrum. The following lemma shows that if T is convex and τ is convex a.s., then T^* is an upper bound to τ^*. Recall that τ^* provides a pathwise upper bound to f (see Lemma 2.5) which in turn bounds the multifractal spectrum from above.

To check if $T(q)$ is convex, one can usually calculate the function explicitly, whereas establishing the convexity of $\tau(q)$ is more subtle. In the previous section, we stated Lemma 2.6 which provides a way to guarantee a.s. convexity.

Lemma 2.7. *If T is convex and τ is convex a.s. in an interval I, then a.s. $\tau(q) \geq T(q)$ for all $q \in I$.*

Proof. Fix $q \in I$ such that $T(q) < \infty$ and take any $\gamma < T(q)$. Since $S_n(q,\gamma)$ is positive,

$$\sum_{n\geq m}\mathbf{E}\left[S_n(q,\gamma)\right] = \mathbf{E}\left[\sum_{n\geq m}S_n(q,\gamma)\right] < \infty, \tag{2.23}$$

and thus $\sum_{n\geq m}S_n(q,\gamma) < \infty$ a.s., that is, $\tau(q) > \gamma$ a.s. Next, choose $\gamma_m \nearrow T(q)$. In conclusion, $\tau(q) \geq T(q)$ a.s., for any countable set of $q \in I$. Since τ and T are a.s. convex, the statement holds a.s. for all $q \in I$. □

Combining lemmas 2.4, 2.5, and 2.7, we can now order the multifractal spectra.

Theorem 2.8. *If $T(q)$ is convex and $\tau(q)$ is convex a.s. for $q > 0$, and $\lim_{n\to\infty}\max_k|J_k^n| = 0$ a.s., then the multifractal spectra are ordered as follows: with probability one, for all α,*

$$\dim^{\mathcal{C}}\left(E_\alpha^{\mathcal{C}}\right) \le \dim^{\mathcal{C}}\left(V_\alpha^{\mathcal{C}}\right) \le f_{\mathcal{C}}^{+}(\alpha+) \le \inf_{q>0}\left(q\alpha - \tau_{\mathcal{C}}(q)\right) \le \inf_{q>0}\left(q\alpha - T_{\mathcal{C}}(q)\right). \tag{2.24}$$

The basic arguments that lead to Theorem 2.8 are sometimes addressed as "steepest ascent" methods or as large deviation arguments. They are quite standard in multifractal analysis [12, 14–17, 19–22, 24, 30, 34, 47–50] to obtain an upper bound on the multifractal spectrum $\dim(E_\alpha)$ in a setting of interest. Notably, (2.24) is quite generally applicable since a mild condition on the covers asserts that τ is convex without the need of strong assumptions on the measure. Note also that some of the inequalities of (2.24) may be strict [51, 52].

Arbitrary partitions in a deterministic setting have been studied by Brown et al. [53]. Using their results, the pathwise inequalities of Theorem 2.8, that is, the parts not including $T_{\mathcal{C}}$, could have been established under slightly different technical assumptions. However, our setting leads to the assumptions that are more easily verified for processes studied in this paper.

In certain settings, different partition functions might become useful, yet still provide upper bounds akin to the above [20, 53–56]. A very typical setting is to form the partitions $\mathcal{C}$ via dyadic cubes. The formalism (2.24) is most useful only for the increasing part of the spectrum as the right-hand side is convex and increasing. Using similar arguments one can easily obtain an upper bound which uses the negative range of q values and is decreasing and convex, as is done, for example, in [47].

3. Multifractal Jump Processes

3.1. Definition and Basic Properties

We start from T-martingales defined by independent multiplication as in [2]. Consider a family of independent *positive* processes $\Lambda^{(i)}$ which are independent rescaled copies of a stationary mother process Λ defined on $\mathbb{R}^+$. In this paper, we restrict to the class of jump processes satisfying the following assumptions:

(A1) Λ is stationary with $\Lambda > 0$ and $\mathbf{E}[\Lambda] = 1$; there exists $b > 1$ such that $\Lambda^{(i)}(\cdot/b^i)$ are i.i.d. processes, distributed as Λ;

(A2) Λ is a weakly mixing Markov process which is defined by $\Lambda(t) \doteq M_k$ for $T_k \le t < T_{k+1}$, where $T_0 = 0$ and $\{T_{k+1} - T_k\}_k$ are i.i.d. exponential random variables of mean $1/\lambda$, and where the M_k form a stationary positive time series independent of $\{T_k\}_k$ satisfying $\mathbf{E}[M_k] = 1$ for all k.

Recall that a real-valued stationary process is weakly mixing if for all $B_1, B_2 \in \mathcal{B}$ (the σ-algebra generated by the process) $\lim_{T\to\infty} T^{-1}\int_0^T |P(B_1 \cap S^t B_2) - P(B_1)P(B_2)|\, \mathrm{d}t = 0$ where S^t is the shift operator. Examples of interest include the cases where the sequence $\{M_k\}_k$ forms an i.i.d. sequence of random variables (Λ is a renewal reward process) or a finite state irreducible Markov chain. The requirement that the multiplier processes are strictly independent of the partitioning is added for convenience here and more general cases can be studied. For example, if Λ is a finite state Markov process, then analogous results can be derived.

Next define the finite product processes

$$\Lambda_n(t) \doteq \prod_{i=0}^{n-1} \Lambda^{(i)}(t). \tag{3.1}$$

For $t \in [0,1]$, the cumulative processes

$$A_n(t) \doteq \int_0^t \Lambda_n(s)\,\mathrm{d}s = \int_0^t \prod_{i=0}^{n-1} \Lambda^{(i)}(s)\,\mathrm{d}s, \quad n = 1,2,\ldots \tag{3.2}$$

can be associated with positive measures defined on the Borel sets $\mathcal{B}$ of $[0,1]$:

$$\mu_n(B) \doteq \int_B \Lambda_n(s)\,\mathrm{d}s, \quad n = 1,2,\ldots, \; B \in \mathcal{B}. \tag{3.3}$$

We note that the restriction to the unit interval is purely for convenience, and extensions to compact intervals and to the real line are straightforward.

In the context of the martingales of Kahane [2] and in multifractal analysis, we are interested in the limit measure $\mu \doteq \lim_n \mu_n$ and its associated cumulative process A. The existence of the limiting objects (possibly degenerate) is established in [2].

Definition 3.1. Assume (A1) and (A2). Then, the *multifractal jump process* (MJP) is the limit

$$A(t) \doteq \mu([0,t]) \doteq \lim_{n \to \infty} \mu_n([0,t]) = \lim_{n \to \infty} A_n(t) \quad \text{a.s.} \tag{3.4}$$

An MJP is called *degenerate* if $A(1) = 0$ almost surely, and nondegenerate otherwise.

