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Research Article

Bayesian Estimation in Delta and Nabla Discrete Fractional Weibull Distributions

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We use discrete fractional calculus for showing the existence of delta and nabla discrete distributions and then apply time scales for definitions of delta and nabla discrete fractional Weibull distributions. Also, we study the Bayesian estimation of the functions of parameters of these distributions.

1. Introduction

One of the active areas of research in statistics is to model discrete life time data by developing discretized version of suitable continuous lifetime distributions. The discretization of a continuous distribution using different methods has attracted renewed attention of researchers in the last few years; for example, see [1–9]. Recently, these different methods are classified based on different criteria of discretization in detail by Chakraborty (see [10]).

In this article, we present a new method for discretization of most of continuous distributions, where their pdfs consist of the monomial Taylor and exponential function. As an example, we do discretization for Weibull distribution. Our discretization method in comparison with prior methods for discretization of continuous distributions has two main advantages. First, for a given continuous distribution, it is possible to generate two types (delta and nabla types) of corresponding discrete distributions. Second, the main result of this paper is a unification of the continuous distributions and their corresponding discrete distributions, which is at the same time a distribution to the case of so-called time scale. We use discrete fractional calculus for showing the existence of delta and nabla discrete distributions and then apply time scales for definition of delta and nabla discrete distributions and as a unification theory under which continuous and discrete distributions are subsumed. Finally, we study the Bayesian estimation of functions of parameters of these distributions.

2. Preliminaries

In this section, we provide a collection of definitions and related results which are essential and will be used in the next discussions. As mentioned in [11, 12] the definitions and theorem are as follows.

A time scale $\mathbb{T}$ is an arbitrary nonempty closed subset of the real numbers $\mathbb{R}$. The most well-known examples are $\mathbb{T} = \mathbb{R}$ and $\mathbb{T} = \mathbb{Z}$. The forward (backward) jump operator is defined by $\sigma(t) := \inf\{s \in \mathbb{T} : s > t\}$ ($\rho(t) := \sup\{s \in \mathbb{T} : s < t\}$), where $\inf \emptyset := \sup \mathbb{T}$ and $\sup \emptyset := \inf \mathbb{T}$. A point $t \in \mathbb{T}$ is said to be right-dense if $t < \sup \mathbb{T}$ and $\sigma(t) = t$ (left-dense if $t > \inf \mathbb{T}$ and $\rho(t) = t$) and right-scattered if $\sigma(t) > t$ (left-scattered if $\rho(t) < t$). The forward (backward) graininess function $\mu : \mathbb{T} \to [0, \infty)$ ($\nu : \mathbb{T} \to [0, \infty)$) is defined by $\mu(t) := \sigma(t) - t$ ($\nu(t) := t - \rho(t)$). More generally, we denote all $\rho(t)$, $\sigma(t)$, and t with $\eta(t)$.

Definition 1. A function $f : \mathbb{T} \to \mathbb{R}$ is called regulated if its right-sided limits exist at all right-dense points in $\mathbb{T}$ and its left-sided limits exist at all left-dense points in $\mathbb{T}$.

Definition 2. A function $f : \mathbb{T} \to \mathbb{R}$ is called rd-continuous (ld-continuous) if it is continuous at right-dense (left-dense) points in $\mathbb{T}$ and its left-sided (right-sided) limits exist at left-dense (right-dense) points in $\mathbb{T}$.

The set $\mathbb{T}_k$ ($\mathbb{T}_k^*$) is derived from the time scale $\mathbb{T}$ as follows: If $\mathbb{T}$ has a left-scattered maximum (right-scattered minimum)

m, then $\mathbb{T}_k := \mathbb{T} - \{m\}$ ($\mathbb{T}_k^* := \mathbb{T} - \{m\}$). Otherwise, $\mathbb{T}_k := \mathbb{T}$ ($\mathbb{T}_k^* := \mathbb{T}$).

Definition 3. A function $f : \mathbb{T} \to \mathbb{R}$ is said to be delta (nabla) differentiable at a point $t \in \mathbb{T}_k$ ($t \in \mathbb{T}_k^*$) if there exists a number $f^\Delta(t)$ ($f^\nabla(t)$) with the property that, given any $\epsilon > 0$, there exists a neighborhood U of t such that

$$\begin{gathered}\left|f(\sigma(t)) - f(s) - f^\Delta(t)(\sigma(t) - s)\right| \le \epsilon\left|\sigma(t) - s\right|, \\ \left(\left|f(\rho(t)) - f(s) - f^\nabla(t)(\rho(t) - s)\right| \le \epsilon\left|\rho(t) - s\right|\right)\end{gathered} \tag{1}$$

for all $s \in U$.

For a function $f : \mathbb{T} \to \mathbb{R}$ it is possible to introduce a derivative $f^\Delta(t)$ ($f^\nabla(t)$) and an integral $\int_a^b f(t)\Delta t$ ($\int_a^b f(t)\nabla t$) in such a manner that $f^\Delta(t) = f'$ ($f^\nabla(t) = f'$) and $\int_a^b f(t)\Delta t = \int_a^b f(t)dt$ ($\int_a^b f(t)\nabla t = \int_a^b f(t)dt$) in the case $\mathbb{T} = \mathbb{R}$ and $f^\Delta(t) = \Delta f$ ($f^\nabla(t) = \nabla f$) and $\int_a^b f(t)\Delta t = \sum_a^{b-1} f(t)$ ($\int_a^b f(t)\nabla t = \sum_{a+1}^b f(t)$) in the case $\mathbb{T} = \mathbb{Z}$, where the forward and backward difference operators are defined by $\Delta f = f(t+1) - f(t)$ and $= f(t) - f(t-1)$, respectively. Also, we define the iterated operators $\Delta^n = \Delta(\Delta^{n-1})$ and $\nabla^n = \nabla(\nabla^{n-1})$ for $n \in \mathbb{N}$.

