

## Research Article

# Analytically Simple and Computationally Efficient Results for the $GI^X/Geo/c$ Queues

Mohan L. Chaudhry,<sup>1</sup> James J. Kim ,<sup>2</sup> and Abhijit D. Banik<sup>3</sup>

<sup>1</sup>Department of Mathematics and Computer Science, Royal Military College of Canada, P.O. Box 17000, STN Forces, Kingston Ont, Canada K7K 7B4

<sup>2</sup>Royal Canadian Air Force (RCAF), 86 Moreuil Wood Blvd, Petawawa Ont, Canada K8H 1A5

<sup>3</sup>School of Basic Sciences, Indian Institute of Technology Bhubaneswar, Permanent Campus Argul, Jatni, Khurda-725 050, Odisha, India

Correspondence should be addressed to James J. Kim; s25412@rmc.ca

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A simple solution to determine the distributions of queue-lengths at different observation epochs for the model  $GI^X/Geo/c$  is presented. In the past, various discrete-time queueing models, particularly the multiserver bulk-arrival queues, have been solved using complicated methods that lead to incomplete results. The purpose of this paper is to use the roots method to solve the model  $GI^X/Geo/c$  that leads to a result that is analytically elegant and computationally efficient. This method works well even for the case when the inter-batch-arrival times follow heavy-tailed distributions. The roots of the underlying characteristic equation form the basis for all distributions of queue-lengths at different time epochs.

## 1. Introduction

The study of discrete-time queues is relatively recent compared to its continuous-time counterpart. Over the last few decades, the field quickly gained value among queueing theorists and researchers due to the digitization of information technology, particularly in the area of signal processing devices, microcomputers, and computer networks. In analyzing the discrete-time queues, many researchers have recognized that the continuous-time models are no longer suitable to accurately measure the performance of systems in which the basic operational units are digital, such as machine cycle time, bits, and packets (see Miyazawa and Takagi [1] for details). For further details on the discrete-time models and telecommunications, one may see Bruneel and Kim [2]. In this connection, see also Takagi [3].

In discrete-time queueing theory, various techniques have been introduced by many researchers to analyze the  $GI/Geo/1$  queues. Some of their techniques include the imbedded Markov chain, supplementary variable, semi-

Markov process, birth and death process, matrix-analytic, combinatorial, and numerical methods. What is pervasive across these techniques is in their final product: The  $GI/Geo/1$  queue has three distinct queue-length distributions at three different time epochs. Though each queue-length distribution has its own importance, they all have other purposes as well. For instance, the queue-length distribution at a pre-arrival time epoch is used to compute the actual waiting-time distribution, and in the case of queues with finite capacity, it is used to evaluate the blocking probabilities (see Chaudhry and Gupta [4]). The queue-length distribution at a random time epoch is needed to compute the virtual waiting-time distribution, whereas the queue-length distribution at an outside observer's time epoch is used to obtain the mean waiting-time-in-queue using Little's law. Moreover, relations between the queue-length distributions at different time epochs involve interesting mathematical derivations. For some of the earlier work on the discrete-time single-server queueing models, see Zukerman and Rubin [5].

In comparing the  $GI/Geo/1$  queue with its continuous-time counterpart  $GI/M/1$ , the main difference between the two models is in the measurement of time. While a continuous-time model has a time parameter that is a real number, a discrete-time model has a time parameter that is an integer. In the model  $GI/Geo/1$ , the time axis is divided into individual time slots where the duration of one time slot is a single unit of time. In each individual time slot, two events may occur: arrival and departure. When an arrival occurs before a departure, it is called  $GI/Geo/1$  with early arrival system (EAS), and when a departure occurs before an arrival, it is called  $GI/Geo/1$  with late arrival system (LAS). Further, in  $GI/Geo/1$  with LAS, if a server is idle and a customer arrives, then either his service can start immediately or in the next time slot. In the former case, it is known as an immediate access (IA), whereas in the latter case, it is known as a delayed access (DA). When discussing the  $GI/Geo/1$  type queues, there exist six queue-length distributions: pre-arrival, random, and outside observer's time epochs (three for EAS and three for LAS-DA).