It can be shown that A is non-degenerate if and only if $\mathbf{E}[A(1)] = 1$ (see [39]). The convergence of $A_n(1)$ in $\mathcal{L}_p$ for some $p > 1$ naturally implies nondegeneracy since then $\mathbf{E}[A(1)] = \lim_n \mathbf{E}[A_n(1)] = 1$. To provide criteria for convergence in $\mathcal{L}_p$ is thus of central importance. As pointed out in [2], criteria for convergence in $\mathcal{L}_2$ are particularly manageable and useful. In the following theorem, which is a straightforward adaption of [39, Corollary 3], the $\mathcal{L}_2$ conditions are stated for MJPs.

Theorem 3.2 (see [39, Corollary 3]). *Assume (A1), (A2), and* $\mathbf{E}[M^2] < \infty$. *If there are positive* γ_1, γ_2 *and* C *with*

$$\sigma^2 e^{-\gamma_1 |x|} \leq \operatorname{Cov}(\Lambda(0), \Lambda(x)) = \sum_{j=0}^{\infty} \frac{(\lambda x)^j}{j!} \operatorname{Cov}(M_0, M_j) e^{-\lambda x} \leq C|x|^{-\gamma_2}, \tag{3.5}$$

then $A_n(t)$ *converges in* $\mathcal{L}_2$ *if and only if* $b > 1 + \sigma^2$ *where* $\sigma^2 = \operatorname{Var}(M)$.

Condition (3.5) allows a very large family of time series M_k. For example, a sufficient condition for convergence is

$$\operatorname{Cov}(M_0, M_j) \leq \frac{C}{j+1}, \quad j = 0, 1, 2, \ldots, \tag{3.6}$$

for some constant $C > 0$.

The majority of the following results are stated assuming a converging MJP in some $\mathcal{L}_p$, $p > 0$. The conditions for $p = 2$ are, of course, sufficient for $p \in (0, 2]$. In principle, setting explicit conditions for $p > 2$ would be possible. However, even for the simplest possible case when the M_j are i.i.d. and the $\mathcal{L}_3$ convergence is considered, long and tedious calculations are needed. Thus the general convergence considerations are out of the scope of this paper.

The assumption of weak mixing in (A2) is needed to show that the limit measure μ is ergodic with respect to the time shift operator. Ergodicity of μ, together with the positivity of the multipliers M_k, guarantees that the random variable $\mu([0,t])$ does not have an atom at zero. This result, in turn, will be used in the multifractal decomposition.

Proposition 3.3. *The associated measure μ on $\mathbb{R}^+$ of an MJP is ergodic.*

Proof. Consider the spaces $\Omega_i = (X_i, \mathcal{B}_i, P_i)$, $i = 0, 1, \ldots$, and $\Omega = \prod_{i=0}^{\infty} \Omega_i = (X, \mathcal{B}, P)$, where X_i is the set of piece-wise continuous functions, $\mathcal{B}_i$ its Borel-algebra, and P_i is the law of $\Lambda^{(i)}$. The shift operator is weakly mixing in Ω_i and thus it is also weakly mixing in $\Omega_0 \times \cdots \times \Omega_{n-1}$ for any fixed n. Since the $B_0 \times \cdots \times B_{n-1} \times \prod_{i=n}^{\infty} \Omega_i$ form a semialgebra generating $\mathcal{B}$, it follows that the shift operator is also weakly mixing in the infinite product space Ω (see e.g., [57, Theorem 1.7]). Weak mixing implies ergodicity which is then trivially inherited to the subsystem determining the random measure μ. $\square$

The following measures are instrumental in our analysis:

$$\mu^{(n)}(B) \doteq \int_B \prod_{i=0}^{n-1} \frac{1}{\Lambda^{(i)}(\tau)} \mathrm{d}\mu(\tau), \tag{3.7}$$

that is, the measure where n first terms of the product are neglected. Thus

$$\mu^{(n)}(B) \overset{\text{dist}}{=} b^{-n} \mu(b^n B). \tag{3.8}$$

Proposition 3.4. *Assume that A is a non-degenerate MJP. Then, with probability one, $\mu([0,t]) > 0$ for all $t \in (0,1]$.*

Proof. First consider an arbitrary $t > 0$ and make a change of variable to get

$$\mu([0,t]) = \int_0^t \Lambda_n(s) \mathrm{d}\mu^{(n)}(s) \overset{\text{dist}}{=} b^{-n} \int_0^{tb^n} \Lambda_n(sb^{-n}) \mathrm{d}\tilde{\mu}(s), \tag{3.9}$$

where μ and $\tilde{\mu}$ are identically distributed and $\tilde{\mu}$ independent of Λ_n. Then by the positivity of Λ_n and ergodicity of μ, we have

$$\begin{aligned} \mathbf{P}(\mu([0,t]) = 0) &= \mathbf{P}\left(b^{-n}\int_0^{tb^n} \Lambda_n(sb^{-n})\mathrm{d}\tilde{\mu}(s) = 0\right) = \mathbf{P}(b^{-n}\mu([0,tb^n]) = 0) \\ &\leq \mathbf{P}\left(b^{-n}\mu([0,tb^n]) \leq \frac{t}{2}\right) \longrightarrow 0 \quad \text{as } n \longrightarrow \infty. \end{aligned} \tag{3.10}$$

Thus $\mu([0,t]) > 0$ a.s. and μ being a positive measure, trivially $\mu([0,s]) > 0$ for all $s \geq t$ a.s.

Now denote $U_t = \{\omega : \ \mu[0,t] = 0\}$. Then

$$\mathbf{P}(\mu([0,t]) = 0 \ \forall t \in (0,1]) = \mathbf{P}\left(\bigcup_{n=1}^{\infty} U_{1/n}\right) \leq \sum_n \mathbf{P}(U_{1/n}) = 0, \tag{3.11}$$

which completes the proof. □

3.2. Random Partition and Multifractal Envelope

In order to determine the prevalence of scaling exponent in an entire interval rather than one single point, we use the formalism developed in Section 2. In analogy with related multiplicative processes [15], we should expect moments of the multipliers to affect the multifractal properties through the *structure function*

$$\beta(q) \doteq q - 1 - \log_b \mathbf{E}[\Lambda^q]. \tag{3.12}$$

Indeed, as a direct consequence of [39, Propositions 7 and 5], the structure function gives the deterministic envelope of the multifractal formalism based on the dyadic partitions $\mathfrak{D}$.

Corollary 3.5 (see [39]). *If* $\mathbf{E}[\Lambda^p] = \mathbf{E}[M^p] < \infty$ *and* $A_n(t)$ *converges in* $\mathcal{L}_p$ *for some* $p > 1$*, then, for* $0 < q \leq p$*,*

$$\underline{C}t^{\beta(q)+1} \leq \mathbf{E}\left[A(t)^q\right] \leq \overline{C}t^{\beta(q)+1} \quad \forall t \in [0,1], \tag{3.13}$$

$$T_{\mathfrak{D}}(q) = \beta(q) = q - 1 - \log_b \mathbf{E}[M^q]. \tag{3.14}$$

Note that under the assumption (3.5) of Theorem 3.2, convergence of the MJP $A(t)$ in $\mathcal{L}_2$ is equivalent with $\beta(2) > 0$.

