Supplementary Material

for

An Alternative Sensitivity Approach For Longitudinal Analysis with Dropout

Amal Almohisen,[1*] Robin Henderson,[2] Arwa M. Alshingiti[1]

[1]Dept of Statistics and Operations Research, College of Science, King Saud University, Riyadh, Saudi Arabia.

[2]School of Mathematics and statistics, Newcastle University, Newcastle Upon Tyne, UK.

*To whom correspondence should be addressed; E-mail: amalmoh@ksu.edu.sa

---

1 Least False Estimate

In equation (14) we need to calculate the expressions of $E[R]$, $E[Rx]$, $E[Rx^2]$, $E[RY_1]$, $E[RY_2]$, $E[RY_1x]$ and $E[RY_2x]$ under MAR and MNAR. To illustrate we show here the calculations for $E[R]$. As we know $E[R] = E[E[R|x]]$, thus we have to calculate $E[R|x]$ first, then we can find $E[R]$. Note that under MAR, R does not depend on ϵ_2 as it depends only on x, U and ϵ_1. We have

$$\begin{aligned} E[R|x] &= \int_{-\infty}^{\infty} P(R=1|x,\epsilon_1,U) f(\epsilon_1,U)\,\mathrm{d}\epsilon_1\,\mathrm{d}U \\ &= \int_{-\infty}^{\infty} \mathrm{expit}\{\theta_0+\theta_1 Y_1\} f(\epsilon_1,U)\,\mathrm{d}\epsilon_1\,\mathrm{d}U \\ &= \int_{-\infty}^{\infty} \mathrm{expit}\{\theta_0+\theta_1(\beta_1+\beta_2 x)+\theta_1(U+\epsilon_1)\} f(\epsilon_1,U)\,\mathrm{d}\epsilon_1\,\mathrm{d}U \\ &= \int_{-\infty}^{\infty} \mathrm{expit}\{K_1+\theta_1 w_1\} f(w_1)\,\mathrm{d}w_1, \text{ say,} \end{aligned}$$

where

$$w_1 = \epsilon_1 + U \tag{1}$$

$$K_1 = \theta_0 + \theta_1(\beta_1+\beta_2 x). \tag{2}$$

We now use an approximation of the expit to the cumulative normal, see [2]:

$$\mathrm{expit(z)} \approx \Phi(\mathrm{c\,z}),\ \mathrm{c} = \frac{16\sqrt{3}}{15\pi}. \tag{3}$$

Therefore:

$$E[R|x] \approx \int_{-\infty}^{\infty} \Phi\{c(K_1+\theta_1 w_1)\} f(w_1)\,\mathrm{d}w_1.$$

From equation (1) $w_1 = \epsilon_1 + U$ and we assumed in our model that $\epsilon_1 \sim N(0,\sigma^2_{\epsilon_1})$ and $U \sim N(0,\sigma^2_U)$. Hence we can say that w_1 is normally distributed with mean 0 and variance $\sigma^2_{w_1}$, i.e $w_1 \sim N(0,\sigma^2_{w_1})$, where $\sigma^2_{w_1} = \sigma^2_{\epsilon_1} + \sigma^2_U$, which allows us to replace $f(w_1)$ with $\phi(w_1;0,\sigma^2_{w_1})$ and

$$E[R|x] \approx \int_{-\infty}^{\infty} \Phi\{c(K_1+\theta_1 w_1)\}\phi(w_1;0,\sigma^2_{w_1})\,\mathrm{d}w_1.$$

The integral is now equivalent to the numerator of the pdf of the Extended Skew Normal Distribution (ESN). The ESN distribution is described in [2]. We will consider the definition and notation that was given in [1]. Thus

$$
\begin{aligned}
E[R|x] &\approx \Phi\left[\frac{cK_1}{\sqrt{1+c^2\theta_1^2\sigma_{w_1}^2}}\right] \\
&= \Phi\left[\frac{c(\theta_0+\theta_1(\beta_1+\beta_2 x))}{\sqrt{1+c^2\theta_1^2\sigma_{w_1}^2}}\right] \\
&= \Phi(A_1+A_2 x) \qquad (4)
\end{aligned}
$$

where we have used equation (2) to replace K_1 and we have defined $A_1 = \frac{c}{\sqrt{1+c^2\theta_1^2\sigma_{w_1}^2}}(\theta_0 + \theta_1\beta_1)$ and $A_2 = \frac{c}{\sqrt{1+c^2\theta_1^2\sigma_{w_1}^2}}(\theta_1\beta_2)$.