Definition 4. A function $p : \mathbb{T} \to \mathbb{R}$ is called μ-regressive (ν-regressive) provided that $1 + \mu(t)p(t) \neq o$ ($1 - \nu(t)p(t) \neq o$) for all $t \in \mathbb{T}_k$ ($t \in \mathbb{T}_k^*$).

The set $\mathbb{R}_\mu$ ($\mathbb{R}_\nu$) of all μ-regressive and rd-continuous (ν-regressive and ld-continuous) functions forms an Abelian group under the circle plus addition $\oplus$ defined by $(p \oplus q)(t) := p(t) + q(t) + \mu(t)p(t)q(t)$ ($(p \oplus q)(t) := p(t) + q(t) - \nu(t)p(t)q(t)$) for all $t \in \mathbb{T}_k$ ($t \in \mathbb{T}_k^*$). The additive inverse $\ominus p$ of $p \in \mathbb{R}_\mu$ ($p \in \mathbb{R}_\nu$) is defined by

$$\begin{gathered}(\ominus p)(t) := -\frac{p(t)}{1 + \mu(t)p(t)} \\ \left((\ominus p)(t) := -\frac{p(t)}{1 - \nu(t)p(t)}\right)\end{gathered} \tag{2}$$

for all $t \in \mathbb{T}_k$ ($t \in \mathbb{T}_k^*$).

For real numbers a and b we denote $\mathbb{N}_a = \{a, a+1, \ldots\}$ and ${}_b\mathbb{N} = \{b, b-1, \ldots\}$.

Theorem 5. *Let $p \in \mathbb{R}_\mu$ ($p \in \mathbb{R}_\nu$) and $t_0 \in \mathbb{T}$ be a fixed point. Then the delta (nabla) exponential function $e_p(\cdot, t_0)$ ($e_p^*(\cdot, t_0)$) is the unique solution of the initial value problem*

$$\begin{gathered}y^\Delta = p(t)y, \quad y(t_0) = 1 \\ \left(y^\nabla = p(t)y, \ y(t_0) = 1\right).\end{gathered} \tag{3}$$

If $\mathbb{T} = \mathbb{N}_a$, when $p(t) \equiv p$, where $p \in \mathbb{R}_\mu$ ($p \in \mathbb{R}_\nu = \mathbb{C} \setminus \{1\}$) and $t_0 = a$, it is easy to see that $e_p(t, a) = (1+p)^{t-a}$ ($e_p^*(t, a) = (1-p)^{a-t}$) and if $\mathbb{T} = \mathbb{R}$, $e_p(t, a) = e^{p(t-a)}$ ($e_p^*(t, a) = e^{p(t-a)}$), where "e" is ordinary exponential function. Moreover, in the special case, $e_1(t, 0) = 2^t$ ($e_{1/2}^*(t, 0) = 2^t$). More generally, we will denote all $e_p(t, a)$, $e_p^*(t, a)$, and $e^{p(t-a)}$ with $\widehat{e}_p(t, a)$.

Definition 6. The delta (nabla) Taylor monomials are the functions $h_n : \mathbb{T} \times \mathbb{T} \to \mathbb{R}$, $n \in \mathbb{N}_0$, and are defined recursively as follows:

$$\begin{gathered}h_0(t, s) = 1, \\ h_{n+1}(t, s) = \int_s^t h_n(\tau, s)\Delta\tau, \\ \forall t, s \in \mathbb{T}, \\ \left(h_0^*(t, s) = 1, \ h_{n+1}^*(t, s) = \int_s^t h_n^*(\tau, s)\nabla\tau, \ \forall t, s \in \mathbb{T}\right).\end{gathered} \tag{4}$$

We consider three cases for the time scale $\mathbb{T}$.

(a) If $\mathbb{T} = \mathbb{R}$, then $\sigma(t) = \rho(t) = t$ and the Taylor monomials can be written explicitly as

$$h_n(t, s) = h_n^*(t, s) = \frac{(t-s)^n}{n!}, \quad t, s \in \mathbb{R}, \ n \in \mathbb{N}_0. \tag{5}$$

For each $\alpha \in \mathbb{R} \setminus \{-\mathbb{N}\}$, define the αth Taylor monomial to be

$$h_\alpha(t, s) = \frac{(t-s)^\alpha}{\Gamma(\alpha+1)}, \tag{6}$$

and Γ denoted the special gamma function.

In this paper, we only consider the special case, $h_\alpha(t) := h_\alpha(t, 0) = t^\alpha/\Gamma(\alpha+1)$ as Taylor monomial (tm).