In order to analyze the MJP, it is natural to consider the nested sequence of partitions J^n induced by Λ_n. By assumptions (A1) and (A2),

$$\Lambda^{(i)}(t) = M_k^{(i)}, \quad T_{k-1}^{(i)} \leq t < T_k^{(i)}, \tag{3.15}$$

where $\{T_k^{(i)}\}_{k=1}^{\infty}$ forms a Poisson point process of intensity λb^i and $\{M_k^{(i)}\}_{k=1}^{\infty}$ is a stationary positive time series independent of $\{T_k^{(i)}\}_k$. Consequently, the product process

$$\Lambda_n(t) = \prod_{i=0}^{n-1} \Lambda^{(i)}(t) \tag{3.16}$$

is constant on intervals $J_k^n \doteq [T_{k-1}^n, T_k^n)$, where the T_k^n are defined by a Poisson point process of intensity $\lambda_n = \lambda \sum_{i=0}^{n-1} b^i = O(b^n)$. From the construction of $\{T_k^n\}_k$, it follows that the partitions $J^n = \{J_k^n : \ k = 1, \ldots\}$ are nested. We denote the resulting collection of *nested Poisson partitions* J^n by $\mathcal{P}$.

Lemma 3.6. *Consider a nested Poisson partition $\mathcal{P}$. Let t_o be arbitrary. With probability one, as $n \to \infty$,*

$$\frac{\log|J^n(t_o)|}{n} \longrightarrow -\log b, \qquad \frac{\log\ \max_k |J_k^n|}{n} \longrightarrow -\log b, \qquad \frac{\log \min_k |J_k^n|}{n} \longrightarrow -2\log b. \tag{3.17}$$

Proof. Without loss of generality, assume $\lambda_n = \lambda b^n$ and $t_0 \in [0,1)$.

(1°): Clearly

$$\mathbf{P}\big(\text{No points in } [t_0 - x, t_0 + x]\big) \le \mathbf{P}(|J^n(t_0)| > x) \le 2\ \mathbf{P}\big(\text{No points in}[t_0, t_0 + x/2]\big). \tag{3.18}$$

Thus for n large enough,

$$\begin{aligned} \mathbf{P}\big(|J^n(t_0)| > b^{-n}e^{n\varepsilon}\big) &\le 2e^{(-1/2)\lambda_n b^{-n} e^{n\varepsilon}} \le 2e^{(-1/2)n\varepsilon\lambda}, \\ \mathbf{P}\big(|J^n(t_0)| < b^{-n}e^{n\varepsilon}\big) &\le 1 - e^{-2\lambda_n b^{-n} e^{-n\varepsilon}} \le 2\lambda e^{-n\varepsilon}. \end{aligned} \tag{3.19}$$

(2°): Since the maximal interval is always longer than a fixed interval, for n large enough,

$$\mathbf{P}\left(\max_k |J_k^n| < b^{-n}e^{-n\varepsilon}\right) \le \mathbf{P}\big(|J^n(t_0)| < b^{-n}e^{-n\varepsilon}\big) \le 2\lambda e^{-n\varepsilon}. \tag{3.20}$$

On the other hand, if the maximal interval is longer than a given x, then at least one of the subintervals $[(i-1)(x/2), i(x/2)]$, $i = 1, \ldots, \lfloor 2/x \rfloor$, has to be empty. Thus

$$\mathbf{P}\left(\max_k |J_k^n| > b^{-n}e^{n\varepsilon}\right) \le \sum_{i=1}^{\lfloor 2b^n e^{-n\varepsilon} \rfloor} e^{(-1/2)\lambda_n b^{-n} e^{n\varepsilon}} \le 2e^{-n((\lambda/4)n\varepsilon^2 - \log b)}. \tag{3.21}$$

(3°): Let $\{U_k^m\}_{k=1}^{m+1}$ be a partition generated by a uniform random sample of size m on interval $[0,1]$. It is straightforward to show, for example, by induction, that

$$\mathbf{P}\left(\min_k \left|U_k^m\right| < x\right) = \begin{cases} 1-(1-(m+1)x)^m, & 0 < x < \dfrac{1}{1+m}, \\ 1, & x \geq \dfrac{1}{m+1}. \end{cases} \tag{3.22}$$

Let N_n denote the number of points of a Poisson process with density λb^n located in $[0,1]$. Applying (3.22) gives

$$\begin{aligned} \mathbf{P}\left(\min_k \left|J_k^n\right| < b^{-2n}e^{-n\varepsilon}\right) &= \sum_{m=0}^{\infty} \mathbf{P}\left(\min_{k=1,\dots,N_n+1} \left|J_k^n\right| < b^{-2n}e^{-n\varepsilon} \mid N_n = m\right)\mathbf{P}(N_n = m) \\ &= \sum_{m=0}^{\lfloor b^{2n}e^{n\varepsilon}\rfloor -1} \left[1-\left(1-(m+1)b^{-2n}e^{-n\varepsilon}\right)^m\right]\frac{(\lambda b^n)^m}{m!}e^{-\lambda b^n} + \sum_{m=\lfloor b^{2n}e^{n\varepsilon}\rfloor}^{\infty} \frac{(\lambda b^n)^m}{m!}e^{-\lambda b^n} \\ &\leq \sum_{m=0}^{\infty}\left(m^2+m\right)b^{-2n}e^{-n\varepsilon}\frac{(\lambda b^n)^m}{m!}e^{-\lambda b^n} + \frac{(\lambda b^n)^{\lfloor b^{2n}e^{n\varepsilon}\rfloor}}{\lfloor b^{2n}e^{n\varepsilon}\rfloor !} \\ &\leq e^{-n\varepsilon}\left(2\lambda b^{-n}+\lambda^2\right) + e^{-n\varepsilon}, \end{aligned} \tag{3.23}$$

where the last inequality holds when n is large enough. On the other hand, if the minimal interval is larger than some given x, then there is at most one point in each subinterval $[(i-1)x, ix]$, $i = 1,\dots,\lfloor 1/x\rfloor$. Thus

$$\begin{aligned} \mathbf{P}\left(\min_k \left|I_k^n\right| > b^{-2n}e^{n\varepsilon}\right) &\leq \left(e^{-\lambda b^{-n}}\left(1+\lambda b^{-n}e^{n\varepsilon}\right)\right)^{b^{2n}e^{-n\varepsilon}-1} \\ &= \exp\left(-\lambda b^n + \lambda b^{-n}e^{n\varepsilon} + \left(b^{2n}e^{-n\varepsilon}-1\right)\log\left(1+\lambda b^{-n}e^{n\varepsilon}\right)\right) \leq e^{-c_1 n\varepsilon}. \end{aligned} \tag{3.24}$$

(4°): Apply Borel-Cantelli to the above three cases to complete the proof. □

This result sheds some light on the use of such Poisson partitions in the context of computing the Hausdorff dimensions and multifractal spectra. They do not satisfy the homogeneous scaling assumption of [46]; yet, being nested they allow still to treat sets for which all points satisfy (2.5), as for such sets $\dim(\cdot) = \dim^{\mathcal{P}}(\cdot)$. In addition, by lemma 2.6 and theorem 2.8, the multifractal envelope $T_{\mathcal{P}}^*$ provides an upper bound on the multifractal spectrum $\dim^{\mathcal{P}}(E_\alpha)$, almost surely.