The queueing model  $GI^X/Geo/1$  considers the model  $GI/Geo/1$  with a batch arrival. The earliest work on  $GI^X/Geo/1$  appears to be that of Vinck and Bruneel [6] who use the complex contour integration technique. Though they provide an analytical solution to  $GI^X/Geo/1$  with EAS at different time epochs, they do not provide the corresponding numerical results. Furthermore, in their work, the analysis of  $GI^X/Geo/1$  with LAS-DA is missing, yet it is deemed to be an important aspect of  $GI^X/Geo/1$  when considering applications in telecommunications (see Takagi [3]). In response to this, Chaudhry and Gupta [7] provide the first complete solution to  $GI^X/Geo/1$  using the supplementary variable technique. One of the main contribution of their work is that they do not stop after finding the probability generation function (p.g.f.) which was a common way to conclude the analysis of a queueing model at that time, but perform the inversion of the p.g.f. Doing so enables the finding of steady-state queue-length distributions in terms of the roots of the model's characteristic equation. This technique is referred to as the roots method.

Historically, the roots method was dismissed by some queueing theorists due to perceived difficulties in computing the roots of a model's characteristic equation. Neuts states (see Stidham [8]) "in discussing matrix-analytic solutions, I had pointed out that when the Rouché roots coincide or are close together, the method of roots could be numerically inaccurate. When I finally got copies of Crommelin's papers, I was elated to read that, for the same reasons as I, he was concerned about the clustering of roots. In 1932, Crommelin knew; in 1980, many of my colleagues did not. ..." Readers can refer to Neuts [9] for his other comments on the roots method. In the 1980s, commercial computing software such as MAPLE were not able to find the roots (they do now). To address the issue of root-finding in queueing theory, the QROOT software was developed by M. L. Chaudhry in the early 1990s and demonstrated that such roots can be found (see Chaudhry [10]). The roots method was then successfully adopted to solve a wide variety of queueing problems as noted by Janssen and van Leeuwen [11] who write

"initially, the potential difficulties of root-finding were considered to be a slur on the unblemished transforms since the determination of the roots can be numerically hazardous and the roots themselves have no probabilistic interpretation. However, Chaudhry et al. [12] have made every effort to dispel the skepticism towards root-finding in queueing theory. ..."

While the roots method is simply another way of solving queueing problems, there are added advantages to it as well. Gouweleeuw [13] states "it is more efficient to use the roots method to get explicit expressions for probabilities from generating functions." Furthermore, a recent paper by Maity and Gupta [14] compares the spectral theory approach against the roots method. Maity and Gupta [14] identify several difficulties in getting results using the spectral theory approach, an approach which may be simpler than the matrix-geometric approach as stated in several papers by Chakka (see, e.g., Chakka [15]). As well, Daigle and Lucantoni [16] state "whenever the roots method works, it works blindingly fast."

However, while the roots method historically only dealt with queues with light-tailed distributions, more recent research by Harris et al. [17] conclude that the roots method cannot solve queues with heavy-tailed inter-arrival times. The heavy-tailed distributions constitute a class of probability distributions that are characterized by slower decay rate than the exponential or geometric distribution. When considering the heavy-tailed distributions as an inter-arrival time distribution, the consensus among some researchers is that the roots method cannot be applied due to the unique probabilistic properties of the heavy-tailed random variables. In sharing this view, Harris et al. [17] state that "the standard root-finding problem gets complicated particularly when the inter-arrival time distribution possesses a complicated non-closed form or non-analytic Laplace-Stieltjes transform (L-ST)." The same difficulty persists in discrete-time queues since the discrete heavy-tailed probability distributions such as Weibull and Log-Normal distributions do not have a closed form of probability generating function (p.g.f.). In addition, the discrete Pareto distribution, for certain values of its parameter, has an infinite mean just like the continuous Pareto distribution.

Nevertheless, the heavy-tailed distributions are useful tools in modeling real-life examples (see Willinger and Paxon [18], Leland et al. [19], Park et al. [20, 21], and Pitkow [22]). In particular, the heavy-tailed distributions (or synonymously referred to as the power, long, or fat-tailed distribution) are particularly useful when modeling the inter-arrival times of network packets and connection sizes under heavy traffic congestion (see Harris et al. [17]). While the use of heavy-tailed distributions has predominantly been in the continuous-time realm, the use of discrete heavy-tailed distributions in probabilistic modeling is evident in a number of synchronous Time Division Multiplexed (TDM) communication systems such as slotted wireless communications channels, asynchronous transfer mode, and optical packet switching (see Hernández et al. [23]). In the field of medical science and biostatistics, Para and Jan [24] have recently discovered that the Burr-type XIII and Lomax

distributions are helpful in modeling a discrete data which exhibits heavy tails.

In this paper, we show that the model  $GI^X/Geo/c$  including the heavy-tailed inter-batch-arrival times can be solved through roots. In doing so, we express all queue-length distributions of  $GI^X/Geo/c$  in terms of the roots of the model's characteristic equation.