(b) If $\mathbb{T} = \mathbb{Z}$, then $\sigma(t) = t + 1$ and the Taylor monomials can be written explicitly as

$$h_n(t, s) = \frac{(t-s)^{\underline{n}}}{n!}, \quad t, s \in \mathbb{Z}, \ n \in \mathbb{N}_0, \tag{7}$$

where

$$t^{\underline{n}} = \prod_{j=0}^{n-1}(t - j) = \frac{\Gamma(t+1)}{\Gamma(t+1-j)} \tag{8}$$

and product is zero when $t + 1 - j = 0$ for some j. More generally, for α arbitrary define $t^{\underline{\alpha}} = \Gamma(t+1)/\Gamma(t+1-\alpha)$, where the convention of that division at pole yields zero. This generalized falling function allows us to extend (7) to define a general Taylor monomial that will serve us well in the probability distributions setting.

For each $\alpha \in \mathbb{R} \setminus \{-\mathbb{N}\}$, define the delta αth Taylor monomial to be

$$h_\alpha(t, s) = \frac{(t-s)^{\underline{\alpha}}}{\Gamma(\alpha+1)}. \tag{9}$$

In this paper, we only consider the special case $h_{\underline{\alpha}}(t) := h_\alpha(t, 0) = t^{\underline{\alpha}}/\Gamma(\alpha+1)$ as delta Taylor monomial (dtm).

(c) If $\mathbb{T} = \mathbb{Z}$, then $\rho(t) = t - 1$ and the Taylor monomials can be written explicitly as

$$h_n(t, s) = \frac{(t-s)^{\overline{n}}}{n!}, \quad t, s \in \mathbb{Z}, \ n \in \mathbb{N}_0, \tag{10}$$

where

$$t^{\overline{n}} = \prod_{j=0}^{n-1} (t+j) = \frac{\Gamma(t+n)}{\Gamma(t)}. \tag{11}$$

More generally, for any $\alpha \in \mathbb{R} \setminus \{-\mathbb{N}\}$, the α rising function is defined as $t^{\overline{\alpha}} = \Gamma(t+\alpha)/\Gamma(t)$ and $0^{\overline{\alpha}} = 0$. The rising $t^{\overline{\alpha}}$ and falling $t^{\underline{\alpha}}$ are related by $t^{\overline{\alpha}} = (t+\alpha-1)^{\underline{\alpha}}$.

This rising function allows us to extend (10) in order to define a general Taylor monomial that will serve us well in the probability distributions setting.

For each $\alpha \in \mathbb{R} \setminus \{-\mathbb{N}\}$ define the nabla αth Taylor monomial to be

$$h_\alpha^*(t,s) = \frac{(t-s)^{\overline{\alpha}}}{\Gamma(\alpha+1)}. \tag{12}$$

In this paper, we only consider the special case $h_{\overline{\alpha}}(t) := h_\alpha^*(t,0) = t^{\overline{\alpha}}/\Gamma(\alpha+1)$ as nabla Taylor monomial (ntm).

More generally, we will denote all $h_{\underline{\alpha}}(t)$, $h_{\overline{\alpha}}(t)$, and $h_\alpha(t)$ with $\widehat{h}_\alpha(t)$.

Let b be a real number and $f : {}_b\mathbb{N} \to \mathbb{R}$. The delta Riemann right fractional sum of order $\alpha > 0$ is defined by Abdeljawad [13] as

$$\Delta^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t+\alpha}^{b} (\rho(s)-t)^{\underline{\alpha-1}} f(s), \quad t \in {}_{b-\alpha}\mathbb{N}. \tag{13}$$

We define the nabla Riemann right fractional sum of order $\alpha > 0$ as

$$\nabla^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \sum_{s=t}^{b-1} (\sigma(s)-t)^{\overline{\alpha-1}} f(s), \quad t \in {}_{b-1}\mathbb{N}. \tag{14}$$

The delta Riemann right fractional difference of order $\alpha > 0$ is defined by Abdeljawad [13] as

$$\Delta^{\alpha} f(t) = (-1)^n \nabla^n \Delta^{-(n-\alpha)} f(t), \tag{15}$$

for $t \in {}_{b-(n-\alpha)}\mathbb{N}$ and $n = [\alpha]+1$, where $[\alpha]$ is the greatest integer less than α. Also, the nabla Riemann right fractional difference of order $\alpha > 0$ is defined by

$$\nabla^{\alpha} f(t) = (-1)^n \Delta^n \nabla^{-(n-\alpha)} f(t), \tag{16}$$

for $t \in {}_{b-n}\mathbb{N}$.

In [14], authors have obtained the following alternative definition for delta Riemann right fractional difference

$$\Delta^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=t-\alpha}^{b} (\rho(s)-t)^{\underline{-\alpha-1}} f(s). \tag{17}$$

Similarly, we can prove the following formula for nabla Riemann right fractional difference:

$$\nabla^{\alpha} f(t) = \frac{1}{\Gamma(-\alpha)} \sum_{s=t}^{b-1} (\sigma(s)-t)^{\overline{-\alpha-1}} f(s). \tag{18}$$

For an introduction to discrete fractional calculus, the reader is referred to [15–18].

3. Generating Discrete Distributions by Discrete Fractional Calculus

The following results show the relationship between continuous and discrete fractional calculus and statistics and also allow us to define different types of discrete distributions.

