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Research Article

Moderate and Large Deviations for the Smoothed Estimate of Sample Quantiles

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We derive the moderate and large deviations principle for the smoothed sample quantile from a sequence of independent and identically distributed samples of size *n*.

1. Introduction

As it is known, the quantiles can be used for describing some properties of random variables without the restriction of moment conditions. Quantiles play a fundamental role in statistics; they are the critical values we use in hypothesis testing and interval estimation and often are the characteristics of distributions we wish most to estimate. The use of quantiles as primary measure of performance has gained prominence, particularly in microeconomic, financial, and environmental analyses and so on.

To be more specific, let F denote the unknown cumulative distributions function (c.d.f.). In terms of the inverse c.d.f., the p-quantile is given by $\xi_p = F^{-1}(p)$, where

$$F^{-1}(u) = \inf\{t; F(t) \geq u\}, \quad u \in (0,1). \tag{1}$$

Let $F_n(t)$ be the empirical distributions based on the sample $\{X_i; i = 1, 2, \ldots, n\}$; that is,

$$F_n(x) = \frac{1}{n}\sum_{i=1}^{n} 1_{\{X_i \leq x\}}, \quad -\infty < x < \infty. \tag{2}$$

Then the sample p-quantile based on the empirical distribution function can be represented as

$$\hat{\xi}_p = \inf\{x; F_n(x) \geq p\}, \quad p \in (0,1). \tag{3}$$

The limit properties of $\hat{\xi}_p$ have been studied in numerous literatures. Lahiri and Sun [1] gave Berry-Esseen theorems for samples of strongly mixing random variables under a polynomial mixing rate. Wu [2] established the Bahadur representation for the sample p-quantile for dependent sequences. Miao et al. [3] and Xu et al. [4] studied some asymptotic properties of the deviation between p-quantile and the estimator, including the moderate deviations, large deviations, and Bahadur representation. Ma et al. [5] gave the definition of sample p-quantile based on mid-distribution functions to provide a unified framework for asymptotic properties of sample p-quantile from discrete distributions.

However, F_n does not take into account the smoothness of F, that is, the existence of the density function f. Then some investigators proposed several smoothed quantile estimates. Based on a kernel function K, one of the smoothed estimators for F is defined as

$$\hat{F}_n(x) = \int_{\mathbb{R}} K\left(\frac{x-t}{h_n}\right) dF_n(t) = \frac{1}{n}\sum_{i=1}^{n} K\left(\frac{x-X_i}{h_n}\right), \tag{4}$$

where $\{h_n\}$ is a positive sequence of bandwidths with $h_n \to 0$ as $n \to \infty$. Then, the smoothed sample quantile estimate of ξ_p, $\hat{\xi}_{pn}$ is defined by

$$\hat{\xi}_{pn} = \hat{F}_n^{-1}(p) = \inf\left\{x; \hat{F}_n(x) \geq p\right\}, \quad p \in (0,1). \tag{5}$$

Asymptotic properties for different forms of sample quantile have been investigated extensively in the literature. The kernel-type estimate of the quantile ξ_p early work on the estimators of the quantile function includes Nadaraya [6] and

Parzen [7]. Reiss [8] showed that the asymptotic relative deficiency of the sample quantile with respect to a linear combination of finitely many order statistics diverges to infinity as the sample size increases. Falk [9] also examined the asymptotic relative deficiency of the sample quantile compared to kernel-type quantile estimators. Yang [10] studied the asymptotic properties of kernel-type quantile estimators. Padgett [11] extended the previous works to handle right-censored data. Cai and Roussas [12] established pointwise consistency, asymptotic normality with rates, and weak convergence of the smoothed estimates.

In this paper, we will derive the pointwise moderate and large deviations principle for $\widehat{\xi}_{pn} - \xi_p$. There exists extensive large deviation literature involving many areas of probability and statistics. We refer to the book of Dembo and Zeitouni [13] and the references therein for an account of results and applications. In nonparametric function estimation setting, several results have been stated these last years. We refer to Louani [14], Gao [15], He and Gao [16], and Korbe Diallo and Louani [17], where results related to the kernel density estimator are obtained.