Our analytical proof on the existence of roots (see Appendix A) and the root-finding algorithm (see Section 11) form the foundation of our contribution. In our numerical analysis, we demonstrate how the roots of the characteristic equation behave within the contour of a unit circle. In doing so, several interesting results appear as we vary the traffic intensity, the batch size distribution, inter-batch-arrival times, and the maximum batch size.

While solving the model  $GI^X/Geo/c$  through roots directly addresses some statements made by several authors, there are added benefits to it as well. Queueing theory, under the scope of algorithmic efficiency, places heavy emphasis on achieving a simpler and more efficient solution procedure as it saves memory usage and computing-time. In our method, the roots are quickly found and the queue-lengths (which are all in terms of roots) can be computed efficiently. Efficiency equates to higher engineering productivity, whereas in a purely mathematical sense, a new approach to  $GI^X/Geo/c$  through roots reveals something new in the literature. For instance, the real root from the characteristic equation of  $GI^X/Geo/c$  can also be used to approximate the asymptotic loss probability in  $GI^X/Geo/c/S$  with a moderately sized  $S$ , where  $S$  is the maximum capacity of the system. Also, the characteristic equation of  $GI^X/Geo/c$  can be transformed into that of  $GI^X/M/c$ . Doing so enables us to compute the roots of the characteristic equation of  $GI^X/M/c$  with an inter-batch-arrival time distribution that has nonclosed and nonexistent L-STs. Lastly, it is identified that that some of the preexisting derivations in a model with light-tailed distributions are no longer applicable if the mean inter-batch-arrival time is infinite.

The roots method significantly simplifies and extends the similar work done by others using different techniques: In solving the non-bulk-arrival queue  $GI/Geo/c$ , Chan and Maa [25] use the imbedded Markov chain technique to derive the queue-length distribution for EAS at the pre-arrival epoch only. In this connection, one may also see Gao et al. who use the generating function approach to solve their non-bulk queues  $GI/Geo/c$  [26, 27] and  $GI/D/c$  [28, 29]. We have avoided that approach by using the difference-equation approach. Besides, none of the above authors consider heavy-tailed inter-arrival times. Similarly, Wittevrongel et al. [30] solve using complex analysis and contour integration while only considering the EAS with no numerical results. Lastly, Chaudhry et al. [31] solve using the supplementary variable technique that consider the light-tailed inter-batch-arrival times only. The results by Chaudhry et al. [31] are in terms of iterative relations between different queue-length distributions which can be analytically laborious and numerically inefficient. Our work also responds to the comments made by several mathematicians such as Kendall, Kleinrock, and Neuts [9]. Kendall [32] made a famous

remark that queueing theory wears the Laplacian curtain. Kleinrock [33] states "one of the most difficult parts of this method of spectrum factorization is to solve for the roots."

## 2. Model Description

Consider the model  $GI^X/Geo/c$  in which customers arrive in batches of size  $X$  with a maximum size  $r$  ( $0 < r < \infty$ ). The probability mass function (p.m.f.) of  $X$  is  $b_h = P(X = h)$  ( $1 \leq h \leq r$ ) with mean  $\bar{b} = \sum_{h=1}^r hb_h$  and p.g.f.  $B(z) = \sum_{h=1}^r b_h z^h$  ( $|z| \leq 1$ ). The  $n$ -th inter-batch-arrival time, say  $T_n$  ( $n \geq 1$ ), is a discrete-time period that is measured from the moment just before the  $n$ -th batch-arrival (say at time  $t$ ) to the moment just before the  $(n+1)$ -th batch-arrival (say at time  $t+m$ ). Inter-batch-arrival times are independent identically distributed random variables (i.i.d.r.v.'s) distributed as  $T$  which is divided into  $m$  time slots. The random variable  $T$  has a p.m.f.  $a_m = P(T = m)$  ( $a_0 = 0, m \geq 1$ ), mean  $\bar{a} = \sum_{m=1}^{\infty} ma_m = 1/\lambda$ , and p.g.f.  $A(z) = \sum_{m=1}^{\infty} a_m z^m$  ( $|z| \leq 1$ ). The batch size and the inter-batch-arrival times are independent. There are  $c$  servers in the model where each has service time  $Y$  that is independent to one another and geometrically distributed as

$$P(Y = y) = (1 - \mu)^{y-1} \mu, \quad 0 < \mu < 1, y \geq 1. \quad (1)$$

The mean service time of one server is  $E[Y] = (1/\mu)$  and the traffic intensity of the model is  $\rho = (\bar{b}\lambda/c\mu) < 1$ . In addition, let  $\omega(j|i)$  be the probability of an event that  $j$  customers are served out of  $i$  customers within a single time slot. The conditional probability  $\omega(j|i)$  then follows the binomial distribution

$$\omega(j|i) = \begin{cases} \binom{i}{j} \mu^j (1 - \mu)^{i-j}, & 0 \leq j \leq i < c, \\ \binom{c}{j} \mu^j (1 - \mu)^{c-j}, & 0 \leq j \leq c \leq i, \end{cases} \quad (2)$$

