

## Research Article

# The Half-Logistic Lomax Distribution for Lifetime Modeling

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We introduce a new two-parameter lifetime distribution called the half-logistic Lomax (HLL) distribution. The proposed distribution is obtained by compounding half-logistic and Lomax distributions. We derive some mathematical properties of the proposed distribution such as the survival and hazard rate function, quantile function, mode, median, moments and moment generating functions, mean deviations from mean and median, mean residual life function, order statistics, and entropies. The estimation of parameters is performed by maximum likelihood and the formulas for the elements of the Fisher information matrix are provided. A simulation study is run to assess the performance of maximum-likelihood estimators (MLEs). The flexibility and potentiality of the proposed model are illustrated by means of real and simulated data sets.

## 1. Introduction

The commonly used lifetime distributions (exponential, gamma, Weibull, Lomax, lognormal, etc.) have a limited range of behavior and do not provide adequate fit to complex data sets in different sciences. Generalizations of these distributions offer more flexibility and provide reasonable parametric fits to complex data sets. Motivated by the various applications of Lomax and half-logistic distributions in areas of income and wealth inequality, firm size, size of cities, queuing problems, actuarial science, medical and biological sciences, and engineering, we propose a two-parameter continuous lifetime distribution by compounding the half-logistic and the Lomax distribution called *half-logistic Lomax (HLL) distribution*.

The Lomax [1] (or Pareto Type-II) distribution was introduced to model business failure data. For more detail about the Lomax distribution, we refer the readers to Rady et al. [2], Tahir et al. [3], and the references therein. In literature, there are several generalizations of the Lomax distribution. Abdul-Moniem [4] developed the exponentiated Lomax distribution, and Al-Awadhi and Ghitany [5] introduced the discrete Poisson–Lomax distribution by using the Lomax distribution as a mixing distribution for the Poisson parameter.

Asgharzadeh et al. [6] proposed the Pareto Poisson–Lindely distribution, and Cordeiro et al. [7] investigated the gamma-Lomax distribution and studied its properties. Ghitany et al. [8] and Gupta et al. [9] considered the Marshal–Olkin approach and extended the Lomax distribution, and Lemonte and Cordeiro [10] proposed and studied the McDonald-Lomax, the beta Lomax, and the Kumaraswamy Lomax distributions. Other models constitute flexible family of distributions in terms of the variates of shapes and hazard functions; see, for example, Al-Zahrani and Sagor [11], El-Bassiouny et al. [12], Rady et al. [2], Kilany [13], and Tahir et al. [3]. These generalizations of the Lomax distribution are considered to be useful life distribution models.

The cumulative distribution function (cdf) of Lomax distribution is given by

$$H(x; \alpha, \beta) = 1 - (1 + \beta x)^{-\alpha}, \quad x > 0, \quad (1)$$

where  $\alpha > 0$  is a shape parameter and  $\beta > 0$  is a scale parameter. The probability density function (pdf) corresponding to (1) is

$$h(x; \alpha, \beta) = \alpha\beta(1 + \beta x)^{-(\alpha+1)}, \quad x > 0. \quad (2)$$

Cordeiro et al. [14] define the cdf of the new *type I half-logistic-G* (TIHL-G) family of distributions by

$$F(x; \lambda, \theta) = \int_0^{-\ln(1-G(x;\theta))} \frac{2\lambda e^{-\lambda t}}{(1 + e^{-\lambda t})^2} dt \tag{3}$$

$$= \frac{1 - [1 - G(x; \theta)]^\lambda}{1 + [1 - G(x; \theta)]^\lambda},$$

where  $G(x; \theta)$  is the baseline cdf depending on a parameter vector  $\theta$  and an additional shape parameter  $\lambda > 0$ . As a special case, if  $\lambda = 1$ , then the TIHL-G is the half-logistic-G (HL-G) distribution with cdf

$$F(x; \theta) = \frac{G(x; \theta)}{1 + \overline{G}(x; \theta)}, \tag{4}$$

where  $\overline{G}(x; \theta) = 1 - G(x; \theta)$ .

The corresponding pdf to (4) is given by

$$f(x; \theta) = \frac{2g(x; \theta)}{[1 + \overline{G}(x; \theta)]^2}, \quad \text{where } g(x) = \frac{dG(x)}{dx}. \tag{5}$$

This paper aims to provide a new lifetime model with a minimum number of parameters by compounding the half-logistic and the Lomax distribution called half-logistic Lomax (HLL) distribution. The proposed distribution is heavy-tailed and has a decreasing or upside-down bathtub (or unimodal) shaped hazard rates depending on its parameters. Upside-down bathtub shaped hazard rates are common in reliability, engineering, and survival analysis. The HLL distribution can also be applied in engineering as the Lomax [1] distribution and can be a useful alternative to other well-known densities in lifetime applications. It is interesting to note that the HLL distribution is a special case of Marshall-Olkin-Lomax distribution introduced by Ghitany et al. [8]. We obtain some mathematical properties of the proposed distribution and parameters of the model are estimated by the maximum-likelihood estimation method.

The rest of this paper is organized as follows. In Section 2, we introduce the half-logistic Lomax distribution and provide plots of its density function. In Section 3, we investigate various mathematical properties of the HLL distribution including survival and hazard rate function, quantile function, moments, mean residual life function, mean deviation from the mean and the mean deviation from the median, entropies, and order statistics. In Section 4, estimation of parameters is given by MLE method and the asymptotic distribution of the estimators is studied via Fisher's information matrix. Simulation results on the behavior of the MLEs are presented in Section 5. A real data application is conducted in Section 6. Finally, in Section 7, we conclude that the HLL distribution is

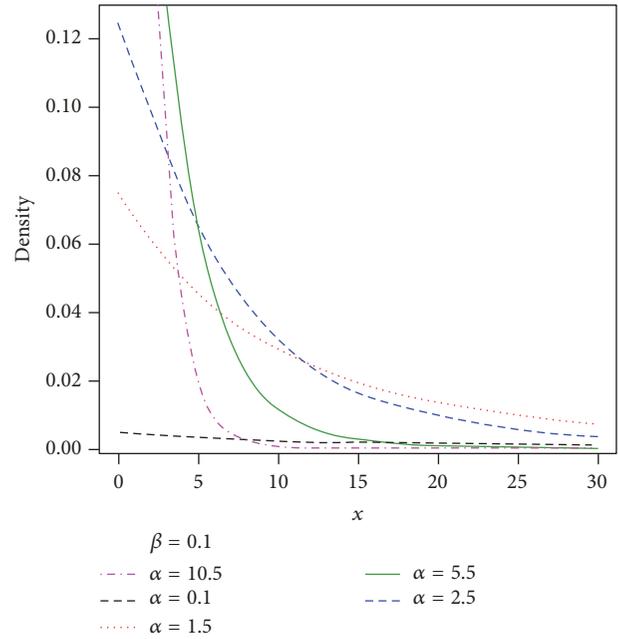


FIGURE 1: Plots of the HLL pdf for some parameter values.

the best model as compared to other competing models.