Suppose that X is a positive continuous random variable. The expectation of the tm function, $h_{\alpha-1}(X)$, coincides with Riemann-Liouville right fractional integral of the pdf at the origin for $\alpha > 0$ and Marchaud fractional derivative of the pdf at the origin for $-1 < \alpha < 0$; that is, we have

$$E[h_{\alpha-1}(X)] = \begin{cases} (I_-^{\alpha} f)(0), & \alpha > 0 \\ (D_-^{\alpha} f)(0), & -1 < \alpha < 0, \end{cases} \tag{19}$$

where

$$(I_-^{\alpha} f)(t) = \frac{1}{\Gamma(\alpha)} \int_0^{\infty} x^{\alpha-1} f(x+t)\, dx \tag{20}$$

is the Riemann-Liouville right fractional integral, while

$$(D_-^{\alpha} f)(t) = \frac{1}{\Gamma(-\alpha)} \int_0^{\infty} x^{-\alpha-1} \{f(x+t) - f(t)\}\, dx \tag{21}$$

is the Marchaud fractional derivative [19]. The definitions of fractional operators can be found in [20].

It can be seen that the limits of the above integrals are equal to the support of random variable X. Considering this point, we present the following theorems for discrete random variable X.

Theorem 7. *Suppose that X is a discrete random variable. The expectation of the dtm function, $h_{\underline{\alpha-1}}(X)$, coincides with delta Riemann right fractional sum of the pmf at -1 for $\alpha > 0$ and delta Riemann right fractional difference of the pmf at -1 for $\alpha < 0$, $\alpha \notin \{-\mathbb{N}\}$; that is,*

$$E\left[h_{\underline{\alpha-1}}(X)\right] = \begin{cases} (\Delta^{-\alpha} f)(-1), & \alpha > 0 \\ (\Delta^{\alpha} f)(-1), & \alpha < 0,\ \alpha \notin \{\ldots,-2,-1\}, \end{cases} \tag{22}$$

where

$$(\Delta^{-\alpha} f)(t) = \sum_{x=\alpha-1}^{b-1-t} \frac{x^{\underline{\alpha-1}}}{\Gamma(\alpha)} f(x+t+1) \tag{23}$$

is the delta Riemann right fractional sum, while

$$(\Delta^{\alpha} f)(t) = \sum_{x=-\alpha-1}^{b-t-1} \frac{x^{\underline{-\alpha-1}}}{\Gamma(-\alpha)} f(x+t+1) \tag{24}$$

is the delta Riemann right fractional difference.

Proof. For $\alpha > 0$, substitute $x = \rho(s) - t$ in expression (13) and also, for $\alpha < 0$ and $\alpha \notin \{-1,-2,\ldots\}$, in expression (17). □

Here, considering the limits of summation we can define the discrete distributions with the support $\mathbb{N}_{\alpha-1}$ or a finite subset of it. In this case, we will call X delta discrete random variable. In this work, we will define the delta discrete fractional Weibull distribution. Another example is the delta discrete uniform distribution, $\mathrm{DU}\{\alpha-1,\alpha,\ldots,\alpha+\beta\}$, where $\alpha\in\mathbb{R}$ and $\beta\in\mathbb{N}_{-1}$.

Theorem 8. *We suppose that X is a discrete random variable. The expectation of the ntm function, $h_{\overline{\alpha-1}}(X)$, coincides with nabla Riemann right fractional sum of the pmf at 1 for $\alpha>0$ and nabla Riemann right fractional difference of the pmf at 1 for $\alpha<0$, $\alpha\notin\{-\mathbb{N}\}$; that is,*

$$E\left[h_{\overline{\alpha-1}}(X)\right]=\begin{cases}\left(\nabla^{-\alpha}f\right)(1), & \alpha>0\\ \left(\nabla^{\alpha}f\right)(1), & \alpha<0,\ \alpha\notin\{\ldots,-2,-1\},\end{cases}\tag{25}$$

where

$$\left(\nabla^{-\alpha}f\right)(t)=\sum_{x=1}^{b-t}\frac{x^{\overline{\alpha-1}}}{\Gamma(\alpha)}f(x+t-1)\tag{26}$$

is the nabla Riemann right fractional sum, while

$$\left(\nabla^{\alpha}f\right)(t)=\sum_{x=1}^{b-t}\frac{x^{\overline{-\alpha-1}}}{\Gamma(-\alpha)}f(x+t-1)\tag{27}$$

is the nabla Riemann right fractional difference.

Proof. For $\alpha>0$, substitute $x=\sigma(s)-t$ in expression (14) and also for $\alpha<0$ and $\alpha\notin\{-1,-2,\ldots\}$, in expression (18). □

Then, considering the limits of summation in recent theorem we can define the discrete distributions with support $\mathbb{N}_1$ or a finite subset of it. In this case, we will call X nabla discrete random variable. In this work, we will define the nabla discrete fractional Weibull distribution. Another example is the nabla discrete uniform distribution, $\mathrm{DU}\{1,2,\ldots,\alpha-\beta+1\}$, where $\alpha\in\mathbb{R}$ and $\beta\in{}_{\alpha}\mathbb{N}$.

4. The Delta and Nabla Discrete Fractional Weibull Distributions

In this section, we will introduce delta and nabla discrete fractional Weibull distributions, by substituting continuous Taylor monomial and exponential functions with their corresponding discrete types (on the discrete time scale) in continuous Weibull distribution.