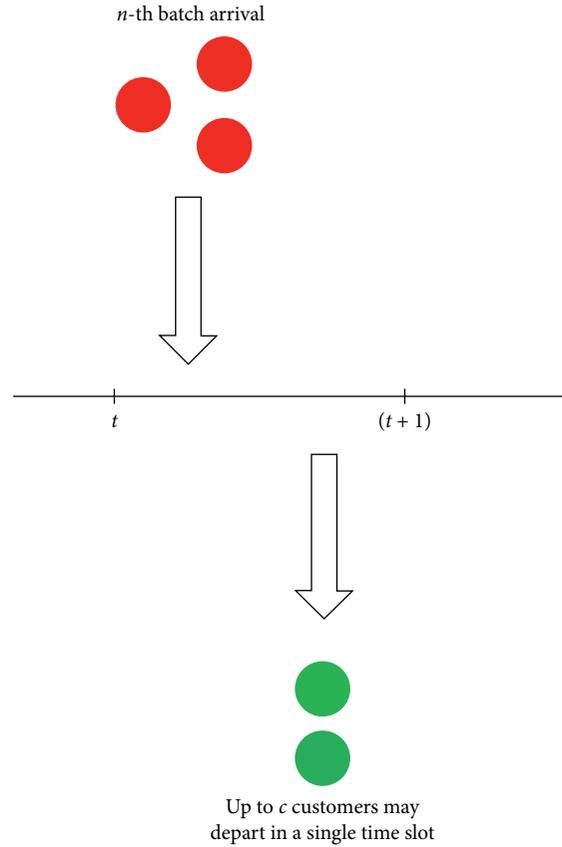
with  $\omega(0|0) = 1$  and  $(a/b) = 0$  for  $a < b$  or  $a < 0$ . Lastly, given that  $GI^X/Geo/c$  is a discrete-time queueing model, it has two different but related aspects (EAS and LAS-DA). Since the model is solved under the steady-state condition, the queue-length distribution of the Late Arrival System with Immediate Access (LAS-IA) is identical to that of the EAS.

## 3. $GI^X/Geo/c$ with EAS at a Pre-Arrival Time Epoch

In  $GI^X/Geo/c$  with EAS, the  $n$ -th batch arrival occurs before the departures in the  $n$ -th inter-batch arrival time. This is illustrated in Figure 1.

Let  $N_n^-$  be the number of customers in the system, including those, if any, receiving service just before the  $n$ -th batch arrival (i.e., at the  $n$ -th pre-arrival time epoch). The stochastic process  $\{N_n^-, n \geq 1\}$  forms a homogenous Markov chain:

$$N_{n+1}^- = N_n^- + X_n - D_n, \quad N_n^-, D_n \geq 0, X_n \geq 1, \quad (3)$$

FIGURE 1:  $GI^X/Geo/c$  with EAS.

where  $D_n$  is the number of customer departures during  $T_n$  and  $X_n$  is the size of the  $n$ -th batch arrival. In considering the steady-state aspect of  $GI^X/Geo/c$  with EAS,  $N_n^-$  becomes  $N^-$  as  $n \rightarrow \infty$  with a p.m.f.  $Q_j^- = \lim_{n \rightarrow \infty} P(N_n^- = j)$  ( $j \geq 0$ ) and  $X_n$  becomes  $X$  as  $n \rightarrow \infty$ . Let  $P_{i,j}(n) = P[N_n^- = j | N_0^- = i]$  ( $i, j \geq 0, n \geq 1$ ) be the  $n$ -step transition probabilities of  $\{N_n^-, n \geq 1\}$ . We define  $P_{i,j} = \lim_{n \rightarrow \infty} P_{i,j}(n)$  and then using the imbedded Markov chain analysis, we have the following transition probabilities:

$$P_{i,j} = \begin{cases} \sum_{h=1}^r b_h k_{i+h-j}, & i \geq 0, j \geq 1, \\ 1 - \sum_{k=1}^{\infty} P_{i,k}, & i \geq 0, j = 0, \end{cases} \quad (4)$$

where  $b_h = 0$  for  $h \leq 0$ . As  $n \rightarrow \infty$ ,  $D_n$  becomes  $D$  with p.m.f.  $k_j = \lim_{n \rightarrow \infty} P(D_n = j)$  ( $j \geq 0$ ), which is defined as