### 2. The HLL Distribution

We define the *half-logistic-Lomax* (HLL) density function by inserting (1) and (2) into (5). So, we obtain

$$f(x; \alpha, \beta) = \frac{2\alpha\beta(1 + \beta x)^{-(\alpha+1)}}{[1 + (1 + \beta x)^{-\alpha}]^2}, \quad x > 0; \alpha, \beta > 0. \tag{6}$$

The corresponding cumulative density function (cdf) follows from (1) and (4) and is given by

$$F(x) = \frac{[1 - (1 + \beta x)^{-\alpha}]}{[1 + (1 + \beta x)^{-\alpha}]}. \tag{7}$$

Hereafter, we will denote a random variable  $X$  having pdf (6) by  $X \sim \text{HLL}(\alpha, \beta)$ . The limit of the HLL density (6) as  $x \rightarrow \infty$  is 0 and the limit as  $x \rightarrow 0$  is  $\alpha\beta/2$ . Figure 1 depicts some of the possible shapes of density (6) for selected parameter values. The mode of density (6) is obtained from solving  $[d \log[f(x)]/dx]_{x=x_0} = 0$ , which is given by

$$\text{Mode} = x_0 = \frac{1}{\beta} \left[ \left( \frac{\alpha - 1}{\alpha + 1} \right)^{1/\alpha} - 1 \right]. \tag{8}$$

### 3. Mathematical Properties

This section describes mathematical properties of the HLL distribution.

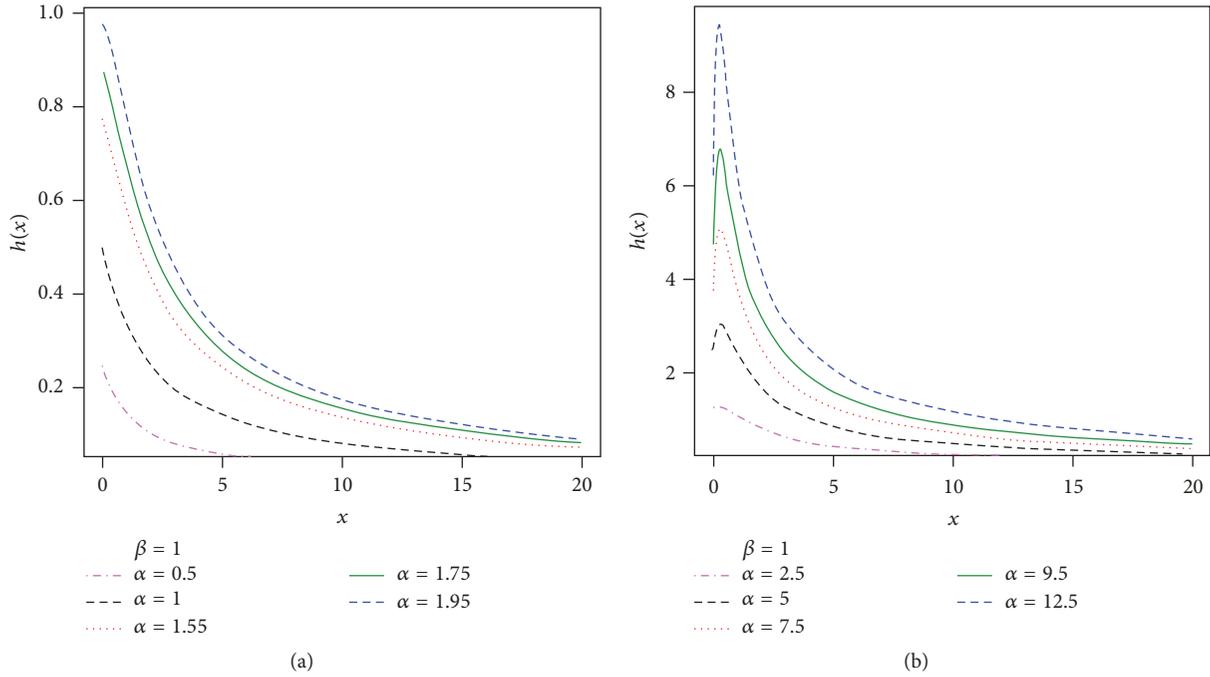


FIGURE 2: Plots of the HLL hrf for some parameter values.

3.1. *The Survivor and Hazard Rate Functions.* Using (6) and (7), the survival function (sf) and hazard rate function (hrf) of  $X$  are, respectively, given by

$$S(x) = \frac{2}{[1 + (1 + \beta x)^\alpha]}, \tag{9}$$

$$h(x) = \frac{\alpha\beta}{(1 + \beta x)[1 + (1 + \beta x)^{-\alpha}]}.$$

The limit of  $h(x)$  as  $x \rightarrow 0^+$  is  $\alpha\beta/2$ ; that is,  $h(0)$  is bounded from below and continuous in its parameters. The

limit of  $h(x)$  as  $x \rightarrow \infty$  is 0. According to Glaser [15], we determine the parameter intervals for which the hazard rate function of the HLL distribution is decreasing or upside-down bathtub. Let

$$\eta(x) = -\frac{d}{dx} \ln f(x)$$

$$= \frac{\beta [(\alpha + 1) - (\alpha - 1)(1 + \beta x)^{-\alpha}]}{(1 + \beta x)[1 + (1 + \beta x)^{-\alpha}]}.$$
(10)

Then its first derivative is given by

$$\eta'(x) = \frac{\beta^2 [2(\alpha^2 - 1)(1 + \beta x)^{\alpha-2} + (\alpha - 1)(1 + \beta x)^{-2} - (\alpha + 1)(1 + \beta x)^{2(\alpha-1)}]}{[1 + (1 + \beta x)^\alpha]^2}.$$
(11)

If  $0 < \alpha \leq 2$ , then  $\eta'(x) < 0$  for all  $x$ . Then the hazard rate function is decreasing. By Glaser's theorem [15], it is sufficient to show that there exists  $x_0 > 0$  such that  $\eta'(x) > 0$  for all  $x \in (0, x_0)$ .  $\eta'(x_0) = 0$  implies that  $x_0 = (1/\beta)[\{(\alpha - 1)/(\alpha + 1)\}^{1/\alpha} - 1]$  and  $\eta'(x) < 0$  for all  $x > x_0$ , which implies unimodal shape of the hazard rate function. For  $\alpha > 2$ ,  $\eta'(x) > 0$  for all  $x < x_0$  and  $\eta'(x) < 0$  for all  $x > x_0$ . Thus, the hazard rate function is upside-down bathtub-shaped for  $\alpha > 2$ . Figure 2 illustrates some of the possible shapes of hazard rate function for selected values of the parameters.

3.2. *Quantile and Random Number Generation.* The cumulative distribution function is given by (7). Inverting  $F(x) = u$ , we obtain

$$F^{-1}(x) = \frac{1}{\beta} \left[ \left( \frac{1-u}{1+u} \right)^{1/\alpha} - 1 \right], \quad 0 < u < 1. \tag{12}$$

Equation (12) can be used to simulate the HLL variable.

3.3. *Moments.* The following theorem gives moments of the HLL distribution and its mean moments and cumulants.