4.1. The Nabla Discrete Fractional Weibull Distribution

Definition 9. It is said that the random variable X has a nabla discrete fractional Weibull distribution with (α,λ) parameters, denoted by $W^{\nabla}(\alpha,\lambda)$, if its pmf is given by

$$\begin{aligned}\Pr[X=x]&=\frac{\lambda\Gamma(\alpha+1)h_{\overline{\alpha-1}}(x)}{e_{\lambda}^{*}\left(\rho\left(\Gamma(\alpha+1)h_{\overline{\alpha}}(x)\right),0\right)}\\&=\alpha\lambda x^{\overline{\alpha-1}}(1-\lambda)^{x^{\overline{\alpha}}-1},\quad x=\mathbb{N}_1,\end{aligned}\tag{28}$$

where $\alpha>0$, $0<\lambda<1$.

Now we show that

$$\sum_{x=1}^{\infty}\alpha\lambda x^{\overline{\alpha-1}}(1-\lambda)^{x^{\overline{\alpha}}-1}=1.\tag{29}$$

For this purpose, by using Theorems 4.3 and 3.8 (integration by substitution) from [21] and considering $(x^{\overline{\alpha}})^{\nabla}=\alpha x^{\overline{\alpha-1}}$ and under the substitution $u=x^{\overline{\alpha}}$, we have

$$\begin{aligned}\sum_{x=1}^{\infty}\alpha\lambda x^{\overline{\alpha-1}}(1-\lambda)^{x^{\overline{\alpha}}-1}&=\int_0^{\infty}\alpha\lambda x^{\overline{\alpha-1}}(1-\lambda)^{x^{\overline{\alpha}}-1}\nabla x\\&=\int_0^{\infty}\lambda(1-\lambda)^{u-1}\nabla u=\sum_{u=1}^{\infty}\lambda(1-\lambda)^{u-1}=1.\end{aligned}\tag{30}$$

(i) Particular Cases. (a) For $\alpha=1$, $W^{\nabla}(\alpha,\lambda)$ in (28) reduces to a one-parameter nabla discrete fractional Weibull with pmf

$$\Pr[X=x]=\lambda(1-\lambda)^{\rho(x)},\quad x=1,2,\ldots.\tag{31}$$

Obviously, this is the pmf of geometric distribution (the number of independent trials required for first success) or nabla discrete exponential distribution.

(b) For $\alpha=n$, $n\in\mathbb{N}$, $W^{\nabla}(\alpha,\lambda)$ in (28) is a new delta discrete distribution with pmf

$$\Pr[X=x]=n!\binom{x+n-2}{x-1}\lambda(1-\lambda)^{n!\binom{x+n-1}{x-1}-1},\quad x=1,2,\ldots.\tag{32}$$

If we substitute $\rho(x)=x$, (31) and (32) are given by

$$\Pr[X=x]=\lambda(1-\lambda)^{x},\quad x=0,1,2,\ldots,\tag{33}$$

$$\Pr[X=x]=n!\binom{x+n-1}{x}\lambda(1-\lambda)^{n!\binom{x+n}{x}-1},\quad x=0,1,2,\ldots,\tag{34}$$

respectively. It can be seen that (33) is the same geometric distribution (the number of failures for first success). Then (34), as a general case of (33), is a type of negative binomial distribution.

(c) For $\alpha=2$, $W^{\nabla}(\alpha,\lambda)$ in (28) is a nabla discrete distribution with pmf

$$\Pr[X=x]=2x\lambda(1-\lambda)^{x(x+1)-1},\quad x=1,2,\ldots,\tag{35}$$

where we will call it nabla discrete Rayleigh distribution.

(ii) Statistical Properties. If $X\sim W^{\nabla}(\alpha,\lambda)$, then the survival function, the hazard function, and the mean of random variable X are given by

$$s(x)=(1-\lambda)^{(\rho(x))^{\overline{\alpha}}},\tag{36}$$

$$h(x)=\alpha\lambda x^{\overline{\alpha-1}}(1-\lambda)^{2(\rho(x))^{\overline{\alpha}}(\rho(x))^{\overline{\alpha/2}-1}},\tag{37}$$

$$E[X] = \sum_{x=1}^{\infty} (1-\lambda)^{(\rho(x))^{\bar{\alpha}}}, \tag{38}$$

respectively.

To continue, we use the beta type I, beta type II, and Kummer-beta distributions. These distributions can be found in [22–25].

(iii) Bayesian Estimation in Nabla Discrete Fractional Weibull Distribution. In this section, we study the Bayesian estimation of functions of parameter λ of nabla discrete fractional Weibull distribution. The likelihood function of λ, in this case, is given by

$$L(\lambda) \propto \lambda^{n} (1-\lambda)^{\sum x_i^{\bar{\alpha}} - n}. \tag{39}$$

We take a prior distribution given below:

$$\pi(\lambda) = \frac{\lambda^{p-1}(1-\lambda)^{q-1} e^{\mu(1-\lambda)}}{\mathrm{B}(p,q)\ {}_1F_1(q; p+q; \mu)}, \quad 0<\lambda<1,\ p,q>0,\ \mu\in\mathbb{R}, \tag{40}$$

where $\mathrm{B}(p,q) = \Gamma(p)\Gamma(q)/\Gamma(p+q)$ and

$${}_1F_1(a;b;z) = \sum_{k=0}^{\infty} \left(a^{\bar{k}} z^k / b^{\bar{k}} k!\right). \tag{41}$$