$$k_{i+h-j} = \begin{cases} 0, & i+h < j, \\ \sum_{m=1}^{\infty} a_m \left( \frac{cm}{i+h-j} \right) (1-\mu)^{cm-(i+h-j)} (\mu)^{i+h-j}, & c \leq j \leq i+h, \\ \sum_{m=1}^{\infty} a_m \left( \frac{i+h}{j} \right) (1-\mu)^{mj} (1-(1-\mu)^m)^{i+h-j}, & 1 \leq j \leq i+h < c, \\ \sum_{m=1}^{\infty} \left[ \sum_{w=1}^m \sum_{s=j}^c \sum_{l=c-s+1}^c \left( \frac{c(w-1)}{i+h-l-s} \right) \mu^{i+h-l-s} (1-\mu)^{c((w-1)-(i+h-l-s))} \right. \\ \left. \times \left( \frac{c}{l} \right) \mu^l (1-\mu)^{c-l} \left( \frac{s}{j} \right) (1-\mu)^{(m-w)j} (1-(1-\mu)^{(m-w)})^{(s-j)} \right] a_m, & 1 \leq j < c < i+h, \end{cases} \quad (5)$$

for  $1 \leq h \leq r$ . The  $P_{i,j}$ 's given above can be derived by replacing the single-arrival notion with a batch-arrival condition  $h(1 \leq h \leq r)$  in the one-step transition probabilities of  $GI/Geo/c$  which are available in Chan and Maa [25]. Furthermore, the same results are also available in Chaudhry et al. [31]. Lastly, in expressing  $\{Q_j^-, j \geq 0\}$  in terms of roots, the Chapman-Kolmogorov equation,

$$Q_j^- = \sum_{i=0}^{\infty} Q_i^- P_{i,j}, \quad j \geq 0, \quad (6)$$

is used extensively. Since (6) is a  $j$  set of first-order linear difference equations with  $\{Q_j^-\}$  being the unknown functions to be determined, we can assume a solution of a general form

$$Q_j^- = Cz^j, \quad j \geq \max(c, r), \quad (7)$$

where  $C$  is a nonzero constant. Note that  $\{Q_j^-, j \geq \max(c, r)\}$  is the pre-arrival queue-length distribution for the overloaded servers of  $GI^X/Geo/c$  with EAS since  $j \geq \max(c, r)$  indicates that all  $c$  servers are busy throughout  $T$ . Substituting the above into (6) yields the characteristic equation of  $GI^X/Geo/c$

$$1 = B(z^{-1})K(z), \quad (8)$$

where  $K(z) = \sum_{j=0}^{\infty} k_j z^j = \sum_{m=1}^{\infty} a_m (\mu z + 1 - \mu)^{cm} = A\{(1 - \mu + \mu z)^c\}$ . Since (8) has  $r$  roots inside the unit circle  $|z| = 1$  (see Appendix A for proof), let these roots be  $z_1, z_2, \dots, z_r$ . Hence,  $Q_j^-$  for  $j \geq \max(c, r)$  becomes  $r$ -fold and is expressed as

$$Q_j^- = \sum_{h=1}^r C_h z_h^j, \quad j \geq \max(c, r), \quad (9)$$

where  $C_h$  (yet to be evaluated) for  $1 \leq h \leq r$  are the nonzero constants. In completely finding  $\{Q_j^-, j \geq 0\}$ , we replace  $\{Q_j^-, j \geq \max(c, r)\}$  in (6) with (9) such that

$$Q_j^- = \sum_{i=0}^{\max(c-1, r-1)} Q_i^- P_{i,j} + \sum_{i=\max(c, r)}^{\infty} \sum_{h=1}^r C_h z_h^i P_{i,j}, \quad j \geq 0. \quad (10)$$

The normalizing condition

$$\sum_{j=0}^{\max(c-1, r-1)} Q_j^- + \sum_{j=\max(c, r)}^{\infty} \sum_{h=1}^r C_h z_h^j = 1, \quad (11)$$

is used in conjunction with (10) such that when  $c \leq r$  we let  $j = 1, 2, \dots, 2r - 1$  in (10) to have  $2r$  equations that are required to solve for  $\{Q_j^-, 0 \leq j \leq r - 1\}$  and  $\{C_h, 1 \leq h \leq r\}$ . On the contrary, when  $c > r$  we let  $j = 1, 2, \dots, c + r - 1$  in (10), and with the normalizing condition we have  $c + r$  equations that are required to solve for  $\{Q_j^-, 0 \leq j \leq c - 1\}$  and  $\{C_h, 1 \leq h \leq r\}$  (the  $C_h$ 's when  $c > r$  are different from those when  $c \leq r$ ). Solving these equations leads to the solution

$$Q_j^- = \begin{cases} \text{determined above,} & 0 \leq j \leq \max(c - 1, r - 1), \\ \sum_{h=1}^r C_h z_h^j, & j \geq \max(c, r). \end{cases} \quad (12)$$