**Theorem 1.** The  $r$ th moment about the origin of  $X \sim \text{HLL}(\alpha, \beta)$  is given by

$$\mu'_r = E[X^r] = \frac{2\alpha}{\beta^r} \sum_{k=1}^{\infty} \sum_{j=0}^r \binom{r}{j} \frac{k(-1)^{r+j+k-1}}{(k\alpha - j)}, \quad (13)$$

$$r = 1, 2, \dots$$

*Proof.* We have

$$E[X^r] = 2\alpha\beta \int_0^{\infty} x^r \frac{(1 + \beta x)^{-(\alpha+1)}}{[1 + (1 + \beta x)^{-\alpha}]^2} dx. \quad (14)$$

Setting  $y = (1 + \beta x)$ ,  $dx = dy/\beta$  yields

$$\begin{aligned} E[X^r] &= \frac{2\alpha}{\beta^r} \int_1^{\infty} (y-1)^r y^{-(\alpha+1)} (1+y^{-\alpha})^{-2} dy \\ &= \frac{2\alpha}{\beta^r} \int_1^{\infty} \left\{ \sum_{j=0}^r \binom{r}{j} y^{j-\alpha-1} (-1)^{r+j} \right. \\ &\quad \times \left. \sum_{k=1}^{\infty} (-1)^{k-1} k (y^{-\alpha})^{k-1} \right\} dy = \frac{2\alpha}{\beta^r} \\ &\quad \cdot \sum_{k=1}^{\infty} \sum_{j=0}^r \binom{r}{j} (-1)^{r+j+k-1} k \int_1^{\infty} y^{j-k\alpha-1} dy = \frac{2\alpha}{\beta^r} \\ &\quad \cdot \sum_{k=1}^{\infty} \sum_{j=0}^r \binom{r}{j} \frac{k(-1)^{r+j+k-1}}{(k\alpha - j)}. \end{aligned} \quad (15)$$

□

**Corollary 2.** The first four moments of  $X \sim \text{HLL}(\alpha, \beta)$  are given, respectively, by

$$\begin{aligned} \mu'_1 &= E[X] = \frac{2}{\beta} \sum_{k=1}^{\infty} (-1)^{k+1} \times \frac{1}{(k\alpha - 1)}, \\ \mu'_2 &= E[X^2] = \frac{4}{\beta^2} \sum_{k=1}^{\infty} (-1)^{k+1} \times \frac{1}{(k\alpha - 1)(k\alpha - 2)}, \\ \mu'_3 &= E[X^3] = \frac{12}{\beta^3} \sum_{k=1}^{\infty} (-1)^{k+1} \\ &\quad \times \frac{1}{(k\alpha - 1)(k\alpha - 2)(k\alpha - 3)}, \\ \mu'_4 &= E[X^4] = \frac{48}{\beta^4} \sum_{k=1}^{\infty} (-1)^{k+1} \\ &\quad \times \frac{1}{(k\alpha - 1)(k\alpha - 2)(k\alpha - 3)(k\alpha - 4)}. \end{aligned} \quad (16)$$

*Proof.* Putting  $r = 1, 2, 3, 4$  in (13) yields the desired result. □

The mean moments ( $\mu_r$ ) and cumulants ( $\kappa_r$ ) of  $X$  can be obtained by the following relation;

$$\begin{aligned} \mu_r &= \sum_{j=0}^r (-1)^j \binom{r}{j} \mu'_{r-j} (\mu'_1)^j, \\ \kappa_r &= \mu'_r - \sum_{j=1}^{r-1} \binom{r-1}{j-1} \kappa_j \mu'_{r-j}. \end{aligned} \quad (17)$$

Here  $\kappa_1 = \mu'_1$ . We have  $\kappa_2 = \mu'_2 - \mu_1'^2$ ,  $\kappa_3 = \mu'_3 - 3\mu'_2\mu'_1 + 2\mu_1'^3$ ,  $\kappa_4 = \mu'_4 - 4\mu'_3\mu'_1 - 3\mu_2'^2 + 12\mu'_2\mu_1'^2 - 6\mu_1'^4$ , and so on.

The skewness  $\gamma_1 = \kappa_3/\kappa_2^{3/2}$  and kurtosis  $\gamma_2 = \kappa_4/\kappa_2^2$  follow from the second, third, and fourth cumulants.

The variance of  $X$ , denoted by  $\sigma^2 = \mu'_2 - \mu_1'^2$ , is given by

$$\begin{aligned} \sigma^2 &= \frac{4}{\beta^2} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(k\alpha - 1)(k\alpha - 2)} \\ &\quad - \left[ \frac{2}{\beta} \sum_{k=1}^{\infty} (-1)^{k+1} \frac{1}{(k\alpha - 1)} \right]^2. \end{aligned} \quad (18)$$

The  $r$ th descending factorial moment of  $X$  is

$$\begin{aligned} \mu'_{(r)} &= E[X^{(r)}] = E[X(X-1) \times \dots \times (X-r+1)] \\ &= \sum_{j=0}^r s(r, j) \mu'_j, \end{aligned} \quad (19)$$

where

$$s(r, j) = \frac{1}{j!} \left[ \frac{d^j}{dx^j} x^{(r)} \right]_{x=0} \quad (20)$$

is the Sterling number of the first kind which counts the number of ways to permute a list of  $r$  items into  $j$  cycles. □

**Theorem 3.** The moment generating function of  $X \sim \text{HLL}(\alpha, \beta)$  is given by

$$M_X(t) = 2\alpha e^{-t/\beta} \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \left( \frac{t}{\beta} \right)^i \frac{k(-1)^{k-1}}{(k\alpha - i) i!}. \quad (21)$$

*Proof.* We have

$$E[e^{tX}] = 2\alpha\beta \int_0^{\infty} \frac{e^{tx} (1 + \beta x)^{-(\alpha+1)}}{[1 + (1 + \beta x)^{-\alpha}]^2} dx. \quad (22)$$

Setting  $y = (1 + \beta x)$ ,  $dx = dy/\beta$  yields

$$E[e^{tX}] = 2\alpha e^{-t/\beta} \int_1^{\infty} y^{-(\alpha+1)} e^{ty/\beta} [1 + y^{-\alpha}]^{-2} dy. \quad (23)$$

After simplification, we get

$$M_X(t) = 2\alpha e^{-t/\beta} \sum_{k=1}^{\infty} \sum_{i=0}^{\infty} \left( \frac{t}{\beta} \right)^i \frac{k(-1)^{k-1}}{(k\alpha - i) i!}. \quad (24)$$

□

**3.4. Mean Residual Life (MRL) Function.** The MRL function is important in reliability, survival analysis, actuarial sciences, economics, and social sciences for characterizing lifetime distributions. It also plays an important role in repair and replacement strategies and summarizes the entire residual life function.

**Theorem 4.** *The MRL function of  $X \sim HLL(\alpha, \beta)$  is given by*

$$m(x) = E(X - x | X > x) = \frac{1}{\beta} \left[ 1 + (1 + \beta x)^\alpha \right] \sum_{k=1}^{\infty} (-1)^{k-1} \frac{(1 + \beta x)^{-(k\alpha-1)}}{(k\alpha - 1)}. \tag{25}$$

*Proof.* The MRL function is defined as

$$m(x) = E(X - x | X > x) = \frac{1}{1 - F(x)} \int_x^{\infty} [1 - F(t)] dt. \tag{26}$$

Therefore, we have

$$m(x) = \left[ 1 + (1 + \beta x)^\alpha \right] \int_x^{\infty} \frac{(1 + \beta t)^{-\alpha}}{\left[ 1 + (1 + \beta t)^{-\alpha} \right]} dt. \tag{27}$$

Setting  $y = (1 + \beta x), dx = dy/\beta$  yields

$$m(x) = \frac{1}{\beta} \left[ 1 + (1 + \beta x)^\alpha \right] \int_{(1+\beta x)}^{\infty} \frac{y^{-\alpha}}{\left[ 1 + y^{-\alpha} \right]} dy. \tag{28}$$