In order to state our main results, let us introduce the definition of large deviation principle. Let (S, d) be a metric space and let $\{Y_n : n \geq 1\}$ be a sequence of S-valued random variables on probability space $(\Omega, \mathfrak{F}, P)$. Let $\lambda(n)$ be a sequence of positive real numbers satisfying $\lambda(n) \to \infty$ as $n \to \infty$. A function $I(\cdot) : S \to [0, +\infty]$ is said to be a rate function if it is lower semicontinuous and it is said to be a good rate function if its level set $\{x \in S : I(x) \leq l\}$ is compact for all $l \geq 0$. The sequence $\{Y_n, n \geq 1\}$ is said to satisfy a large deviation principle with speed $\lambda(n)$ and with good rate function I if, for any closed set F in S,

$$\limsup_{n\to\infty} \frac{1}{\lambda(n)} \log P\left(Y_n \in F\right) \leq -\inf_{x\in F} I(x) \tag{6}$$

and, for open set G in S,

$$\liminf_{n\to\infty} \frac{1}{\lambda(n)} \log P\left(Y_n \in G\right) \leq -\inf_{x\in G} I(x). \tag{7}$$

2. Assumptions and Main Results

In order to display our results, we introduce some assumptions.

(A1)

$$\lim_{x\to\infty} f(x) = 0. \tag{8}$$

(A2) $K(x) \geq 0$, and $\int_{-\infty}^{\infty} K(x)dx = 1$, $\int_{-\infty}^{\infty} K^2(x)dx < \infty$.

(A3) $I(\lambda) := \int_{-\infty}^{+\infty} (\exp\{-\lambda K(z)\} - 1)dz < \infty$, for any $\lambda < 0$.

(A4) $J(\lambda) := \int_{-\infty}^{+\infty} K(z)\exp\{-\lambda K(z)\}dz < \infty$, for any $\lambda < 0$.

(A5) $\int_{-\infty}^{+\infty} K^2(z)\exp\{-\lambda K(z)\}dz < \infty$, for any $\lambda < 0$.

Firstly, we give the pointwise moderate deviation principle.

Theorem 1. *Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with an absolutely continuous distribution function $F(x)$, and let ξ_p be the p-quantile of F for $p \in (0,1)$. Assume that the conditions (A1) and (A2) hold; corresponding to the sample $\{X_1, X_2, \ldots, X_n\}$, the smoothed sample p-quantile which is denoted by $\widehat{\xi}_{pn}$ is defined as in Section 1. Let $\{b_n\}$ be a positive sequence satisfying*

$$\begin{gathered} b_n \longrightarrow \infty, \\ \frac{b_n}{\sqrt{n}h_n} \longrightarrow 0 \\ \text{as } n \longrightarrow \infty. \end{gathered} \tag{9}$$

Then, for any $r > 0$, we have

$$\begin{aligned} \lim_{n\to\infty} & \frac{1}{h_n b_n^2} \log P\left(\frac{\sqrt{n}}{b_n}\left|\widehat{\xi}_{pn} - \xi_p\right| \geq r\right) \\ & = -\frac{r^2}{2f\left(\xi_p\right)\left(1 + \int_{-\infty}^{\infty} K^2(z)\,dz\right)}. \end{aligned} \tag{10}$$

The following result establishes a pointwise large deviation principle.