#### 4. $GI^X/Geo/c$ with EAS at a Random Time Epoch

In the steady-state  $GI^X/Geo/c$  with EAS, a pre-arrival time epoch falls just before a batch-arrival, whereas a random time epoch falls at any instant between two consecutive batch arrivals. Let  $R_n$  be the discrete-time period measured from  $t$  to  $t + i (1 \leq i \leq m)$ . In addition,  $R_n$  are i.i.d.r.v.'s distributed as  $R$  with p.m.f.  $a_i^* = P(R = i) (a_0^* = 0, i \geq 1)$ . Visual illustration of  $R_n$  and  $T_n$  for  $GI^X/Geo/c$  with EAS are provided in Figure 2, where  $R_n$  is the duration of time measured from the  $n$ -th pre-arrival time epoch to the  $(n + 1)$ -th random time epoch, whereas  $T_n$  is the duration of time measured from the  $n$ -th pre-arrival time epoch to the  $(n + 1)$ -th pre-arrival time epoch.

Based on discrete renewal theory and the length-biased sampling phenomenon (see Chaudhry and Templeton [34] for definition and Kim [35] for similar application in continuous-time queues), if  $R$  consists of  $i$  time slots, then its p.m.f.  $\{a_i^*, 1 \leq i \leq m\}$  can be expressed in terms of  $\{a_m, m \geq 1\}$  such that

$$a_i^* = \frac{1}{a} (a_i + a_{i+1} + \dots + a_{m-2} + a_{m-1} + a_m), \quad 1 \leq i \leq m. \quad (13)$$

As  $m \rightarrow \infty$ , the above expression becomes

$$a_i^* = \frac{1}{a} \lim_{m \rightarrow \infty} \sum_{l=i}^m a_l, \quad i \geq 1. \quad (14)$$

In addition, let  $Q_j$  be the steady-state p.m.f. of the number of customers in the system at a random time epoch for  $GI^X/Geo/c$  with EAS. Using the above relation, we can determine  $Q_j$  with the following Chapman-Kolmogorov equation:

$$Q_j = \sum_{i=0}^{\infty} Q_i^- P_{i,j}^*, \quad j \geq 1, \quad (15)$$

where  $\{Q_i^-, i \geq 0\}$  are already known and  $P_{i,j}^*$ 's are the transition probabilities from Section 3, except that  $\{a_m\}$  is replaced with  $\{a_m^*\}$ . In addition,  $Q_0 = 1 - \sum_{j=1}^{\infty} \sum_{i=0}^{\infty} Q_i^- P_{i,j}^*$  holds. The GASTA property (Geometric Arrivals See Time Averages; see Takagi [3]) holds if  $a_l = \lambda(1 - \lambda)^{l-1} (l \geq 1)$  is substituted into the relation between  $a_m$  and  $a_m^*$ . Doing so leads to  $a_i^* = a_i (i \geq 1)$ ; hence, due to this property,  $Q_j = Q_j^- (j \geq 0)$  in  $Geo^X/Geo/c$ .

#### 5. $GI^X/Geo/c$ with EAS at an Outside Observer's Time Epoch

By definition, an outside observer's time epoch in  $GI^X/Geo/c$  with EAS falls anywhere just after a batch-arrival and

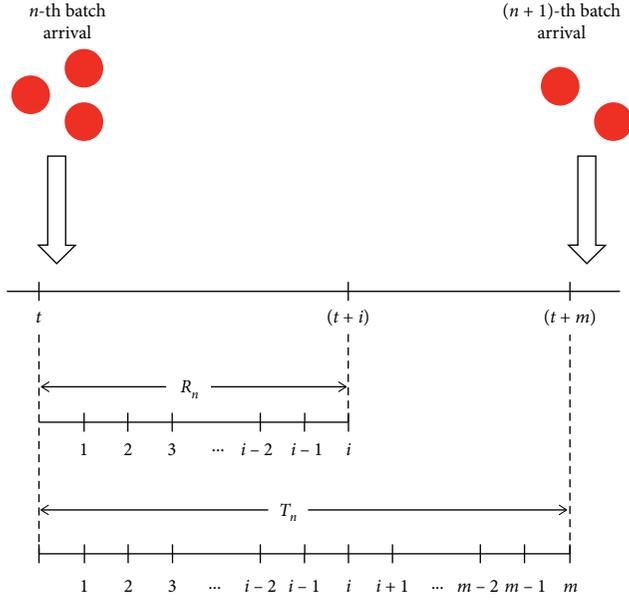


FIGURE 2:  $GI^X/Geo/c$  with EAS at pre-arrival and random time epochs.

immediately before an event of departures. Let  $\{Q_j^o, j \geq 0\}$  be the steady-state queue-length distribution of  $GI^X/Geo/c$  with EAS at an outside observer's time epoch.