The connection to the measures $\mu^{(n)}$ defined in (3.7) is especially practical when considering sets $J^n(t)$ because then

$$\mu(J^n(t)) = \mu^{(n)}(J^n(t))\prod_{i=0}^{n-1}\Lambda^{(i)}(t) \overset{\text{dist}}{=} b^{-n}\tilde{\mu}(b^n J^n(t))\prod_{i=0}^{n-1}\Lambda^{(i)}(t), \tag{3.25}$$

where $\tilde{\mu}$ is distributed as μ and is independent of J^n and $\Lambda^{(i)}$, $i = 0,\ldots,n-1$. On the other hand, by corollary 3.5 and(3.8)

$$C_1 b^{-n(q-1-\beta(q))}|I|^{1+\beta(q)} \le \mathbf{E}\Big[\mu^{(n)}(I)^q\Big] \le C_2 b^{-n(q-1-\beta(q))}|I|^{1+\beta(q)} \tag{3.26}$$

for a fixed interval I. We find that $T_{\mathcal{P}} = T_{\mathfrak{D}}$ under mild conditions.

Proposition 3.7. *If* $\mathbf{E}[M^p] < \infty$ *and* $A_n(t)$ *converges in* $\mathcal{L}_p$ *for some* $p > 1$*, then*

$$T_{\mathcal{P}}(q) = \beta(q) = q - 1 - \log_b \mathbf{E}[M^q] \quad \text{for } 0 < q \le p. \tag{3.27}$$

Proof. Consider $\mathbf{E}[\sum_{k=1}^{N_n} |J_k^n|^{1+\varepsilon}]$ with $\varepsilon > 0$. Denote by $\{U_k^j : k = 1,\ldots,j+1\}$ the partition resulting from j uniformly distributed points on $[0,1]$. Then

$$\mathbf{E}\Big[\Big|U_k^j\Big|^\alpha\Big] = \int_0^1 j s^\alpha (1-s)^{j-1}\, \mathrm{d}s = \frac{j!\Gamma(1+\alpha)}{\Gamma(j+1+\alpha)}. \tag{3.28}$$

Using the above result and conditioning with respect to the number of intervals gives

$$\begin{aligned}
\mathbf{E}\left[\sum_{k=1}^{N_n}|J_k^n|^{1+\varepsilon}\right] &= \sum_{j=0}^{\infty}\mathbf{E}\left[\sum_{k=1}^{j+1}\Big|U_k^j\Big|^{1+\varepsilon}\right]\mathbf{P}(N_n = j+1) = \sum_{j=0}^{\infty}\frac{(j+1)j!\Gamma(2+\varepsilon)}{\Gamma(j+2+\varepsilon)}\frac{(\lambda_n)^j e^{-\lambda_n}}{j!}\\
&\le \Gamma(2+\varepsilon)(\lambda_n)^{-\varepsilon}e^{-\lambda_n}\sum_{j=0}^{\infty}\frac{(\lambda_n)^{j+\varepsilon}}{\Gamma(j+1+\varepsilon)} = \Gamma(2+\varepsilon)(\lambda_n)^{-\varepsilon}\left(1-\frac{\Gamma(\varepsilon,\lambda_n)}{\Gamma(\varepsilon)}\right)\\
&\le Cb^{-n\varepsilon}.
\end{aligned} \tag{3.29}$$

Now assume $\gamma = \beta(q) - \epsilon$ with $\epsilon > 0$. Conditioning with respect to the partition J^n helps us to split the addends into three factors, that is,

$$\mathbf{E}\left[\sum_{k=1}^{N_n}\mu(J_k^n)^q|J_k^n|^{-\gamma}\right] = \mathbf{E}\mathbf{E}\left[\sum_{k=1}^{N_n}\Lambda_n(T_{k-1}^n)^q|J_k^n|^{-\gamma}\mu^{(n)}(J_k^n)^q \mid J^n\right]. \tag{3.30}$$

Use the scaling law (3.26) together with the independence of $\mu^{(n)}$ with respect to J^n and that

$$\mathbf{E}[\Lambda_n(T_k^n)^q \mid J^n] = \mathbf{E}[\Lambda_n(T_k^n)^q] = \mathbf{E}[M^q]^n = b^{-n(\beta(q)+1-q)} \tag{3.31}$$

in order to estimate

$$\mathbf{E}\left[\sum_{k=1}^{N_n}\mu(J_k^n)^q|J_k^n|^{-\gamma}\right] \le Cb^{-n(q-1-\beta(q))}\ \mathbf{E}[M^q]^n\ \mathbf{E}\left[\sum_{k=1}^{N_n}|J_k^n|^{1+\beta(q)-\gamma}\right] = C\ \mathbf{E}\left[\sum_{k=1}^{N_n}|J_k^n|^{1+\varepsilon}\right] \le \tilde{C}b^{-n\varepsilon}. \tag{3.32}$$

Next, assume that $\gamma = \beta(q) + \varepsilon$. Condition again with respect to the partition J^n and apply the lower bound of (3.13) to get

$$\mathbf{E}\left[\sum_{k=1}^{N_n}\mu(J_k^n)^q\left|J_k^n\right|^{-\gamma}\right] \geq C\ \mathbf{E}\left[\sum_{k=1}^{N_n}\left|J_k^n\right|^{1-\varepsilon}\right] \geq C\ \mathbf{E}\left[\sum_{k=1}^{N_n}\left|J_k^n\right|\right] = C. \tag{3.33}$$

Thus, we have found that $\sum_{n\geq m}\mathbf{E}[S_n(q,\gamma)] \to 0$ whenever $\gamma < \beta(q)$ and $\sum_{n\geq m}\mathbf{E}[S_n(q,\gamma)] \nrightarrow 0$ whenever $\gamma > \beta(q)$. $\square$

3.3. *Information Dimension*

This section is devoted to analyzing $E_{\alpha_1}^{\mathcal{P}}$, the set of points with scaling exponent

$$\alpha_1^{\mathcal{P}} \doteq \alpha_1 \doteq \beta'(1) = 1 - \mathbf{E}[\Lambda\log_b\Lambda] = 1 - \mathbf{E}[M\log_b M] \tag{3.34}$$

which is the most relevant of all $E_\alpha^{\mathcal{P}}$.