After simplification, we get (25). □

**3.5. Mean Deviations.** The mean deviations about mean and median are, respectively, defined by

$$\delta_1(\mu) = 2\mu F(\mu) - 2\mu + 2 \int_{\mu}^{\infty} xf(x) dx, \tag{29}$$

$$\delta_2(M) = 2MF(M) - M - \mu + 2 \int_M^{\infty} xf(x) dx. \tag{30}$$

**Theorem 5.** *The mean deviations about mean and mean deviation about median of  $X \sim HLL(\alpha, \beta)$  are given by*

$$\delta_1(\mu) = \frac{-4\mu}{\left[ 1 + (1 + \beta\mu)^\alpha \right]} + \frac{4\alpha}{\beta} \sum_{k=1}^{\infty} k(-1)^{k-1} \cdot \left[ \frac{1}{(k\alpha - 1)(1 + \beta\mu)^{k\alpha-1}} - \frac{1}{k\alpha(1 + \beta\mu)^{k\alpha}} \right], \tag{31}$$

$$\delta_2(M) = M \frac{\left[ (1 + \beta M)^\alpha - 3 \right]}{\left[ (1 + \beta M)^\alpha + 1 \right]} - \frac{2}{\beta} \sum_{k=1}^{\infty} \frac{1}{(k\alpha - 1)} + \frac{4\alpha}{\beta} \cdot \sum_{k=1}^{\infty} k(-1)^{k-1} \cdot \left[ \frac{1}{(k\alpha - 1)(1 + \beta M)^{k\alpha-1}} - \frac{1}{k\alpha(1 + \beta M)^{k\alpha}} \right]. \tag{32}$$

*Proof.* Let  $I = \int_{\mu}^{\infty} xf(x)dx$ .

By inserting pdf of the HLL into (29) and setting  $y = (1 + \beta x), dx = dy/\beta$ , we get

$$I = \frac{2\alpha}{\beta} \int_{(1+\beta\mu)}^{\infty} (y^{-\alpha} - y^{-(\alpha+1)}) (1 + y^{-\alpha})^{-2} dy. \tag{33}$$

Using the series representations  $(1 + w)^{-2} = \sum_{k=1}^{\infty} k(-1)^{k-1} w^{k-1}$ , where  $k$  is positive integer, we get

$$I = \frac{2\alpha}{\beta} \sum_{k=1}^{\infty} k(-1)^{k-1} \cdot \left[ \frac{1}{(k\alpha - 1)(1 + \beta\mu)^{k\alpha-1}} - \frac{1}{k\alpha(1 + \beta\mu)^{k\alpha}} \right]. \tag{34}$$

Setting  $I$  into (29) and after some manipulations we get the desired result.

Similarly, the measure  $\delta_2(M)$  can be obtained. □

**3.6. Entropies.** The entropy of a random variable  $X$  is a measure of variation of uncertainty. Two popular entropies are the Rényi and Shannon entropies.

The Rényi entropy is defined as

$$I_R(\gamma) = \frac{1}{1 - \gamma} \log \left\{ \int_0^{\infty} f^\gamma(x) dx \right\}, \tag{35}$$

where  $\gamma > 0, \gamma \neq 1$ .

The Rényi entropy for the HLL distribution is given by

$$\int_0^{\infty} f^\gamma(x) dx = (2\alpha\beta)^r \cdot \int_0^{\infty} (1 + \beta x)^{-\gamma(\alpha+1)} \left[ 1 + (1 + \beta x)^{-\alpha} \right]^{-2\gamma} dx. \tag{36}$$

Setting  $y = (1 + \beta x), dx = dy/\beta$  yields

$$\int_0^{\infty} f^\gamma(x) dx = \frac{(2\alpha\beta)^r}{\beta} \int_1^{\infty} y^{-\gamma(\alpha+1)} \left[ 1 + y^{-\alpha} \right]^{-2\gamma} dy = \frac{(2\alpha\beta)^r}{\beta} \sum_{j=0}^{\infty} \frac{(-1)^j (2\gamma)_j}{j! (\gamma\alpha + j\alpha + \gamma - 1)}. \tag{37}$$

Thus, we have

$$I_R(\gamma) = \frac{1}{(1 - \gamma)} \log \left[ \frac{(2\alpha\beta)^r}{\beta} \sum_{j=0}^{\infty} \frac{(-1)^j (2\gamma)_j}{j! (\gamma\alpha + j\alpha + \gamma - 1)} \right]. \tag{38}$$

Another entropy measure defined by  $E\{-\log[f(x)]\}$  is known as the Shannon entropy and plays a similar role as the kurtosis measure in comparing the shapes of densities and measuring tail heaviness. We have

$$E\{-\log[f(x)]\} = -\log(2\alpha\beta) + (\alpha + 1) E[\log(1 + \beta x)] + 2E[\log\{1 + (1 + \beta x)^{-\alpha}\}]. \tag{39}$$

By using the binomial series expansion and (2.727) of Gradshteyn and Ryzhik [16], it follows that

$$E[\log(1 + \beta x)] = \frac{\log(4)}{\alpha}, \quad (40)$$

$$E\left[\log\left\{1 + (1 + \beta x)^{-\alpha}\right\}\right] = [1 - \log(2)].$$

Hence, the Shannon entropy can be expressed in the form

$$E\{-\log[f(x)]\} = \frac{\log(4)}{\alpha} - \log(2\alpha\beta) + 2. \quad (41)$$

**3.7. Order Statistics.** Let  $X_1, X_2, \dots, X_n$  be a random sample of size  $n$  from a distribution with pdf  $f(x)$  and cdf  $F(x)$ . Let  $X_{1:n}, X_{2:n}, \dots, X_{n:n}$  denote the corresponding order statistics. Then the pdf and cdf of  $X_{r:n}$  are

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} f(x) [F(x)]^{r-1} \cdot [1 - F(x)]^{n-r} \quad (42)$$

$$= \frac{1}{B(r, n-r+1)} f(x) \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i [F(x)]^{i+r-1},$$

$$F_{r:n}(x) = \sum_{k=r}^n \binom{n}{k} [F(x)]^k [1 - F(x)]^{n-k}, \quad (43)$$

$$1 \leq r \leq n,$$

where  $B(a, b)$  is the complete beta function.

**Theorem 6.** Let  $f(x)$  and  $F(x)$  be the pdf and cdf of  $X \sim \text{HLL}(\alpha, \beta)$ . Then the pdf and cdf of  $X_{r:n}$  are

$$f_{r:n}(x) = \frac{1}{B(r, n-r+1)} \frac{2\alpha\beta(1+\beta x)^{-(\alpha+1)}}{[1+(1+\beta x)^{-\alpha}]^2} \cdot \sum_{i=0}^{n-r} \binom{n-r}{i} (-1)^i \left\{ \frac{1-(1+\beta x)^{-\alpha}}{1+(1+\beta x)^{-\alpha}} \right\}^{i+r-1}, \quad (44)$$

$$F_{r:n}(x) = \sum_{k=r}^n \binom{n}{k} \left\{ \frac{1-(1+\beta x)^{-\alpha}}{1+(1+\beta x)^{-\alpha}} \right\}^k \cdot \left[ \frac{2}{1+(1+\beta x)^{-\alpha}} \right]^{n-k}.$$

*Proof.* Inserting (6) and (7) into (44), we get the result.  $\square$

#### 4. Estimation and Asymptotic Distribution

We consider the maximum-likelihood estimation of the parameters  $\alpha, \beta$ . The distributional properties of  $\hat{\alpha}, \hat{\beta}$  are obtained using Fisher information matrix.