In Figure 3, the  $n$ -th inter-batch-arrival time period is a single time slot (i.e.,  $m = 1$ ); hence, the  $(n + 1)$ -th random time epoch and the  $(n + 1)$ -th pre-arrival time epoch coincide at  $t + 1$  (this can be checked by letting  $m = 1$  in Figure 2).

In Figure 3, if there are  $j$  customers in the system at an outside observer's time epoch (with probability  $Q_j^o$ ) and there are  $l$  customers in the system at a random time epoch (with probability  $Q_l$ ), then there must be an event of  $j - l$  customer departures (with probability  $\omega(j - l | j)$ ) between an outside observer's time epoch and a random time epoch. Based on this notion, we form the relation

$$Q_l = \sum_{j=l}^{l+c} Q_j^o \omega(j - l | j), \quad l \geq 0. \quad (16)$$

Since (16) is an  $l$  set of first-order linear difference equations with  $\{Q_j^o\}$  being the unknown functions to be determined, we assume that  $Q_j^o$  for  $j \geq \max(c, r)$  is a geometric sum that consists of the same roots of (8) but with different constant coefficients (hence they are unknown). Let these unknown constant terms be  $E_h$  ( $1 \leq h \leq r$ ), then we have

$$Q_j^o = \sum_{h=1}^r E_h z_h^j, \quad j \geq \max(c, r), \quad (17)$$

which is the p.m.f. of the steady-state number of customers in the system for the overloaded servers of  $GI^X/Geo/c$  with EAS at an outside observer's time epoch. By replacing  $\{Q_j^o, j \geq \max(c, r)\}$  in (16) with the above geometric sum, we have

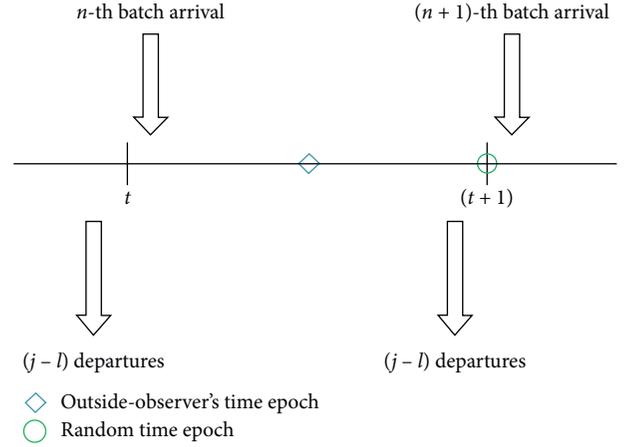


FIGURE 3: Outside observer's and random time epochs in  $GI^X/Geo/c$  with EAS.

$$Q_l = \sum_{j=l}^{\max(c-1, r-1)} Q_j^o \omega(j - l | j) + \sum_{j=\max(c, r)}^{l+c} \sum_{h=1}^r E_h z_h^j \omega(j - l | j), \quad l \geq 0, \quad (18)$$

with the normalizing condition

$$\sum_{j=0}^{\max(c-1, r-1)} Q_j^o + \sum_{j=\max(c, r)}^{\infty} \sum_{h=1}^r E_h z_h^j = 1. \quad (19)$$