The upper bound of the multifractal spectrum $\dim^{\mathcal{P}}(E_\alpha^{\mathcal{P}}) \leq T_{\mathcal{P}}^*$ a.s. follows from lemma 2.6, lemma 3.6, and theorem 2.8. While $T_{\mathcal{P}} = T_{\mathfrak{D}}$ under mild conditions (see above), we need to introduce the auxiliary set

$$K_\alpha^{\mathcal{P}} = \left\{x : \lim_n \alpha_n^{\mathcal{P}}(x) = \alpha,\ \lim_n \frac{\log_b|J^n(x)|}{-n} = 1\right\}, \tag{3.35}$$

a subset of $E_\alpha^{\mathcal{P}}$ for which $\dim(K_\alpha^{\mathcal{P}}) = \dim^{\mathcal{P}}(K_\alpha^{\mathcal{P}})$ due to Lemma 2.1 (or due to Lemma 2.2 as we will see).

To perform a local analysis of path properties and establish facts which hold for almost all paths at almost all locations t, it is convenient to introduce a measure Q which is referred to as the "Peyrière measure" by Kahane. The approach is to apply the LLN to Q which provides a pathwise measure ν which allows us to bound $\dim K_\alpha^{\mathcal{P}}$, and thus $\dim E_\alpha^{\mathcal{P}}$, from below almost surely using Lemma 2.2. The measure Q lives on the space $\Omega \times [0,1]$, defined as the unique probability measure which satisfies

$$Q(\varphi) = \mathbf{E}\left[\int_0^1 \varphi(t,\omega)\mathrm{d}\mu(t)\right] \tag{3.36}$$

for all positive measurable functions $\varphi(t,\omega)$.

Lemma 3.8. *Assume that $A(t)$ is a non-degenerate MJP. If (ω, T) is picked according to the Peyrière measure Q, then*

$$\lim_{n\to\infty}\frac{\log|J^n(T)|}{n} = -\log b \quad Q\text{-}a.s. \tag{3.37}$$

Moreover, if $A_n(t)$ converges in $\mathcal{L}_q$ for some $q > 1$ and $b^{q-1} > \mathbf{E}[\Lambda^q]$, that is, $\beta(q) > 0$, then

$$\frac{\log \mu^{(n)}(J^n(T))}{n} \longrightarrow -\log b \quad Q\text{-}a.s. \tag{3.38}$$

Proof. Recall first that $\mathbf{E}[\mu^{(n)}(J_k^n) \mid J^n] = |J_k^n|$ and that $\Lambda_n(T_k^n)$ is independent of J^n so that

$$\mathbf{E}\left[\Lambda_n(T_k^n) \mid J^n\right] = \mathbf{E}[\Lambda_n] = 1. \tag{3.39}$$

(1°) Condition with respect to the partition J^n in order to get

$$\begin{aligned}
\mathbf{P}_Q(|J^n(T)| < x) = \mathbf{E}\left[\int_0^1 \mathbf{1}\{|J^n(t)| < x\}\mathrm{d}\mu(t)\right] &= \mathbf{E}\mathbf{E}\left[\sum_{k=1}^{N_n}\mathbf{1}\left\{\left|J_k^n\right| < x\right\}\Lambda_n(T_{k-1}^n)\mu^{(n)}\left(J_k^n\right) \mid J^n\right] \\
&= \mathbf{E}\left[\sum_{k=1}^{N_n}\mathbf{1}\left\{\left|J_k^n\right| < x\right\}\ \left|J_k^n\right|\right] = \mathbf{E}\left[\int_0^1 \mathbf{1}\{|J^n(t)| < x\}\mathrm{d}t\right] \\
&= \int_0^1 \mathbf{P}(|J^n(t)| < x)\mathrm{d}t \le C_1 x\lambda_n,
\end{aligned} \tag{3.40}$$

where C_1 is independent of n. Here, the last inequality follows by estimating the probability that a (truncated) exponential random variable is less than x, with $x < 1/2$.

The other direction is easy as well. First we condition as above, and represent the sum as an integral:

$$\mathbf{P}_Q(|J^n(T)| > x) = \int_0^1 \mathbf{P}(\mathbf{1}\{|J^n(t)| > x\})\mathrm{d}t \le C_2 e^{-\lambda_n x}, \tag{3.41}$$

where C_2 is independent of n. Here, the last inequality follows from estimating the probability that a sum of two exponential random variables exceeds x.

Since $\lambda_n = O(b^n)$, we have

$$\sum_{n=1}^{\infty}\mathbf{P}_Q\left(|J^n(T)| < b^{-n}e^{-n\varepsilon}\right) < \infty, \qquad \sum_{n=1}^{\infty}\mathbf{P}_Q\left(|J^n(T)| > b^{-n}e^{n\varepsilon}\right) < \infty. \tag{3.42}$$

(2°) Let $\varepsilon > 0$. Write the Q-probability as a sum

$$\mathbf{P}_Q\left(\mu^{(n)}(J^n(T)) < b^{-n}e^{-n\varepsilon}\right) = \mathbf{E}\left[\sum_{k=1}^{N_n}\mathbf{1}\left\{\mu^{(n)}(J_k^n) < b^{-n}e^{-n\varepsilon}\right\}\Lambda_n(T_k^n)\mu^{(n)}(J_k^n)\right]. \tag{3.43}$$

Scaling the $\mu^{(n)}(J_k^n)$ up by $b^{-n}e^{-n\varepsilon}$ and applying (3.39) lead to

$$\mathbf{P}_Q\left(\mu^{(n)}(J^n(T)) < b^{-n}e^{-n\varepsilon}\right) \leq b^{-n}e^{-n\varepsilon}\mathbf{E}\mathbf{E}\left[\sum_{k=0}^{N_n}\Lambda_n(T_k^n) \mid J^n\right] \leq b^{-n}e^{-n\varepsilon}\mathbf{E}[N_n]. \tag{3.44}$$

This implies $\mathbf{P}_Q(\mu^{(n)}(J^n(T))) \leq Ce^{-n\varepsilon}$, because $\mathbf{E}[N_n] = \lambda_n = O(b^n)$.

On the other hand, by (3.26) and (3.29),

$$\begin{aligned}\mathbf{E}_Q\left[\mu^{(n)}(J^n(T))^{q-1}\right] &= \mathbf{E}\mathbf{E}\left[\sum_{k=1}^{N_n}\Lambda_n(T_{k-1}^n)\ \mu^{(n)}(J_k^n)^q \mid J^n\right] \\ &\leq \widetilde{C}b^{-n(q-1-\beta(q))}\mathbf{E}\left[\sum_{k=1}^{N_n}\left|J_k^n\right|^{1+\beta(q)}\right] \leq Cb^{-n(q-1)}.\end{aligned} \tag{3.45}$$

Then apply the Markov inequality to get

$$\mathbf{P}_Q\left(\mu^{(n)}(J^n(T)) \geq b^{-n}e^{n\varepsilon}\right) \leq Ce^{-n\varepsilon(q-1)}. \tag{3.46}$$

(3°) Applying Borel-Cantelli to (1°) and (2°) completes the proof. □

To establish the Q-independence required for the LLN, we need the following result.