**4.1. Maximum-Likelihood Estimation.** Let  $x_1, x_2, \dots, x_n$  be a random sample of size  $n$  from  $\text{HLL}(\alpha, \beta)$  distribution. Then the corresponding likelihood function is

$$L(\alpha, \beta) = (2\alpha\beta)^n \prod_{i=1}^n \frac{(1+\beta x_i)^{-(\alpha+1)}}{[1+(1+\beta x_i)^{-\alpha}]^2}. \quad (45)$$

The log-likelihood function is given by

$$\ell(x; \alpha, \beta) = n \log(2\alpha\beta) - (\alpha+1) \sum_{i=1}^n \log(1+\beta x_i) - 2 \sum_{i=1}^n \log[1+(1+\beta x_i)^{-\alpha}]. \quad (46)$$

Calculating the first-order partial derivatives of (46) with respect to  $\alpha, \beta$  and equating them to zero, we get the following nonlinear equations:

$$\frac{\partial \ell}{\partial \alpha} = \frac{n}{\alpha} - \sum_{i=1}^n \log(1+\beta x_i) + 2 \sum_{i=1}^n \frac{\log(1+\beta x_i)}{[1+(1+\beta x_i)^{-\alpha}]} = 0, \quad (47)$$

$$\frac{\partial \ell}{\partial \beta} = \frac{n}{\beta} - (\alpha+1) \sum_{i=1}^n \frac{x_i}{(1+\beta x_i)} + 2\alpha \sum_{i=1}^n \frac{x_i}{(1+\beta x_i)[1+(1+\beta x_i)^{-\alpha}]} = 0.$$

To find out the maximum-likelihood estimates (MLEs) of  $\alpha, \beta$ , we have to solve the above nonlinear equations. Apparently, there is no closed form solution in  $\alpha, \beta$ . We have to use a numerical technique method, such as Newton-Raphson method, to obtain the solution.

**4.2. Asymptotic Distribution.** The normal approximation for the large-sample for the MLE can be used to compute confidence intervals (CIs). For finding the observed information matrix of  $\alpha, \beta$ , we compute the second-order partial derivatives of  $l$  (46) in the appendix.

The variance-covariance matrix may be approximated by  $\Sigma = I^{-1}$ , where  $I$  is the observed information matrix. Since  $\Sigma$  involves the parameters  $\alpha, \beta$ , we replace the parameters by the corresponding MLEs in order to obtain an estimate of  $\Sigma$ , which is denoted by  $\hat{\Sigma} = \hat{I}^{-1}$ , where  $\hat{I}_{ij} = I_{ij}$ , when  $(\hat{\alpha}, \hat{\beta})$  is replaced by  $\alpha, \beta$ . Using these results, an approximate 100(1 -  $\vartheta$ )% CI for  $\alpha, \beta$ , respectively, is given by

$$\hat{\alpha} \pm z_{\vartheta/2} \sqrt{\hat{\Sigma}_{11}}, \quad (48)$$

$$\hat{\beta} \pm z_{\vartheta/2} \sqrt{\hat{\Sigma}_{22}},$$

where  $z_{\vartheta/2}$  is the upper ( $\vartheta/2$ )th percentile of the standard normal distribution.

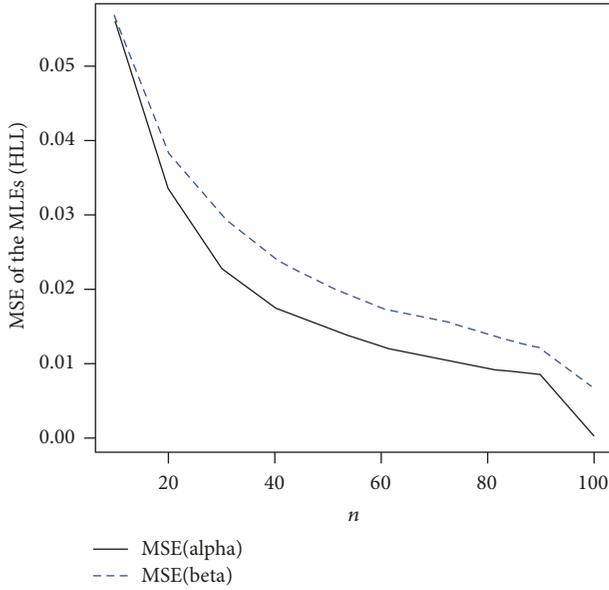


FIGURE 3: Plots of the MSE( $\hat{\alpha}$ ) and MSE( $\hat{\beta}$ ) for  $\alpha = 2, \beta = 1$  based on 10,000 replications.

### 5. Simulation Study

The assessment of the maximum-likelihood estimates is based on a simulation study. The following steps were followed:

- (1) Generate 5,000 samples of size  $n$  from (6). The HLL variables are generated using  $X = (1/\beta)[((1-U)/(1+U))^{1/\alpha} - 1]$  given in (12), where  $U \sim \text{Uniform}(0, 1)$ .
- (2) Compute the MLEs for the 5,000 samples, say  $(\hat{\alpha}_i, \hat{\beta}_i)$  for  $i = 1, 2, \dots, 5000$ .
- (3) Compute the mean square errors (MSE) for each parameter.

We repeat these steps for  $n = 10, 20, \dots, 100$  with  $\alpha = 2, \beta = 1$ . The comparison is based on MSEs. Figure 3 plots the MSEs of the MLEs of two parameters. The assessment based on this simulation study is that the MSEs for each parameter decrease to zero with increasing sample size.

### 6. Application

6.1. *Cancer Data.* We have considered a real data set, which records the remission times (in months) of a random sample of 128 bladder cancer patients reported in [17]. We have fitted the HLL distribution to the data set using MLE and compared the proposed distribution with the following distributions:

- (i) The Lomax distribution, introduced by Lomax [1], with pdf is

$$f_{\text{Lomax}}(x) = \alpha\beta(1 + \beta x)^{-(\alpha+1)}, \quad \text{where } x, \alpha, \beta > 0. \quad (49)$$

- (ii) The McLomax distribution, introduced by Lemonte and Cordeiro [10], with pdf is

$$f_{\text{McLomax}}(x) = \frac{c\alpha\beta^\alpha(\beta+x)^{-(\alpha+1)}}{B(ac^{-1}, \eta+1)} \left\{1 - \left(\frac{\beta}{\beta+x}\right)^\alpha\right\}^{a-1} \cdot \left[1 - \left\{1 - \left(\frac{\beta}{\beta+x}\right)^\alpha\right\}^c\right]^\eta, \quad (50)$$

where  $x, \alpha, \beta, a, c > 0, 0 \leq \eta$ .

- (iii) The beta Lomax (BLomax) distribution is a submodel of McLomax for  $c = 1$  with pdf

$$f_{\text{BLomax}}(x) = \frac{\alpha\beta^\alpha(\beta+x)^{-(\alpha+1)}}{B(a, \eta+1)} \left\{1 - \left(\frac{\beta}{\beta+x}\right)^\alpha\right\}^{a-1} \cdot \left[1 - \left\{1 - \left(\frac{\beta}{\beta+x}\right)^\alpha\right\}\right]^\eta, \quad (51)$$

where  $x, \alpha, \beta, a > 0, 0 \leq \eta$ .

- (iv) The Kumaraswamy Lomax (KwLomax) distribution is a submodel of McLomax for  $a = c$ , with pdf

$$f_{\text{KwLomax}}(x) = \frac{\alpha\beta^\alpha(\beta+x)^{-(\alpha+1)}}{(\eta+1)} \left\{1 - \left(\frac{\beta}{\beta+x}\right)^\alpha\right\}^{a-1} \cdot \left[1 - \left\{1 - \left(\frac{\beta}{\beta+x}\right)^\alpha\right\}^a\right]^\eta, \quad (52)$$

where  $x, \alpha, \beta, a > 0, 0 \leq \eta$ .

- (v) The Exponentiated Standard Lomax (ESLomax) distribution, considered by Gupta et al. [18], with pdf is

$$f_{\text{ESLomax}}(x) = \frac{\alpha}{a}(1+x)^{-(\alpha+1)} \{1 - (1+x)^{-\alpha}\}^{a-1}, \quad (53)$$

where  $x, \alpha, a > 0$ .