This prior density is known as the Kummer-beta density and denoted by $\mathrm{KB}(p,q,\mu)$. The posterior probability density function of λ, corresponding to $\pi(\lambda)$, is given by

$$\begin{aligned} \pi(\lambda \mid x) &\propto L(\lambda)\pi(\lambda) \\ &\propto \lambda^{n+p-1}(1-\lambda)^{\sum x_i^{\bar{\alpha}} - n + q - 1} e^{\mu(1-\lambda)} \Longrightarrow \\ \lambda \mid x &\sim \mathrm{KB}\left(n+p, \sum x_i^{\bar{\alpha}} - n + q, \mu\right). \end{aligned} \tag{42}$$

$\pi(\lambda)$ is a natural conjugate prior density. Note that, for $\mu = 0$, the above prior density simplifies to a beta type I density with parameters p and q. Under the squared error loss function (SELF) given by $\mathfrak{L}(\gamma(\lambda),\delta) = (\gamma(\lambda)-\delta)^2$, where $\gamma(\lambda)$ is a function of λ and δ is a decision, the Bayesian estimate $\widehat{\gamma}_{\mathrm{B}}$ of $\gamma(\lambda) = \lambda^r$, corresponding to posterior density $\pi(\lambda \mid x)$, is given by

$$\begin{aligned} \widehat{\lambda}_{\mathrm{B}}^{r} &= \frac{\int_0^1 \lambda^{n+r+p-1}(1-\lambda)^{\sum x_i^{\bar{\alpha}} - n + q - 1} e^{\mu(1-\lambda)} d\lambda}{\mathrm{B}\left(n+p, \sum x_i^{\bar{\alpha}} - n + q\right)\ {}_1F_1\left(\sum x_i^{\bar{\alpha}} - n + q; \sum x_i^{\bar{\alpha}} + p + q; \mu\right)} \\ &= \frac{\mathrm{B}\left(n+p+r, \sum x_i^{\bar{\alpha}} - n + q\right)\ {}_1F_1\left(\sum x_i^{\bar{\alpha}} - n + q; \sum x_i^{\bar{\alpha}} + p + q + r; \mu\right)}{\mathrm{B}\left(n+p, \sum x_i^{\bar{\alpha}} - n + q\right)\ {}_1F_1\left(\sum x_i^{\bar{\alpha}} - n + q; \sum x_i^{\bar{\alpha}} + p + q; \mu\right)}. \end{aligned} \tag{43}$$

Similarly, under the weighted squared error loss function (WSELF) given by $\mathfrak{L}(\gamma(\lambda),\delta) = w(\lambda)(\gamma(\lambda)-\delta)^2$, where $w(\lambda)$ is a function of λ, the Bayesian estimate $\widehat{\gamma}_w$ of $\gamma(\lambda)$, for two different forms of $w(\lambda)$, is given below.

(a) Let $w(\lambda) = \lambda^{-2}$. The Bayesian estimate $\widehat{\gamma}_m$ of $\gamma(\lambda)$, known as the minimum expected loss (MEL) estimate, for $\gamma(\lambda) = \lambda^r$, corresponding to posterior density $\pi(\lambda \mid x)$, is given by

$$\begin{aligned} \widehat{\lambda}_{m}^{r} &= \frac{\int_0^1 \lambda^{(n+r+p-2)-1}(1-\lambda)^{\sum x_i^{\bar{\alpha}} - n + q - 1} e^{\mu(1-\lambda)} d\lambda}{\int_0^1 \lambda^{(n+p-2)-1}(1-\lambda)^{\sum x_i^{\bar{\alpha}} - n + q - 1} e^{\mu(1-\lambda)} d\lambda} \\ &= \frac{\mathrm{B}\left(n+r+p-2, \sum x_i^{\bar{\alpha}} - n + q\right)\ {}_1F_1\left(\sum x_i^{\bar{\alpha}} - n + q; \sum x_i^{\bar{\alpha}} + p + q + r - 2; \mu\right)}{\mathrm{B}\left(n+p-2, \sum x_i^{\bar{\alpha}} - n + q\right)\ {}_1F_1\left(\sum x_i^{\bar{\alpha}} - n + q; \sum x_i^{\bar{\alpha}} + p + q - 2; \mu\right)}. \end{aligned} \tag{44}$$

This loss function was used by Tummala and Sathe [26] for estimating the reliability of certain life time distributions and by Zellner and Park [27] for estimating functions of parameters of some econometric models.

(b) Let $w(\lambda) = \lambda^{-2} e^{\nu(1-\lambda)}$, $\nu < 0$. The Bayesian estimate $\widehat{\gamma}_e$ of $\gamma(\lambda)$, known as the exponentially weighted minimum expected loss (EWMEL) estimate, for $\gamma(\lambda) = \lambda^r$, corresponding to posterior density $\pi(\lambda \mid x)$, is given by

$$\begin{aligned} \widehat{\lambda}_{e}^{r} &= \frac{\int_0^1 \lambda^{(n+r+p-2)-1}(1-\lambda)^{\sum x_i^{\bar{\alpha}} - n + q - 1} e^{(\mu+\nu)(1-\lambda)} d\lambda}{\int_0^1 \lambda^{(n+p-2)-1}(1-\lambda)^{\sum x_i^{\bar{\alpha}} - n + q - 1} e^{(\mu+\nu)(1-\lambda)} d\lambda} \\ &= \frac{\mathrm{B}\left(n+r+p-2, \sum x_i^{\bar{\alpha}} - n + q\right)\ {}_1F_1\left(\sum x_i^{\bar{\alpha}} - n + q; \sum x_i^{\bar{\alpha}} + p + q + r - 2; \mu+\nu\right)}{\mathrm{B}\left(n+p-2, \sum x_i^{\bar{\alpha}} - n + q\right)\ {}_1F_1\left(\sum x_i^{\bar{\alpha}} - n + q; \sum x_i^{\bar{\alpha}} + p + q - 2; \mu+\nu\right)}. \end{aligned} \tag{45}$$