Theorem 2. *Let $X_1, X_2, \ldots, X_n$ be independent identically distributed random variables with an absolutely continuous distribution function $F(x)$, and let ξ_p be the p-quantile of F for $p \in (0,1)$. Assume that the conditions (A1)–(A5) hold; $\widehat{\xi}_{pn}$ is defined as in Theorem 1; then, for any $r > 0$, we have*

$$\lim_{n\to\infty} \frac{1}{nh_n} \log P\left(\widehat{\xi}_{pn} - \xi_p \geq r\right) = -\inf_{x\geq -p} \Lambda_+^*(x), \tag{11}$$

$$\lim_{n\to\infty} \frac{1}{nh_n} \log P\left(\widehat{\xi}_{pn} - \xi_p \leq -r\right) = -\inf_{x\geq p} \Lambda_-^*(x), \tag{12}$$

where

$$\Lambda_+^*(x) = \begin{cases} J^{-1}\left(\dfrac{f\left(\xi_p + r\right) - x}{f\left(\xi_p + r\right)}\right)\left(x - f\left(\xi_p + r\right)\right) + f\left(\xi_p + r\right) I\left(J^{-1}\left(\dfrac{f\left(\xi_p + r\right) - x}{f\left(\xi_p + r\right)}\right)\right), & \text{if } x < f\left(\xi_p + r\right), \\ +\infty, & \text{if } x \geq f\left(\xi_p + r\right), \end{cases}$$

$$\Lambda_-^*(x)=\begin{cases}J^{-1}\left(\dfrac{f(\xi_p-r)-x}{f(\xi_p-r)}\right)\left(f(\xi_p+r)-x\right)+f(\xi_p-r)\,I\left(-J^{-1}\left(\dfrac{f(\xi_p-r)-x}{f(\xi_p-r)}\right)\right), & \text{if } x<f(\xi_p-r),\\ +\infty, & \text{if } x\geq f(\xi_p-r).\end{cases} \tag{13}$$

Remark 3. As it is known, whatever estimates are obtained by way of the smooth cumulative distribution function (c.d.f); they exhibit weaker rate of convergence. We can compare our moderate deviation result with that of the Xu and Miao [18], in which the estimation of the sample quantile was based on the c.d.f. From Theorem 1 in this paper, for n large enough,

$$P\left(\frac{\sqrt{n}}{b_n}\left|\hat{\xi}_{pn}-\xi_p\right|\geq r\right)\approx e^{-C_1h_nr^2b_n^2}. \tag{14}$$

At the same time, we can derive from Xu and Miao [18] that

$$P\left(\frac{\sqrt{n}}{b_n}\left|\hat{\xi}_{pn}-\xi_p\right|\geq r\right)\approx e^{-C_2r^2b_n^2}, \tag{15}$$

where C_1, C_2 are some constants.

3. Proof of the Main Results

3.1. Proof of Theorem 1. For any $r>0$, we have

$$\begin{aligned}&P\left(\frac{\sqrt{n}}{b_n}\left|\hat{\xi}_{pn}-\xi_p\right|\geq r\right)\\&\quad=P\left(\hat{\xi}_{pn}\geq\xi_p+\frac{b_n}{\sqrt{n}}r\right)+P\left(\hat{\xi}_{pn}\leq\xi_p-\frac{b_n}{\sqrt{n}}r\right).\end{aligned} \tag{16}$$

Then,

$$\begin{aligned}&P\left(\hat{\xi}_{pn}\geq\xi_p+\frac{b_n}{\sqrt{n}}r\right)\\&\quad=P\left(\hat{F}_n\left(\xi_p+\frac{b_n}{\sqrt{n}}r\right)<p\right)\\&\quad=P\left(\sum_{i=1}^{n}\left(V_{in}-EV_{in}\right)\geq n\left[(1-p)-EV_{1n}\right]\right),\end{aligned} \tag{17}$$

where $V_{in}=1-K((\xi_p+(b_n/\sqrt{n})r-X_i)/h_n)$.