Now we can find $E[R]$ by integrating the above expectation in (4) over x. Recall that $x \sim N(0, \sigma_x^2)$ so

$$
\begin{aligned}
E[R] &= E_x[E[R|x]] \\
&\approx \int_{-\infty}^{\infty} \Phi(A_1+A_2 x)\phi(x;0,\sigma_x^2)\mathrm{d}x
\end{aligned}
$$

then we use (ESN) to obtain

$$
\begin{aligned}
E[R] &\approx \Phi\left[\frac{A_1}{\sqrt{1+(A_2\sigma_x)^2}}\right] \\
&= \Phi(\nu_1) \qquad (5)
\end{aligned}
$$

where $\nu_1 = \frac{A_1}{\sqrt{1+(A_2\sigma_x)^2}}$.

2 Copas and Eguchi Bias

We will show the calculation of the terms needed for the bias expression (18). We give an example for the bias of β_3. To calculate the bias $I_Y^{-1}E_f[\varepsilon u_Y s_Y]$ we need the score and the information matrix for the MNAR model f_Y. The score is $s_Y = \partial\{\log(f_Y)\}/\partial\beta_3$ and $I_Y = E\left[-\partial^2\{\log(f_Y)\}/\partial\beta_3^2\right]$.

Let $Y_1 \sim N(\mu_1, \sigma_1)$, $Y_2 \sim N(\mu_2, \sigma_2)$ and $Y_{21} \sim N(\mu_{21}, \sigma_{21})$ is the conditional distribution of Y_2 given Y_1, where $\mu_{21} = \mu_2 + \rho\,\sigma_2\left(\frac{Y_1-\mu_1}{\sigma_1}\right)$, and $\sigma_{21} = \sqrt{\sigma_2^2(1-\rho^2)} = \sigma_2\sqrt{(1-\rho^2)}$.

The likelihood under MNAR is

$$
\begin{aligned}
L &= \left(f(Y_2|Y_1)f(Y_1)P(R=1|Y_1,Y_2)\right)^R \left(f(Y_1)P(R=0|Y_1)\right)^{1-R} \\
\ell &= \log L \\
&= \log\left(f(Y_2|Y_1)f(Y_1)P(R=1|Y_1,Y_2)\right)^R + \log\left(f(Y_1)P(R=0|Y_1)\right)^{1-R} \\
&= R\{\log f(Y_2|Y_1) + \log f(Y_1) + \log P(R=1|Y_1,Y_2)\} + (1-R)\{\log f(Y_1) + \log P(R=0|Y_1)\}
\end{aligned}
$$

To calculate the bias of β_3, the terms needed are $f(Y_2|Y_1)$ and $P(R=1|Y_1,Y_2)$. Hence we need to calculate the first and second derivatives of $\log f(Y_2|Y_1)$ and $\log P(R=1|Y_1,Y_2)$. The first derivative is:

$$
\frac{\partial \ell}{\partial \beta_3} = \frac{\partial \ell}{\partial \mu_{21}}\frac{\partial \mu_{21}}{\partial \beta_3}
$$