- (vi) The Lomax-Logarithmic (LL) distribution, introduced by Al-Zahrani and Sagor [11], with pdf is

$$f_{\text{LL}}(x) = \frac{(\eta-1)\alpha\beta(1+\beta x)^{-(\alpha+1)}}{\{1 - (1-\eta)(1+\beta x)^{-\alpha}\} \ln(\eta)}, \quad (54)$$

where  $x, \alpha, \beta, \eta > 0$ .

For identification of the shape of the hazard function of the cancer data set, we use a graphical method based on the Total Time on Test (TTT) plot [19]. Let  $T$  be a random variable with nonnegative values, which represents the survival time. The empirical TTT curve is constructed by plotting  $T(r/n) = [\sum_{i=1}^r Y_{i:n} + (n-r)Y_{r:n}]/(\sum_{i=1}^n Y_{i:n})$ , against  $r/n$ , where  $r = 1, 2, \dots, n$  and  $Y_{i:n}$  ( $i = 1, 2, \dots, n$ ) are the order statistics of the sample. If the empirical TTT transform is straight

TABLE 1: Descriptive statistics.

| Mean | Median | Mode | SD    | Variance | Skewness | Kurtosis | Min. | Max.  |
|------|--------|------|-------|----------|----------|----------|------|-------|
| 9.37 | 6.40   | 5.0  | 10.51 | 110.43   | 3.287    | 15.48    | 0.08 | 79.05 |

TABLE 2: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC, and CAIC for cancer data.

| Model   | Estimates           |                       |                   |                     |                   | Measures      |               |               |               |
|---------|---------------------|-----------------------|-------------------|---------------------|-------------------|---------------|---------------|---------------|---------------|
|         | $\alpha$            | $\beta$               | $a$               | $\eta$              | $c$               | AIC           | BIC           | HQIC          | CAIC          |
| HLL     | 5.2814<br>(2.0847)  | 0.0367<br>(0.0177)    |                   |                     |                   | <b>827.69</b> | <b>833.40</b> | <b>830.01</b> | <b>827.79</b> |
| Lomax   | 13.9384<br>(15.368) | 121.0225<br>(142.714) |                   |                     |                   | 831.67        | 837.37        | 833.98        | 831.76        |
| McLomax | 0.8085<br>(3.364)   | 11.2929<br>(15.818)   | 1.5060<br>(0.234) | 4.1886<br>(25.029)  | 2.1046<br>(3.079) | 829.82        | 844.09        | 835.62        | 830.31        |
| BLomax  | 3.9191<br>(18.192)  | 23.9281<br>(27.338)   | 1.5853<br>(0.280) | 0.1572<br>(5.024)   |                   | 828.14        | 839.55        | 832.78        | 828.46        |
| KwLomax | 0.3911<br>(2.386)   | 12.2973<br>(17.316)   | 1.5162<br>(0.228) | 11.0323<br>(87.144) |                   | 827.88        | 839.29        | 832.52        | 828.21        |
| ESLomax | 1.0877<br>(0.086)   |                       | 4.6575<br>(0.686) |                     |                   | 856.61        | 862.31        | 833.98        | 856.71        |
| LL      | 4.8696<br>(2.1317)  | 20.1652<br>(18.3747)  |                   | 7.6938<br>(11.8362) |                   | 830.88        | 839.44        | 834.36        | 831.07        |

diagonal, hrf is constant; if it is convex, hrf is decreasing; if it is concave, hrf is increasing; if it is convex and then concave, hrf is bathtub; and if it is concave and then convex, hrf is upside-down bathtub [19]. Figure 4 shows the empirical TTT plot for the data set. The shape of the data set is unimodal, as its TTT plot is concave and then convex. The model selection is carried out using Akaike information criterion (AIC), the Bayesian information criterion (BIC), the consistent Akaike information criterion (CAIC), and the Hannan-Quinn information criterion (HQIC).

$$\begin{aligned}
 \text{AIC} &= -2l(\hat{\theta}) + 2q, \\
 \text{BIC} &= -2l(\hat{\theta}) + q \log(n), \\
 \text{HQIC} &= -2l(\hat{\theta}) + 2q \log(\log(n)), \\
 \text{CAIC} &= -2l(\hat{\theta}) + \frac{2qn}{n - q - 1},
 \end{aligned}
 \tag{55}$$

where  $l(\hat{\theta})$  denotes the log-likelihood function evaluated at the MLEs,  $q$  is the number of model parameters, and  $n$  is the sample size. The model with the lowest values for these statistics could be chosen as the best model to fit the data. First, we describe the data set in Table 1. Then, we report the maximum-likelihood estimators (and the corresponding standard errors in parentheses) of the parameters and values of AIC, BIC, HQIC, and CAIC in Table 2. In Table 2, we provide some results reported in [10, 11]. We conclude that our new model HLL has the smallest AIC, BIC, HQIC, and CAIC values among all fitted models, and so it could be chosen as the best model. These results are obtained using

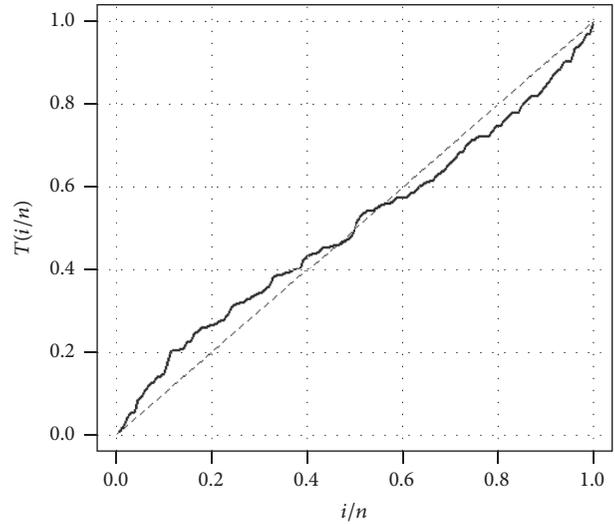


FIGURE 4: Empirical TTT plot for the cancer data.

the AdequacyModel package version 1.0.8 available for the programming language R [20].

6.2. *Simulated Data.* Here, we used a simulation study to check the flexibility of the proposed distribution. We have generated a sample of size  $n = 100$  from the six distributions and compared the fit of the HLL distribution with competing models.