For any $\lambda\in\mathbb{R}$, by Taylor's expansion,

$$\begin{aligned}\Lambda(\lambda)&:=\lim_{n\to\infty}\frac{1}{h_nb_n^2}\log E\\&\quad\cdot\left(\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\sum_{i=1}^{n}\left(V_{in}-EV_{in}\right)\right\}\right)=\lim_{n\to\infty}\frac{1}{h_nb_n^2}\\&\quad\cdot\log\left(E\left(\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\left(V_{in}-EV_{in}\right)\right\}\right)\right)^n\\&=\lim_{n\to\infty}\frac{n}{h_nb_n^2}\log\left(E\left(\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\left(V_{in}-EV_{in}\right)\right\}\right)\right)^n\\&=\lim_{n\to\infty}\frac{n}{h_nb_n^2}\\&\quad\cdot\log\left(1+\frac{\lambda^2b_n^2}{n}\frac{E\left(V_{in}-EV_{in}\right)^2}{2}+o\left(\frac{b_n^2}{n}\right)\right).\end{aligned} \tag{18}$$

On the other hand,

$$\begin{aligned}&E\left(V_{in}-EV_{in}\right)^2\\&\quad=EK^2\left(\frac{\xi_p+(b_n/\sqrt{n})r-X_i}{h_n}\right)\\&\qquad-\left(EK\left(\frac{\xi_p+(b_n/\sqrt{n})r-X_i}{h_n}\right)\right)^2\\&\quad=\int_{-\infty}^{+\infty}K^2\left(\frac{\xi_p+(b_n/\sqrt{n})r-x}{h_n}\right)f(x)\,dx\\&\qquad-\left(\int_{-\infty}^{+\infty}K\left(\frac{\xi_p+(b_n/\sqrt{n})r-x}{h_n}\right)f(x)\,dx\right)^2\\&\quad=h_n\int_{-\infty}^{+\infty}K^2(z)\,f\left(\xi_p+\frac{b_n}{\sqrt{n}}r-h_nz\right)dz\\&\qquad-h_n\left(\int_{-\infty}^{+\infty}K(z)\,f\left(\xi_p+\frac{b_n}{\sqrt{n}}r-h_nz\right)dz\right)^2.\end{aligned} \tag{19}$$

By Lemma 2.2 in Gao [15], we can obtain that

$$\begin{aligned}&\lim_{n\to\infty}\frac{1}{h_nb_n^2}\log E\left(\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\sum_{i=1}^{n}\left(V_{in}-EV_{in}\right)\right\}\right)\\&\quad=\frac{\lambda^2f(\xi_p)\left(1+\int_{-\infty}^{\infty}K^2(z)\,dz\right)}{2}.\end{aligned} \tag{20}$$

Then, by Gärtner-Ellis theorem (see Dembo and Zeitouni [13]), we have

$$\lim_{n\to\infty}\frac{1}{h_n b_n^2}\log P\left(\frac{\sqrt{n}}{b_n}\left(\hat{\xi}_{pn}-\xi_p\right)\geq r\right) = -\frac{r^2}{2f\left(\xi_p\right)\left(1+\int_{-\infty}^{\infty}K^2(z)\,dz\right)}. \tag{21}$$

Likewise,

$$\begin{aligned} P&\left(\hat{\xi}_{pn}\leq\xi_p-\frac{b_n}{\sqrt{n}}r\right) \\ &= P\left(\hat{F}_n\left(\xi_p-\frac{b_n}{\sqrt{n}}r\right)\geq p\right) \\ &= P\left(\sum_{i=1}^{n}\left(Q_{in}-EQ_{in}\right)\geq n\left[p-EQ_{1n}\right]\right), \end{aligned} \tag{22}$$

where $Q_{in}=K((\xi_p-(b_n/\sqrt{n})r-X_i)/h_n)$.