4.2. The Delta Discrete Fractional Weibull Distribution

Definition 10. It is said that the random variable X has a delta discrete fractional Weibull distribution with (α, λ) parameters, denoted by $W^{\Delta}(\alpha, \lambda)$, if its pmf is given by

$$\begin{aligned}\Pr[X = x] &= \frac{\lambda\Gamma(\alpha+1)\, h_{\underline{\alpha-1}}(x)}{e_{\lambda}\left(\sigma\left(\Gamma(\alpha+1)\, h_{\underline{\alpha}}(x)\right), 0\right)} \\ &= \frac{\alpha\lambda x^{\underline{\alpha-1}}}{(1+\lambda)^{x^{\underline{\alpha}}+1}}, \quad x = \mathbb{N}_{\alpha-1},\end{aligned} \tag{46}$$

where $\alpha > 0,\ \lambda > 0$.

Now we show that

$$\sum_{x=\alpha-1}^{\infty}\left(\frac{\alpha\lambda x^{\underline{\alpha-1}}}{(1+\lambda)^{x^{\underline{\alpha}}+1}}\right) = 1. \tag{47}$$

For this purpose, we apply Theorem 5.40 (change of variable) from [12]. Considering $(x^{\underline{\alpha}})^{\Delta} = \alpha x^{\underline{\alpha-1}}$ and then under the substitution $u = x^{\underline{\alpha}}$, we have

$$\begin{aligned}\sum_{x=\alpha-1}^{\infty}\frac{\alpha\lambda x^{\underline{\alpha-1}}}{(1+\lambda)^{x^{\underline{\alpha}}+1}} &= \int_{\alpha-1}^{\infty}\frac{\alpha\lambda x^{\underline{\alpha-1}}}{(1+\lambda)^{x^{\underline{\alpha}}+1}}\Delta x \\ &= \int_{0}^{\infty}\frac{\lambda\Delta u}{(1+\lambda)^{u+1}} = \sum_{u=0}^{\infty}\frac{\lambda}{(1+\lambda)^{1+u}} \\ &= 1.\end{aligned} \tag{48}$$

(i) Particular Cases. (a) For $\alpha = 1$, $W^{\Delta}(\alpha, \lambda)$ in (46) reduces to a one-parameter delta discrete fractional Weibull with pmf

$$\Pr[X = x] = \lambda(1+\lambda)^{-\sigma(x)} = \left(\frac{\lambda}{1+\lambda}\right)\left(\frac{1}{1+\lambda}\right)^{x}, \quad x = 0, 1, \ldots. \tag{49}$$

Obviously, this is the pmf of geometric distribution (the number of failures for first success) or delta discrete exponential distribution.

(b) For $\alpha = n$, $n \in \mathbb{N}$, $W^{\Delta}(\alpha, \lambda)$ in (46) is a new delta discrete distribution with pmf

$$\begin{aligned}&\Pr[X = x] \\ &\quad = n!\binom{x}{x-n+1}\left(\frac{\lambda}{1+\lambda}\right)\left(\frac{1}{1+\lambda}\right)^{n!\binom{x}{x-n}}, \\ &\qquad x = \mathbb{N}_{n-1}.\end{aligned} \tag{50}$$

If we substitute $\sigma(x) = x$, (49) and (50) are given by

$$\Pr[X = x] = \left(\frac{\lambda}{1+\lambda}\right)\left(\frac{1}{1+\lambda}\right)^{x-1}, \quad x = 1, 2, \ldots, \tag{51}$$

$$\Pr[X = x] = n!\binom{x-1}{x-n}\left(\frac{\lambda}{1+\lambda}\right)\left(\frac{1}{1+\lambda}\right)^{n!\binom{x-1}{x-n-1}}, \quad x = n, n+1, \ldots, \tag{52}$$

respectively. It can be seen that (51) is the same geometric distribution (the number of independent trials required for first success). Then (52), as a general case of (51), is a type of negative binomial distribution.

(c) For $\alpha = 2$, $W^{\Delta}(\alpha, \lambda)$ in (46) is a delta discrete distribution with pmf

$$\Pr[X = x] = 2x\left(\frac{\lambda}{1+\lambda}\right)\left(\frac{1}{1+\lambda}\right)^{x(x-1)}, \quad x = 1, 2, \ldots, \tag{53}$$

where we will call it delta discrete Rayleigh distribution.

(ii) Statistical Properties. If $X \sim W^{\Delta}(\alpha, \lambda)$, then the survival function, the hazard function, and the mean of random variable X are given by

$$s(x) = \frac{1}{(1+\lambda)^{x^{\underline{\alpha}}}}, \tag{54}$$

$$h(x) = \alpha\left(\frac{\lambda}{1+\lambda}\right)x^{\underline{\alpha-1}}, \tag{55}$$

$$E[X] = \sum_{x=\alpha-1}^{\infty}\frac{1}{(1+\lambda)^{x^{\underline{\alpha}}}}, \quad \alpha \geq 1, \tag{56}$$

respectively.