The details of generated samples with the specified values of the parameters from the six models are as follows:

TABLE 3: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC, and CAIC for simulated data from Lomax distribution with  $\alpha = 1.5$  and  $\beta = 1$ .

| Model   | Estimates          |                     |                    |                     |                   | Measures      |               |               |               |
|---------|--------------------|---------------------|--------------------|---------------------|-------------------|---------------|---------------|---------------|---------------|
|         | $\alpha$           | $\beta$             | $a$                | $\eta$              | $c$               | AIC           | BIC           | HQIC          | CAIC          |
| HLL     | 1.8032<br>(0.4230) | 1.2386<br>(0.4764)  |                    |                     |                   | <b>257.03</b> | <b>262.24</b> | <b>259.14</b> | <b>257.16</b> |
| Lomax   | 2.2441<br>(0.7434) | 0.5337<br>(0.2437)  |                    |                     |                   | 257.39        | 262.60        | 259.50        | 257.52        |
| McLomax | 2.7515<br>(2.4395) | 2.3568<br>(0.2818)  | 0.9418<br>(0.1849) | -0.1016<br>(0.8005) | 2.8189<br>(3.079) | 262.90        | 275.93        | 268.17        | 263.54        |
| BLomax  | 1.4533<br>(8.1920) | 2.2421<br>(0.9818)  | 0.9361<br>(0.1699) | 0.6659<br>(0.0624)  |                   | 260.90        | 271.32        | 265.12        | 261.32        |
| KwLomax | 8.5753<br>(0.7854) | 2.0767<br>(7.8953)  | 0.9188<br>(0.8728) | -0.7243<br>(7.1544) |                   | 260.90        | 271.32        | 265.12        | 261.32        |
| ESLomax | 1.5909<br>(0.0860) |                     | 1.1135<br>(0.6861) |                     |                   | 258.50        | 263.70        | 260.60        | 258.62        |
| LL      | 2.6536<br>(0.3217) | 0.3359<br>(0.23747) |                    | 0.5196<br>(0.08362) |                   | 258.96        | 266.77        | 262.12        | 259.21        |

TABLE 4: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC, and CAIC for simulated data from McLomax distribution with parameters  $\alpha = 5.25$ ,  $\beta = 2.55$ ,  $a = 1.45$ ,  $\eta = 0.67$ , and  $c = 3.81$ .

| Model   | Estimates           |                     |                    |                     |                    | Measures      |               |               |               |
|---------|---------------------|---------------------|--------------------|---------------------|--------------------|---------------|---------------|---------------|---------------|
|         | $\alpha$            | $\beta$             | $a$                | $\eta$              | $c$                | AIC           | BIC           | HQIC          | CAIC          |
| HLL     | 1.2207<br>(2.0847)  | 2.4477<br>(0.0177)  |                    |                     |                    | 268.28        | 273.50        | 270.39        | 268.41        |
| Lomax   | 1.4243<br>(0.4009)  | 1.0314<br>(0.4619)  |                    |                     |                    | 267.49        | 272.70        | 269.60        | 267.61        |
| McLomax | 14.1535<br>(0.1364) | 3.2796<br>(5.3498)  | 0.8091<br>(0.9234) | -0.8197<br>(9.1029) | 2.9511<br>(4.0079) | 266.95        | 279.97        | 272.22        | 267.58        |
| BLomax  | 1.9585<br>(6.0364)  | 3.7068<br>(3.4102)  | 0.6586<br>(0.1077) | 0.3007<br>(4.0353)  |                    | 265.67        | 276.09        | 269.89        | 266.09        |
| KwLomax | 0.3924<br>(0.6603)  | 7.7169<br>(12.8494) | 0.6896<br>(0.0966) | 6.1821<br>(11.7374) |                    | 265.39        | 275.81        | 269.61        | 265.81        |
| ESLomax | 1.2764<br>(0.1748)  |                     | 0.8217<br>(0.1031) |                     |                    | 264.94        | <b>270.15</b> | <b>267.04</b> | 265.06        |
| LL      | 8.8655<br>(15.7662) | 0.03389<br>(0.0695) |                    | 0.0359<br>(0.0278)  |                    | <b>264.02</b> | 271.83        | 267.18        | <b>264.27</b> |

- (1) Lomax distribution with  $\alpha = 1.5$  and  $\beta = 1$  (see Table 3)
- (2) McLomax distribution with parameters  $\alpha = 5.25$ ,  $\beta = 2.55$ ,  $a = 1.45$ ,  $\eta = 0.67$ , and  $c = 3.81$  (see Table 4)
- (3) BLomax distribution with parameters  $\alpha = 0.85$ ,  $\beta = 12.55$ ,  $a = 1.75$ , and  $\eta = 4.67$  (see Table 5)
- (4) KwLomax distribution with parameters  $\alpha = 0.85$ ,  $\beta = 12.55$ ,  $a = 1.75$ , and  $\eta = 4.67$  (see Table 6)
- (5) ESLomax distribution with parameters  $\alpha = 2.5$  and  $a = 1.15$  (see Table 7)
- (6) LL distribution with parameters  $\alpha = 2.75$ ,  $\beta = 0.45$ , and  $\eta = 0.75$  (see Table 8)

The comparison is based upon the formal goodness-of-fit tests given above. The values of best fitted model are

highlighted in the tables. Based on the goodness-of-fit tests, we conclude that the HLL distribution has better fit than the other competing models.

### 7. Concluding Remarks

In this paper, we introduce a two-parameter HLL distribution by compounding the half-logistic and the Lomax distributions. The shape of hazard function of the new compounding distribution can be monotonically decreasing or upside-down bathtub (unimodal). Some mathematical and statistical properties of the new model including moments and moment generating functions, mean deviations from mean and median, quantile function, mean residual life function, mode, median, order statistics, and entropies are studied.

TABLE 5: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC, and CAIC for simulated data from BLomax distribution with parameters  $\alpha = 0.85$ ,  $\beta = 12.55$ ,  $a = 1.75$ , and  $\eta = 4.67$ .

| Model   | Estimates            |                       |                    |                      |                    | Measures      |               |               |               |
|---------|----------------------|-----------------------|--------------------|----------------------|--------------------|---------------|---------------|---------------|---------------|
|         | $\alpha$             | $\beta$               | $a$                | $\eta$               | $c$                | AIC           | BIC           | HQIC          | CAIC          |
| HLL     | 24.4487<br>(13.7405) | 0.0112<br>(0.0065)    |                    |                      |                    | 533.90        | <b>539.11</b> | <b>536.01</b> | 534.02        |
| Lomax   | 23.6200<br>(11.1025) | 0.0081<br>(0.0038)    |                    |                      |                    | 544.04        | 549.25        | 546.15        | 544.17        |
| McLomax | 1.3135<br>(9.1629)   | 18.6449<br>(73.1176)  | 1.4939<br>(0.3102) | 9.3587<br>(78.4737)  | 2.4534<br>(3.8777) | 534.96        | 547.99        | 540.23        | 535.60        |
| BLomax  | 2.0366<br>(26.5471)  | 93.9001<br>(159.3791) | 1.7237<br>(0.2569) | 13.8059<br>(194.682) |                    | 533.49        | 543.91        | 537.71        | 533.91        |
| KwLomax | 0.9258<br>(2.386)    | 35.7112<br>(17.316)   | 1.5236<br>(0.228)  | 21.0737<br>(139.842) |                    | <b>533.05</b> | 543.47        | 537.27        | <b>533.47</b> |
| ESLomax | 1.3483<br>(0.1185)   |                       | 5.0645<br>(0.8554) |                      |                    | 561.43        | 566.64        | 563.53        | 561.55        |
| LL      | 10.6838<br>(5.5325)  | 0.0392<br>(0.0261)    |                    | 16.8989<br>(15.5951) |                    | 535.87        | 543.69        | 539.04        | 536.12        |

TABLE 6: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC, and CAIC for simulated data from KwLomax distribution with parameters  $\alpha = 0.85$ ,  $\beta = 12.55$ ,  $a = 1.75$ , and  $\eta = 4.67$ .