For any $\lambda\in\mathbb{R}$, by using Taylor's expansion again,

$$\lim_{n\to\infty}\frac{1}{h_n b_n^2}\log E\left(\exp\left\{\frac{\lambda b_n}{\sqrt{n}}\sum_{i=1}^{n}\left(Q_{in}-EQ_{in}\right)\right\}\right) = \frac{\lambda^2 f\left(\xi_p\right)\left(1+\int_{-\infty}^{\infty}K^2(z)\,dz\right)}{2}. \tag{23}$$

Applying Gärtner-Ellis theorem, we can obtain that

$$\lim_{n\to\infty}\frac{1}{h_n b_n^2}\log P\left(\frac{\sqrt{n}}{b_n}\left(\hat{\xi}_{pn}-\xi_p\right)\leq -r\right) = -\frac{r^2}{2f\left(\xi_p\right)\left(1+\int_{-\infty}^{\infty}K^2(z)\,dz\right)}. \tag{24}$$

By (21) and (24), we can obtain the result in the theorem.

3.2. Proof of Theorem 2. For any $r>0$, by Serfling [19],

$$\begin{aligned} P&\left(\hat{\xi}_{pn}-\xi_p\geq r\right) \\ &= P\left(\hat{F}_n\left(\xi_p+r\right)<p\right) \\ &= P\left(\sum_{i=1}^{n}\left(W_{in}-EW_{in}\right)\geq n\left[(1-p)-EW_{1n}\right]\right). \end{aligned} \tag{25}$$

And, for any $\lambda\in\mathbb{R}$,

$$\begin{aligned} E&\left(\exp\left\{\lambda\left(W_{in}-EW_{1n}\right)\right\}\right) \\ &= E\left(\exp\left\{\lambda\left(E\left(K\left(\frac{\xi_p+r-X_i}{h_n}\right)\right)-K\left(\frac{\xi_p+r-X_i}{h_n}\right)\right)\right\}\right) \\ &= \exp\left\{\lambda h_n\int_{-\infty}^{\infty}K(z)\,f\left(\xi_p+r-h_n z\right)dz\right\} \\ &\cdot h_n\int_{-\infty}^{+\infty}\exp\left\{-\lambda K(z)\right\}f\left(\xi_p+r-h_n z\right)dz \\ &= \exp\left\{\lambda h_n\int_{-\infty}^{\infty}K(z)\,f\left(\xi_p+r-h_n z\right)dz\right\}\left(1\right. \\ &\left.+h_n\int_{-\infty}^{+\infty}\left(\exp\left\{-\lambda K(z)\right\}-1\right)f\left(\xi_p+r-h_n z\right)dz\right); \end{aligned} \tag{26}$$

then, by Lemma 2.2 in Gao [15],

$$\begin{aligned} \Lambda_+(\lambda) &:= \lim_{n\to\infty}\frac{1}{nh_n}\log E\left(\exp\left\{\lambda n W_{in}\right\}\right) \\ &= \lambda\int_{-\infty}^{\infty}K(z)\,f\left(\xi_p+r\right)dz \\ &\quad+\int_{-\infty}^{+\infty}\left(\exp\left\{-\lambda K(z)\right\}-1\right)f\left(\xi_p+r\right)dz \\ &= \lambda f\left(\xi_p+r\right) \\ &\quad+f\left(\xi_p+r\right)\int_{-\infty}^{+\infty}\left(\exp\left\{-\lambda K(z)\right\}-1\right)dz. \end{aligned} \tag{27}$$

The Fenchel-Legendre transform of $\Lambda_+(\lambda)$ is

$$\begin{aligned} \Lambda_+^*(x) &= \sup_{\lambda\in\mathbb{R}}\left\{\lambda x-\Lambda(\lambda)\right\} = \sup_{\lambda\in\mathbb{R}}\left\{\lambda x-\lambda f\left(\xi_p+r\right)\right. \\ &\quad\left.-f\left(\xi_p+r\right)\int_{-\infty}^{+\infty}\left(\exp\left\{-\lambda K(z)\right\}-1\right)dz\right\}. \end{aligned} \tag{28}$$