Lemma 3.9. *Assume that $A(t)$ is a non-degenerate MJP. Then*

$$\mathbf{E}_Q\left[\prod_{i=0}^{n-1}g_i\left(\Lambda^{(i)}(T)\right)\right] = \prod_{i=0}^{n-1}\mathbf{E}\left[g_i\left(\Lambda^{(i)}(t)\right)\Lambda^{(i)}(t)\right] \tag{3.47}$$

for all $n \in \mathbb{N}$ and positive Borel functions g_i defined on $\mathbb{R}^+$.

Proof. (Note: this is a replication of the proof appearing in [2]). Condition with respect to the partition J^n and use $\mathbf{E}[\mu_n(J_k^n) \mid J^n] = |J_k^n|$. In addition to this, utilize independence (w.r.t. Ω) and stationarity:

$$\begin{aligned}
\mathbf{E}_Q\left[\prod_{i=0}^{n-1} g_i\left(\Lambda^{(i)}(T)\right)\right] &= \mathbf{E}\left[\int_0^1 \prod_{i=0}^{n-1} g_i\left(\Lambda^{(i)}(t)\right) \mathrm{d}\mu(t)\right] \\
&= \mathbf{E}\mathbf{E}\left[\sum_{k=0}^{N_n} \prod_{i=0}^{n-1} g_i\left(\Lambda^{(i)}(T_{k-1}^n)\right) \Lambda^{(i)}(T_{k-1}^n) \mu^{(n)}(I_k^n) \mid J^n\right] \\
&= \mathbf{E}\left[\sum_{k=0}^{N_n} \prod_{i=0}^{n-1} g_i\left(\Lambda^{(i)}\right)(T_{k-1}^n)\, \Lambda^{(i)}(T_{k-1}^n) \left|J_k^n\right|\right] \\
&= \mathbf{E}\left[\int_0^1 \prod_{i=0}^{n-1} g_i\left(\Lambda^{(i)}(t)\right) \Lambda^{(i)}(t) \mathrm{d}t\right] \\
&= \int_0^1 \prod_{i=0}^{n-1} \mathbf{E}\left[g_i\left(\Lambda^{(i)}(t)\right) \Lambda^{(i)}(t)\right] \mathrm{d}t = \prod_{i=0}^{n-1} \mathbf{E}\left[g_i\left(\Lambda^{(i)}(t)\right) \Lambda^{(i)}(t)\right].
\end{aligned} \tag{3.48}$$

□

Corollary 3.10. *Assume that $A(t)$ is a non-degenerate MJP. Then the random variables $\Lambda^{(i)}(T)$, $i = 0, 1, \ldots, n$, are Q-independent.*

Proof. Fix j and set $g_i(x) \equiv 1$ when $i \neq j$, leaving g_j arbitrary. Lemma 3.9 gives

$$\mathbf{E}_Q\left[g_j\left(\Lambda^{(j)}(T)\right)\right] = \mathbf{E}\left[g_j\left(\Lambda^{(j)}(t)\right) \Lambda^{(j)}(t)\right]. \tag{3.49}$$

Thus

$$\mathbf{E}_Q\left[\prod_{i=0}^{n} g_i\left(\Lambda^{(i)}(T)\right)\right] = \prod_{i=0}^{n} \mathbf{E}_Q\left[g_i\left(\Lambda^{(i)}(T)\right)\right] \tag{3.50}$$

for all $n \in \mathbb{N}$ and positive Borel functions g_i defined on $\mathbb{R}^+$. □

Corollary 3.11. *Assume that $A(t)$ is a non-degenerate MJP. Then $\mathbf{E}_Q[\log \Lambda^{(j)}(T)] = \mathbf{E}[\Lambda \log \Lambda]$ for all $j \in \mathbb{N}$.*

Proof. Set $g_i(x) = 1$ if $i \neq j$ and $g_j(x) = \log x$ and apply Lemma 3.9. □

We now show that the pathwise ν to be used in Lemma 2.2 is the realization of μ itself.

Lemma 3.12. *Assume that $A_n(t)$ converges in $\mathcal{L}_q$ for some $q > 1$. If $\beta(q) > 0$, then the set $K_{\alpha_1}^{\rho}$ has full μ-measure Ω a.s., that is,*

$$\mu\left(K_{\alpha_1}^{\rho}\right) = \mu([0,1]) \qquad \Omega\text{-}a.s. \tag{3.51}$$

Proof. Notice first that

$$\begin{aligned}\frac{\log \mu(J^n(t))}{\log|J^n(t)|} &= \frac{\log \Lambda^{(0)}(t) + \cdots + \log \Lambda^{(n-1)}(t)}{\log|J^n(t)|} + \frac{\log \mu^{(n)}(J^n(t))}{\log|J^n(t)|} \\ &= \frac{(1/n)\sum_{i=0}^{n-1} \log \Lambda^{(i)}(t)}{(1/n)\log|J^n(t)|} + \frac{(1/n)\log \mu^{(n)}(J^n(t))}{(1/n)\log|J^n(t)|}.\end{aligned} \tag{3.52}$$

Then by Corollary 3.10, 3.11, and LLN,

$$\frac{1}{n}\sum_{i=0}^{n-1} \log\ \Lambda^{(i)}(T) \longrightarrow \mathbf{E}_Q[\Lambda(T)] = \mathbf{E}\left[\Lambda \log \Lambda\right] = \mathbf{E}\left[M \log\ M\right]. \tag{3.53}$$

On the other hand, by lemma 3.8,

$$\begin{aligned}\lim_{n\to\infty} \frac{1}{n} \log(|J^n(T)|) &= -\log b \quad \text{Q-a.s.} \\ \lim_{n\to\infty} \frac{1}{n} \log\left(\mu^{(n)}(J^n(T))\right) &= -\log b \quad \text{Q-a.s.}\end{aligned} \tag{3.54}$$

□

Lemma 3.13. *Assume that $A_n(t)$ converges in $\mathcal{L}_q$ for some $q > 1$. If $\beta(q) > 0$, then, with probability one, we have either $\mu([0,1]) = 0$ or* $\dim\ (K^{\mathcal{P}}_{\alpha_1}) = \dim^{\mathcal{P}}(K^{\mathcal{P}}_{\alpha_1}) \geq \alpha_1$.

Proof. Condition (i) of lemma 2.2 is obvious; conditions (ii) and (iii) are satisfied by definition of K_{α_1}. Finally, either condition (o) or $\mu([0,1]) = 0$ holds due to lemma 3.12. □

The main result concerning the information dimension is stated in the following theorem.