Note that $\partial\mu_{21}/\partial\beta_3 = 1$. So we need only to calculate $\partial\ell/\partial\mu_{21}$. We will do this in two parts: First, calculate the derivative $\frac{\partial}{\partial\mu_{21}}\log f(Y_2|Y_1)$:

$$
\begin{aligned}
\log f(Y_2|Y_1) &= \log\left\{\frac{1}{\sqrt{2\pi}\sigma_{21}}\exp\{-\frac{1}{2\sigma_{21}^2}(Y_2-\mu_{21})^2\}\right\} \\
&= -\log(\sqrt{2\pi}\sigma_{21}) - \frac{1}{2\sigma_{21}^2}(Y_2-\mu_{21})^2,
\end{aligned}
$$

so,

$$
\frac{\partial}{\partial\mu_{21}}\log f(Y_2|Y_1) = \frac{(Y_2-\mu_{21})}{\sigma_{21}^2}.
$$

Second, calculate the derivative $\frac{\partial}{\partial \mu_{21}} \log P(R=1|Y_1,Y_2)$:

$$
\begin{aligned}
\log P(R=1|Y_1,Y_2) &= \log(\text{expit}\{\theta_0+\theta_1 Y_1\}\exp\{\varepsilon u_Y\}) \\
&= \log \text{expit}\{\theta_0+\theta_1 Y_1\} + \{\varepsilon u_Y\}
\end{aligned}
$$

leading to

$$
\begin{aligned}
\frac{\partial}{\partial \mu_{21}} \log P(R=1|Y_1,Y_2) &= \frac{\partial}{\partial \mu_{21}} \varepsilon u_Y \\
&= \frac{\partial}{\partial \mu_{21}} \theta_2 (Y_2-\mu_{21}) \\
&= -\theta_2
\end{aligned}
$$

where $\varepsilon = \theta_2 \sigma_{21}$ and $u_Y = (Y_2-\mu_{21})/\sigma_{21}$. Hence $\partial \ell / \partial \mu_{21} = (Y_2-\mu_{21})/\sigma_{21}^2 - \theta_2$. Thus the score is:

$$
\begin{aligned}
s_Y &= \partial\{\log(f_Y)\}/\partial \beta_3 \\
&= \frac{\partial \ell}{\partial \mu_{21}} \frac{\partial \mu_{21}}{\partial \beta_3} \\
&= \frac{(Y_2-\mu_{21})}{\sigma_{21}^2} - \theta_2
\end{aligned}
$$

Similarly, we can calculate the second derivatives. The final result is $I^{-1} = \sigma_{21}^2$. Therefore the bias is

$$
\begin{aligned}
\beta_3 - \beta_3^* &= I^{-1} E[\varepsilon u_Y s_Y] \\
&= \sigma_{21}^2 E[\theta_2 (Y_2-\mu_{21}) s_Y] \\
&= \sigma_{21}^2 E\left[\theta_2 (Y_2-\mu_{21}) \left(\frac{(Y_2-\mu_{21})}{\sigma_{21}^2} - \theta_2\right)\right] \\
&= \theta_2 \sigma_{21}^2 E\left[\frac{(Y_2-\mu_{21})^2}{\sigma_{21}^2} - \theta_2 (Y_2-\mu_{21})\right] \\
&= \theta_2 \sigma_{21}^2 \left\{\frac{1}{\sigma_{21}^2} E\left[(Y_2-\mu_{21})^2\right] - \theta_2 E\left[(Y_2-\mu_{21})\right]\right\} \\
&= \theta_2 \sigma_{21}^2
\end{aligned}
$$

where $\varepsilon = \theta_2\sigma_{21}$ and $u_Y = (Y_2 - \mu_{21})/\sigma_{21}$, $E[(Y_2 - \mu_{21})^2]$ is the variance of the conditional normal distribution $= \sigma_{21}^2$, and $E[(Y_2 - \mu_{21})] = 0$.

References

[1] W. K. Ho, J. N. S. Matthews, R. Henderson, D. Farewell, and L.R. Rodgers. Dropouts in the ab/ba crossover design. *Statistics in Medicine*, 31:1675–1687, 2012.

[2] NL. Johnson, S. Kotz, and N. Balakrishnan. *Continuous Univariate Distributions*. Wiley, Chichester, 1995.