By simple calculation, we can obtain

$$\Lambda_+^*(x) = \begin{cases} J^{-1}\left(\dfrac{f\left(\xi_p+r\right)-x}{f\left(\xi_p+r\right)}\right)\left(x-f\left(\xi_p+r\right)\right)+f\left(\xi_p+r\right)I\left(J^{-1}\left(\dfrac{f\left(\xi_p+r\right)-x}{f\left(\xi_p+r\right)}\right)\right), & \text{if } x<f\left(\xi_p+r\right), \\ +\infty, & \text{if } x\geq f\left(\xi_p+r\right). \end{cases} \tag{29}$$

Then, by the Cramér theorem (see Dembo and Zeitouni [13]), we have

$$\lim_{n\to\infty}\frac{1}{nh_n}\log P\left(\hat{\xi}_{pn}-\xi_p\geq r\right)=\lim_{n\to\infty}\frac{1}{nh_n}\log P \cdot\left(\sum_{i=1}^{n}\left(W_{in}-EW_{in}\right)\geq n\left[\left(1-p\right)-EW_{1n}\right]\right) = -\inf_{x\geq -p}\Lambda_+^*\left(x\right). \tag{30}$$

Similarly,

$$\begin{aligned} &P\left(\hat{\xi}_{pn}-\xi_p\leq -r\right) \\ &\quad = P\left(\hat{F}_n\left(\xi_p-r\right)\geq p\right) \\ &\quad = P\left(\sum_{i=1}^{n}\left(U_{in}-EU_{in}\right)\geq n\left[p-EU_{1n}\right]\right). \end{aligned} \tag{31}$$

And, for any $\lambda\in\mathbb{R}$,

$$\begin{aligned} &E\left(\exp\left\{\lambda\left(U_{in}-EU_{1n}\right)\right\}\right) \\ &\quad = E\left(\exp\left\{\lambda\left(K\left(\frac{\xi_p-r-X_i}{h_n}\right)-E\left(K\left(\frac{\xi_p-r-X_i}{h_n}\right)\right)\right)\right\}\right) \\ &\quad = \exp\left\{-\lambda h_n\int_{-\infty}^{\infty}K\left(z\right)f\left(\xi_p-r-h_nz\right)dz\right\} \\ &\quad \cdot h_n\int_{-\infty}^{+\infty}\exp\left\{\lambda K\left(z\right)\right\}f\left(\xi_p-r-h_nz\right)dz \\ &\quad = \exp\left\{-\lambda h_n\int_{-\infty}^{\infty}K\left(z\right)f\left(\xi_p-r-h_nz\right)dz\right\}\left(1\right. \\ &\quad \left.+h_n\int_{-\infty}^{+\infty}\left(\exp\left\{\lambda K\left(z\right)\right\}-1\right)f\left(\xi_p-r-h_nz\right)dz\right). \end{aligned} \tag{32}$$

Then,

$$\begin{aligned} \Lambda_-\left(\lambda\right) &:= \lim_{n\to\infty}\frac{1}{nh_n}\log E\left(\exp\left\{\lambda nU_{in}\right\}\right) \\ &= \lambda f\left(\xi_p-r\right) \\ &\quad + f\left(\xi_p-r\right)\int_{-\infty}^{+\infty}\left(\exp\left\{\lambda K\left(z\right)\right\}-1\right)dz. \end{aligned} \tag{33}$$

The Fenchel-Legendre transform of $\Lambda_-(\lambda)$ is

$$\Lambda_-^*\left(x\right)=\begin{cases} J^{-1}\left(\dfrac{f\left(\xi_p-r\right)-x}{f\left(\xi_p-r\right)}\right)\left(f\left(\xi_p+r\right)-x\right)+f\left(\xi_p-r\right)I\left(-J^{-1}\left(\dfrac{f\left(\xi_p-r\right)-x}{f\left(\xi_p-r\right)}\right)\right), & \text{if } x<f\left(\xi_p-r\right), \\ +\infty, & \text{if } x\geq f\left(\xi_p-r\right). \end{cases} \tag{34}$$

Then, we obtain (12), and we complete the proof of Theorem 2.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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