Theorem 3.14. *Assume that $A_n(t)$ converges in $\mathcal{L}_q$ for some $q > 1$. If $\beta(q) > 0$, then, with probability one,*

$$\dim^{\mathcal{P}}\left(E^{\mathcal{P}}_{\alpha_1}\right) = \dim\left(E^{\mathcal{C}}_{\alpha_1}\right) = \alpha_1 = 1 - \mathbf{E}\left[M \log_b M\right]. \tag{3.55}$$

For example, if $A(t)$ is an MJP satisfying (3.5) and $\beta(2) > 2$, then $\dim^{\mathcal{P}}(E_{\alpha_1}) = \dim(E_{\alpha_1}) = \alpha_1$.

Proof. First, note that $\beta^*(\alpha_1) = \alpha_1$ since $\alpha_1 = \beta'(1)$; also $\beta(q') > 0$ for $1 < q' < q$ since β is convex; also, $\mu \in \mathcal{L}_{q'}$ for $1 < q' < q$. Second, by Lemmas 3.6 and 2.6, $\tau_{\mathcal{P}}(q)$ is convex for $q > 0$ a.s. and theorem 2.8 applies.

By Lemma 3.13 we have

$$\begin{aligned}\alpha_1 &\leq \dim^p\left(K_{\alpha_1}^p\right) = \dim\left(K_{\alpha_1}^p\right) \leq \dim\left(E_{\alpha_1}^p\right) \leq \dim^p(E_{\alpha_1}) \leq \dim^p\left(V_{\alpha_1}^p\right) \\ &\leq \inf_{1<q'<q}\left(q'\alpha_1 - T_P(q')\right) = \inf_{1<q'<q}\left(q'\alpha_1 - \beta(q')\right) = \alpha_1.\end{aligned} \tag{3.56}$$

□

3.4. Multifractal Decomposition

Following the traditional approach in multifractal analysis, we generalize the above analysis of the information dimension of MJP to a larger set of scaling exponents through a change of the measure. Doing so, we are able to compute $\dim V_\alpha^p$ for a range of exponents α.

To this end, we fix $q > 0$ for the remainder of the section. Introduce the auxiliary mother process

$$\overline{\Lambda}(\cdot) \doteq b^{\beta(q)-q+1}\Lambda(\cdot)^q = \frac{\Lambda(\cdot)^q}{\mathbf{E}[\Lambda^q]}, \tag{3.57}$$

the structure function

$$\overline{\beta}(p) \doteq p - 1 - \log_b \mathbf{E}\left[\overline{\Lambda}^p\right] = \beta(pq) - p\beta(q), \tag{3.58}$$

and the scaling exponent

$$\alpha_q \doteq \beta'(q). \tag{3.59}$$

We denote the associated MJP by $\overline{A}$, its measure by $\overline{\mu}$, and the corresponding Peyrière measure by $\overline{Q}$. Let us quickly summarize how our earlier results translate from A to $\overline{A}$. Clearly, $\mathbf{E}[\overline{\Lambda}] = 1$, and by definition of β

$$\mathbf{E}_{\overline{Q}}[\log_b\Lambda] = \mathbf{E}\left[\overline{\Lambda}\log_b\Lambda\right] = \frac{1}{\mathbf{E}[\Lambda^q]}\mathbf{E}\left[\Lambda^q\log_b\Lambda\right] = 1 - \beta'(q) = 1 - \alpha_q. \tag{3.60}$$

From this we find

$$\mathbf{E}_{\overline{Q}}\left[\log_b\overline{\Lambda}\right] = \mathbf{E}\left[\overline{\Lambda}\log_b\overline{\Lambda}\right] = (\beta(q) - q + 1) + q\mathbf{E}\left[\overline{\Lambda}\log_b\Lambda\right] = 1 - (q\alpha_q - \beta(q)). \tag{3.61}$$

It is quite typical in multifractal analysis that the Legendre transform should appear at this point.

By analogy with the previous section, we should expect that under appropriate assumptions with probability one the information dimension of $\overline{\mu}$ is $q\alpha_q - \beta(q)$. We should also expect to find that with probability one, the measure μ assumes the scaling exponent $\alpha(t) = \alpha_q$ for $\overline{\mu}$-almost all t, which is not a simple translation from earlier sections. If established, all this would imply that the dimension of the set E_{α_q} of points with Hölder exponent α_q almost surely has at least Hausdorff dimension $q\alpha_q - \beta(q)$, since $\overline{\mu}$ assigns full mass to it.

We are able to establish such a lower bound for the Hausdorff dimension of a natural subset of V_α^p, that is, the set

$$W_\alpha^p \doteq \left\{ t : \liminf_n \alpha_n^p(t) \le \alpha,\ \lim_n \frac{\log|J^n(t)|}{n} = -\log b \right\} \tag{3.62}$$

for $\alpha = \alpha_q$. Combining this with an upper bound provided by inequality (2.24), we will arrive at a formula for the dimension of V_α^p.

As a first step towards this goal, we mention a sufficient condition for convergence in $\mathcal{L}_p$, $1 \le p \le 2$, which follows directly from theorem 3.2, that is, $\overline{A}_n(t)$ converges in $\mathcal{L}_p$ if $b > 1 + \mathrm{Var}(\overline{\Lambda})$. Being equivalent to $\overline{\beta}(2) > 0$ or $\beta(2q) > 2\beta(q)$, this condition holds by convexity for all $0 < q' < q$ if it holds for q. Next, replicating the results of the previous section gives the following corollary.

Corollary 3.15. *Assume that $\overline{A}_n(t)$ converges in $\mathcal{L}_p$ for some $p > 1$. If $\overline{\beta}(p) > 0$ for some $p > 0$, then $\overline{Q}$-a.s.,*

$$\begin{aligned}
\lim_{n\to\infty} \frac{\log_b|J^n(T)|}{n} &= -1,\\
\lim_{n\to\infty} \frac{\log_b\overline{\mu}(J^n(T))}{n} &= -q\alpha_q + \beta(q),\\
\lim_{n\to\infty} \frac{\log\,\overline{\mu}(J^n(T))}{\log|J^n(t)|} &= 1 - \mathbf{E}\left[\overline{\Lambda}\log_b\overline{\Lambda}\right] = q\alpha_q - \beta(q).
\end{aligned} \tag{3.63}$$

To provide a counterpart to Lemma 3.12, we need the following result concerning the local scaling of μ under the measure $\overline{Q}$.