| Model   | Estimates            |                      |                    |                     |                      | Measures      |               |               |               |
|---------|----------------------|----------------------|--------------------|---------------------|----------------------|---------------|---------------|---------------|---------------|
|         | $\alpha$             | $\beta$              | $a$                | $\eta$              | $c$                  | AIC           | BIC           | HQIC          | CAIC          |
| HLL     | 3.2092<br>(1.0615)   | 0.1872<br>(0.0847)   |                    |                     |                      | <b>454.88</b> | <b>460.09</b> | <b>456.99</b> | <b>455.02</b> |
| Lomax   | 5.2637<br>(3.0135)   | 0.0650<br>(0.0436)   |                    |                     |                      | 455.74        | 460.95        | 457.85        | 455.87        |
| McLomax | 36.3053<br>(49.9841) | 64.3913<br>(87.0837) | 1.4745<br>(0.2732) | -0.6614<br>(0.1299) | 26.4402<br>(21.5904) | 455.68        | 468.70        | 460.95        | 456.32        |
| BLomax  | 20.4375<br>(18.192)  | 8.2358<br>(1.3201)   | 1.6963<br>(0.3709) | -0.8265<br>(0.0270) |                      | 455.46        | 465.89        | 459.68        | 455.89        |
| KwLomax | 1.4823<br>(2.386)    | 3.9164<br>(7.0316)   | 1.5331<br>(0.3464) | 0.8043<br>(8.0144)  |                      | 455.73        | 466.15        | 459.95        | 456.15        |
| ESLomax | 1.3936<br>(0.1429)   |                      | 2.6629<br>(0.4120) |                     |                      | 457.01        | 462.22        | 459.12        | 457.13        |
| LL      | 3.9058<br>(2.9870)   | 0.1236<br>(0.1943)   |                    | 2.3957<br>(4.9588)  |                      | 457.48        | 465.30        | 460.65        | 457.73        |

TABLE 7: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC, and CAIC for simulated data from ESLomax distribution with parameters  $\alpha = 2.5$  and  $a = 1.15$ .

| Model   | Estimates           |                    |                    |                     |                    | Measures     |              |              |              |
|---------|---------------------|--------------------|--------------------|---------------------|--------------------|--------------|--------------|--------------|--------------|
|         | $\alpha$            | $\beta$            | $a$                | $\eta$              | $c$                | AIC          | BIC          | HQIC         | CAIC         |
| HLL     | 2.9230<br>(0.8789)  | 1.4697<br>(0.6175) |                    |                     |                    | 70.15        | 75.35        | 72.25        | 70.27        |
| Lomax   | 4.4945<br>(2.3241)  | 0.5392<br>(0.3345) |                    |                     |                    | 71.43        | 76.64        | 73.54        | 71.55        |
| McLomax | 9.2546<br>(18.1860) | 0.8577<br>(0.7067) | 1.9039<br>(0.9458) | -0.6952<br>(0.5267) | 2.5132<br>(3.9407) | 72.37        | 85.40        | 77.65        | 73.01        |
| BLomax  | 0.5806<br>(1.7619)  | 0.4735<br>(0.3488) | 1.6416<br>(0.4155) | 3.3194<br>(13.6281) |                    | 70.61        | 81.04        | 74.83        | 71.04        |
| KwLomax | 2.4388<br>(1.1386)  | 0.5530<br>(0.6732) | 1.6260<br>(0.1428) | -0.0069<br>(7.9604) |                    | 70.64        | 81.07        | 74.86        | 71.07        |
| ESLomax | 3.4021<br>(0.4127)  |                    | 1.3948<br>(0.1956) |                     |                    | <b>67.33</b> | <b>72.54</b> | <b>69.44</b> | <b>67.46</b> |
| LL      | 3.1584<br>(1.3691)  | 1.2813<br>(1.4460) |                    | 3.5394<br>(6.8140)  |                    | 72.86        | 80.68        | 76.03        | 73.12        |

TABLE 8: MLEs (standard errors in parentheses) and the measures AIC, BIC, HQIC, and CAIC for simulated data from LL distribution with parameters  $\alpha = 2.75$ ,  $\beta = 0.45$ , and  $\eta = 0.75$ .

| Model   | Estimates           |                    |                    |                     |                    | Measures      |               |               |               |
|---------|---------------------|--------------------|--------------------|---------------------|--------------------|---------------|---------------|---------------|---------------|
|         | $\alpha$            | $\beta$            | $a$                | $\eta$              | $c$                | AIC           | BIC           | HQIC          | CAIC          |
| HLL     | 1.8358<br>(0.4189)  | 1.2344<br>(0.4553) |                    |                     |                    | <b>252.97</b> | <b>258.19</b> | <b>255.08</b> | <b>253.10</b> |
| Lomax   | 2.1811<br>(0.6717)  | 0.5669<br>(0.2408) |                    |                     |                    | 253.22        | 258.43        | 255.33        | 253.34        |
| McLomax | 0.7992<br>(2.8691)  | 1.2223<br>(1.7168) | 0.9887<br>(0.1874) | 1.6530<br>(10.1760) | 1.9169<br>(3.9962) | 259.10        | 272.12        | 264.37        | 259.73        |
| BLomax  | 1.6711<br>(94.5364) | 1.7124<br>(1.0299) | 1.0130<br>(0.1929) | 0.2907<br>(73.1088) |                    | 257.21        | 267.63        | 261.43        | 257.63        |
| KwLomax | 8.6899<br>(7.6698)  | 1.7774<br>(0.8502) | 0.9917<br>(0.2342) | -0.7481<br>(0.2298) |                    | 257.22        | 267.64        | 261.44        | 257.64        |
| ESLomax | 1.6418<br>(0.2057)  |                    | 1.1489<br>(0.1530) |                     |                    | 254.10        | 259.31        | 256.21        | 254.23        |
| LL      | 1.9522<br>(0.5707)  | 1.0375<br>(1.7363) |                    | 3.1618<br>(11.1845) |                    | 255.07        | 262.89        | 258.24        | 255.32        |

We estimate the model parameters by maximum likelihood and determined the observed information matrix. We present a simulation study to illustrate the performance of MLEs. The flexibility and potentiality of the proposed model are illustrated by means of a real data set. We hope that the HLL distribution may attract wider range of applications in areas such as engineering, survival and lifetime data, economics, meteorology, hydrology, and others.

### Appendix

#### Elements of Observed Information Matrix

The elements of  $2 \times 2$  observed information matrix ( $I$ ) are given by

$$\begin{aligned}
 I_{11} &= \frac{\partial^2 l}{\partial \alpha^2} = -\frac{n}{\alpha^2} + 2 \sum_{i=1}^n \frac{(1 + \beta x_i)^\alpha [\log(1 + \beta x_i)]^2}{[1 + (1 + \beta x_i)^\alpha]^2}, \\
 I_{12} = I_{21} &= \frac{\partial^2 l}{\partial \alpha \partial \beta} \\
 &= -\sum_{i=1}^n \frac{x_i}{(1 + \beta x_i)} \\
 &\quad + 2 \sum_{i=1}^n \frac{x_i [1 + (1 + \beta x_i)^\alpha] + \alpha x_i \log(1 + \beta x_i)}{(1 + \beta x_i)^{\alpha+1} [1 + (1 + \beta x_i)^\alpha]^2}, \quad (A.1) \\
 I_{22} &= \frac{\partial^2 l}{\partial \beta^2} \\
 &= -\frac{n}{\beta^2} + (\alpha + 1) \sum_{i=1}^n \frac{x_i^2}{(1 + \beta x_i)^2} \\
 &\quad - 2\alpha \sum_{i=1}^n \frac{x_i^2 [(\alpha + 1)(1 + \beta x_i)^\alpha + 1]}{(1 + \beta x_i) [1 + (1 + \beta x_i)^\alpha]}.
 \end{aligned}$$

### Disclosure

The authors acknowledge that the abstract of the manuscript had been presented in a conference paper under the link <http://archive.stat.uconn.edu/lida17/program.pdf>.

### Conflicts of Interest

The authors declare that they have no conflicts of interest.

### Authors' Contributions

The authors have equally contributed to this work. All authors read and approved the final manuscript.

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