(iii) Bayesian Estimation in Delta Discrete Fractional Weibull Distribution. In this section, we study the Bayesian estimation of functions of parameter λ of delta discrete fractional Weibull distribution. The likelihood function of λ, in this case, is given by

$$L(\lambda) \propto \frac{\lambda^{n}}{(1+\lambda)^{\sum x_i^{\underline{\alpha}}+n}}. \tag{57}$$

We take a prior distribution given below:

$$\pi(\lambda) = \frac{\lambda^{p-1}}{\mathrm{B}(p,q)(1+\lambda)^{p+q}}, \quad \lambda > 0,\ p, q > 0. \tag{58}$$

This prior density is known as the beta type II density and denoted by $\mathrm{B}^{\mathrm{II}}(p, q)$.

The posterior probability density function of λ, corresponding to $\pi(\lambda)$, is given by

$$\begin{aligned}&\pi(\lambda \mid x) \propto L(\lambda)\pi(\lambda) \propto \frac{\lambda^{n+p-1}}{(1+\lambda)^{\sum x_i^{\underline{\alpha}}+n+p+q}} \Longrightarrow \\ &\quad \lambda \mid x \sim \mathrm{B}^{\mathrm{II}}\left(n+p, \sum x_i^{\underline{\alpha}}+q\right).\end{aligned} \tag{59}$$

$\pi(\lambda)$ is a natural conjugate prior density. Note that $\pi(\lambda)$ is a special case of inverted hypergeometric function type I density, which is given by Gupta and Nagar [28] and Nagar

and Alvarez [29]. Under SELF, the Bayesian estimate of $\gamma(\lambda) = \lambda^r$, corresponding to posterior density $\pi(\lambda \mid x)$, is given by

$$\begin{aligned}\widehat{\lambda}_{\mathrm{B}}^{r} &= \int_0^\infty \frac{\lambda^{n+r+p-1}}{\mathrm{B}\left(n+p, \sum x_i^{\alpha}+q\right)(1+\lambda)^{\sum x_i^{\alpha}+n+p+q}} d\lambda \\ &= \frac{\mathrm{B}\left(n+p+r, \sum x_i^{\alpha}+q-r\right)}{\mathrm{B}\left(n+p, \sum x_i^{\alpha}+q\right)}.\end{aligned} \tag{60}$$

Similarly, under WSELF, when $w(\lambda) = \lambda^{-2}$, the MEL estimate of $\gamma(\lambda) = \lambda^r$, corresponding to posterior density $\pi(\lambda \mid x)$, is given by

$$\begin{aligned}\widehat{\lambda}_{m}^{r} &= \frac{\int_0^\infty \lambda^{(n+r+p-2)-1}(1+\lambda)^{-(\sum x_i^{\alpha}+n+p+q)} d\lambda}{\int_0^\infty \lambda^{(n+p-2)-1}(1+\lambda)^{-(\sum x_i^{\alpha}+n+p+q)} d\lambda} \\ &= \frac{\mathrm{B}\left(n+p+r-2, \sum x_i^{\alpha}+q-r+2\right)}{\mathrm{B}\left(n+p-2, \sum x_i^{\alpha}+q+2\right)}.\end{aligned} \tag{61}$$

5. Unification of the Continuous and Discrete Weibull Distributions

For a given time scale $\mathbb{T}$, we present the construction of pdf of Weibull distribution, such that the density function on time scales is

$$f_X(x) = \frac{\lambda\Gamma(\alpha+1)\widehat{h}_{\alpha-1}(x)}{\widehat{e}_\lambda\left(\eta\left(\Gamma(\alpha+1)\widehat{h}_{\alpha-1}(x)\right), 0\right)}, \quad x \in \mathbb{T}. \tag{62}$$

In order that the reader sees how continuous Weibull distribution and delta and nabla discrete fractional Weibull distributions follow from (62), it is only at this point necessary to know that

$$\begin{aligned}\widehat{h}_{\alpha-1}(x) &= h_{\alpha-1}(x) = \frac{x^{\alpha-1}}{\Gamma(\alpha)}, \\ \eta(x) &= x, \\ \widehat{e}_\lambda(\eta(X), 0) &= e^{\lambda x}\end{aligned} \tag{63}$$

if $\mathbb{T} = \mathbb{R}^+$,

$$\begin{aligned}\widehat{h}_{\alpha-1}(x) &= h_{\underline{\alpha-1}}(x) = \frac{x^{\underline{\alpha-1}}}{\Gamma(\alpha)}, \\ \eta(x) &= \sigma(x), \\ \widehat{e}_\lambda(\eta(X), 0) &= (1+\lambda)^{\sigma(x)}\end{aligned} \tag{64}$$

if $\mathbb{T} = \mathbb{N}_{\alpha-1}$, $\alpha > 0$, and

$$\begin{aligned}\widehat{h}_{\alpha-1}(x) &= h_{\overline{\alpha-1}}(x) = \frac{x^{\overline{\alpha-1}}}{\Gamma(\alpha)}, \\ \eta(x) &= \rho(x), \\ \widehat{e}_\lambda(\eta(X), 0) &= (1-\lambda)^{-\rho(x)},\end{aligned} \tag{65}$$

if $\mathbb{T} = \mathbb{N}_1$.

Competing Interests

The authors declare that they have no competing interests.

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