Lemma 3.16. *Assume that $A(t)$ is a non-degenerate MJP and that $\overline{A}_n(t)$ converges in $\mathcal{L}_p$ for some $p > 1$. Then*

$$\liminf_{n\to\infty} \frac{\log_b \mu^{(n)}(J^n(T))}{-n} \le 1 \quad \overline{Q}\text{-a.s.} \tag{3.64}$$

Proof. (1°): Let us start with a simple observation on nested Poisson point processes. Given a sequence of independent Poisson point processes of densities $b^{-k}\lambda$, their superposition forms a sequence of Poisson point processes of densities $(1+\cdots+b^{-n})\lambda$. Denote $I^{n,[x,y]} = \{I_k^{n,[x,y]}\}$ the partition of the interval $[x,y]$ determined by a Poisson point process with density $(1+\cdots+b^{-n})\lambda$. Adding extra points at the locations $x = 1,2,\ldots$ makes the intervals even smaller. Thus the partition $\cup_{i=1}^m I^{\infty,[i-1,i]}$ is finer than $I^{n,[0.m]}$ for all $n = 0,1,\ldots$ and m. In other words, for any $m \in \mathbb{Z}_+$,

$$I^{\infty,[i-1,i]}(t) \subseteq I^{n,[0,m]}(t) \quad \forall n \in \mathbb{N},\ \forall i \in [1,m],\ \forall t \in (i,i-1). \tag{3.65}$$

(2°): By a change of variables, and using $\mathbf{E}[\overline{\Lambda_n}(T_k^n) \mid J^n] = 1$ together with the independence of $\overline{\Lambda}_n$ of J^n, $\mu^{(n)}$, and $\overline{\mu}^{(n)}$, we get

$$\begin{aligned}
\mathbf{P}_{\overline{Q}}\left(\mu^{(n)}(J^n(T)) < b^{-n(1+\varepsilon)}\right) &= \mathbf{E}\left[\int_0^1 \mathbf{1}\left\{\mu^{(n)}(J^n(t)) < b^{-n(1+\varepsilon)}\right\} \mathrm{d}\overline{\mu}(t)\right] \\
&= \mathbf{E}\ \mathbf{E}\left[\sum_{k=1}^{N_n} \mathbf{1}\left\{\mu^{(n)}\left(J_k^n\right) < b^{-n(1+\varepsilon)}\right\} \overline{\Lambda}_n(T_k^n) \overline{\mu}^{(n)}\left(J_k^n\right) \mid J^n\right] \\
&= \mathbf{E}\left[b^{-n} \sum_{k=1}^{N_n} \mathbf{1}\left\{\mu\left(b^n \widetilde{J}_k^n\right) < b^{-n\varepsilon}\right\} \overline{\mu}\left(b^n \widetilde{J}_k^n\right)\right],
\end{aligned} \tag{3.66}$$

where $\widetilde{J}^n$ is distributed as J^n and independent of μ and $\overline{\mu}$.

For each fixed n, construct a nested sequence of partitions $I^{i,[0,b^n]}$, $i = 0, 1 \ldots$, which are independent of μ and $\overline{\mu}$ and satisfy $I^{n,[0,b^n]} = b^n \widetilde{J}^n$. Then write (3.66) as an integral split into parts of length 1 and apply (3.65) to get

$$\begin{aligned}
\mathbf{P}_{\overline{Q}}\left(\mu^{(n)}(J^n(T)) < b^{-n(1+\varepsilon)}\right) &= \mathbf{E}\left[b^{-n} \int_0^{b^n} \mathbf{1}\left\{\mu\left(I^{n,[0,b^n]}(t)\right) < b^{-n\varepsilon}\right\} d\overline{\mu}(t)\right] \\
&\le \mathbf{E}\left[b^{-n} \sum_{i=1}^{\lfloor b^n \rfloor} \int_{i-1}^{i} \mathbf{1}\left\{\mu\left(I^{\infty,[i-1,i]}(t)\right) < b^{-n\varepsilon}\right\} d\overline{\mu}(t)\right] + \mathbf{E}\left[b^{-n} \overline{\mu}([\lfloor b^n \rfloor, \lceil b^n \rceil])\right] \\
&\le \mathbf{P}_{\overline{Q}}\left(\mu\left(I^{\infty,[0,1]}(T)\right) < b^{-n\varepsilon}\right) + b^{-n}.
\end{aligned} \tag{3.67}$$

(3°): By proposition 3.4, $\mu(I) > 0$ a.s. for any fixed interval I. Thus,

$$\mathbf{P}_{\overline{Q}}\left(\mu\left(I^{\infty,[0,1]}(T)\right) = 0\right) = \mathbf{E}\mathbf{E}\left[\sum_{k=1}^{N} \mathbf{1}\left\{\mu\left(I_k^{\infty,[0,1]}\right) = 0\right\} \overline{\mu}\left(I_k^{\infty,[0,1]}\right) \mid I^{\infty,[0,1]}\right] = 0 \tag{3.68}$$

and $\overline{Q}$-a.s. $\mu(I^{\infty,[0,1]}(T)) > 0$. Applying this to (3.67) gives

$$\lim_{n \to \infty} \mathbf{P}_{\overline{Q}}\left(\frac{\log_b \mu^{(n)}(J^n(T))}{-n} > 1 + \varepsilon\right) = 0. \tag{3.69}$$

It is easy to see by contradiction that the previous equation implies

$$\liminf_{n} \frac{\log_b\left(\mu^{(n)}(J^n(T))\right)}{-n} \le 1 \quad \overline{Q}\text{-a.s.} \tag{3.70}$$

(4°): By the LLN, by the analogs of Corollaries 3.10 and 3.11 for $\overline{Q}$, and by (3.60), we have $\overline{Q}$-a.s.

$$\frac{1}{n}\sum_{i=0}^{n-1}\log_b \Lambda^{(i)}(T) \longrightarrow \mathbf{E}_{\overline{Q}}[\Lambda(T)] = 1-\alpha_q. \tag{3.71}$$

By Corollary 3.15 we have $(1/n)\log_b(|J^n(T)|) \to -1$ $\overline{Q}$-a.s. Applying these results and (3.70) to (3.52) completes the proof. □

Combining the previous results, we find the dimension of the set $V^{\mathcal{P}}_{\alpha_q}$.

Theorem 3.17. *Assume that $A(t)$ is a non-degenerate MJP and that $\overline{A}_n(t)$ converges in $\mathcal{L}_p$ for some $p>1$. If $\overline{\beta}(p)>0$, then, with probability one,*

$$\dim^{\mathcal{P}}\left(V^{\mathcal{P}}_{\alpha_q}\right) = \dim\left(V^{\mathcal{P}}_{\alpha_q}\right) = q\alpha_q - \beta(q). \tag{3.72}$$

Proof. The proof follows the lines of lemmas 3.12 and 3.13 and theorem 3.14. To get the lower bound, the set K_{α_1} is replaced by W_{α_q} and Lemma 2.2 is applied to $\overline{\mu}$. The upper bound follows directly from inequality (2.24). In summary,

$$\dim^{\mathcal{P}}\left(V^{\mathcal{P}}_{\alpha_q}\right) = \dim\left(V^{\mathcal{P}}_{\alpha_q}\right) = f^{+}_{\mathcal{P}}\left(\alpha^{+}_q\right) = T^{*}_{\mathcal{P}}(\alpha_q) = T^{*}_{\mathfrak{D}}(\alpha_q) = \beta^{*}(\alpha_q) = q\alpha_q - \beta(q). \tag{3.73}$$

□

For example, if $A(t)$ is an MJP satisfying (3.5) and $\beta(4) > 2\beta(2) > 0$, then (3.72) holds for $0 < q \le 2$.

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