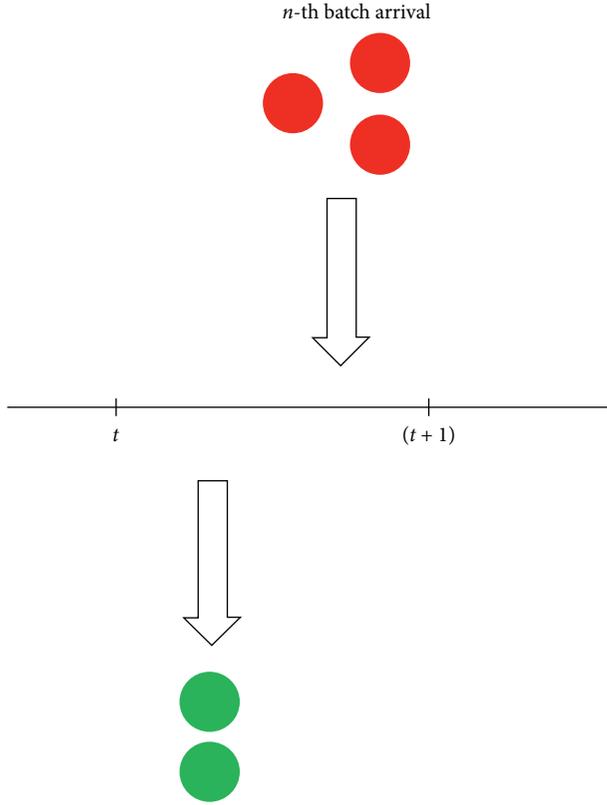
When  $c \leq r$  we let  $l = 1, 2, \dots, 2r - 1$  in (18), and with (19) it leads to  $2r$  equations that are required to solve for  $\{Q_j^o, 0 \leq j \leq r - 1\}$  and  $\{E_h, 1 \leq h \leq r\}$ . On the contrary, when  $c > r$  we let  $l = 1, 2, \dots, c + r - 1$  in (18), and with (19) it leads to  $c + r$  equations that are required to solve for  $\{Q_j^o, 0 \leq j \leq c - 1\}$  and  $\{E_h, 1 \leq h \leq r\}$  (the  $E_h$ 's when  $c > r$  are different from those when  $c \leq r$ ). Solving these equations leads to an expression of  $Q_j^o$  for  $j \geq 0$  as

$$Q_j^o = \begin{cases} \text{determined above,} & 0 \leq j \leq \max(c - 1, r - 1), \\ \sum_{h=1}^r E_h z_h^j, & j \geq \max(c, r). \end{cases} \quad (20)$$

## 6. $GI^X/Geo/c$ with LAS-DA at a Pre-Arrival Time Epoch

In  $GI^X/Geo/c$  with LAS-DA, the  $n$ -th batch arrival occurs after the departures in the  $n$ -th inter-batch arrival time. This is illustrated in Figure 4.

Let  $M_n^-$  be the number of customers in the system, including the ones, if any, in service just before the  $n$ -th batch arrival. The random variable  $M_n^-$  becomes  $M^-$  and has a p.m.f.  $P_j^- = \lim_{n \rightarrow \infty} P(M_n^- = j)$  ( $j \geq 0$ ) as we assume the steady-state condition ( $n \rightarrow \infty$ ). Before proceeding to find  $P_j^-$ , it is worth noting that  $P_j^-$  can be found independently of  $Q_k^-$  ( $0 \leq j \leq k$ ) (as done by Chaudhry and Kim [36] in solving



Up to  $c$  customers may depart in a single time slot

FIGURE 4:  $GI^X/Geo/c$  with LAS-DA.

the model  $GI^X/Geo/1$ ). However, we leverage  $Q_k^-$  from Section 3 to determine  $P_j^-$  for the purpose of demonstrating continuity in our solution procedure. Let  $P_j^-(s)$  be the p.m.f. of there being  $j$  customers in the system  $GI^X/Geo/c$  with LAS-DA at time  $s$ , and let  $Q_k^-(t)$  be the p.m.f. of there being  $k$  ( $0 \leq k \leq j$ ) customers in the system  $GI^X/Geo/c$  with EAS at time  $t$ . Figure 5 illustrates that in EAS, the  $n$ -th batch arrives at the beginning of the time slot  $(t, t + 1)$ , whereas in LAS-DA, the same  $n$ -th batch would arrive at the end of the time slot  $(s, s + 1)$ .

If up to  $c$  customers depart during  $(s, t)$ , then we can develop a relation between  $Q_k^-(t)$  and  $P_j^-(s)$  as

$$Q_k^-(t) = \sum_{j=k}^{n+c} P_j^-(s) \omega(j - k | j), \quad k \geq 0. \quad (21)$$

As a remark,  $0 \leq k \leq j$  indicates that if no customers depart during  $(s, t)$ , then  $k = j$  with probability  $\omega(0 | j)$ , whereas any customer departures during  $(s, t)$  result in  $k < j$  with probability  $\omega(j - k | j)$ . By letting  $Q_n^- = \lim_{n \rightarrow \infty} Q_n^-(t)$  ( $n \geq 0$ ) and  $P_j^- = \lim_{n \rightarrow \infty} P_j^-(s)$  ( $j \geq 0$ ), so that we have the steady-state queue-length distributions, we have

$$Q_k^- = \sum_{j=k}^{n+c} P_j^- \omega(j - k | j), \quad k \geq 0. \quad (22)$$

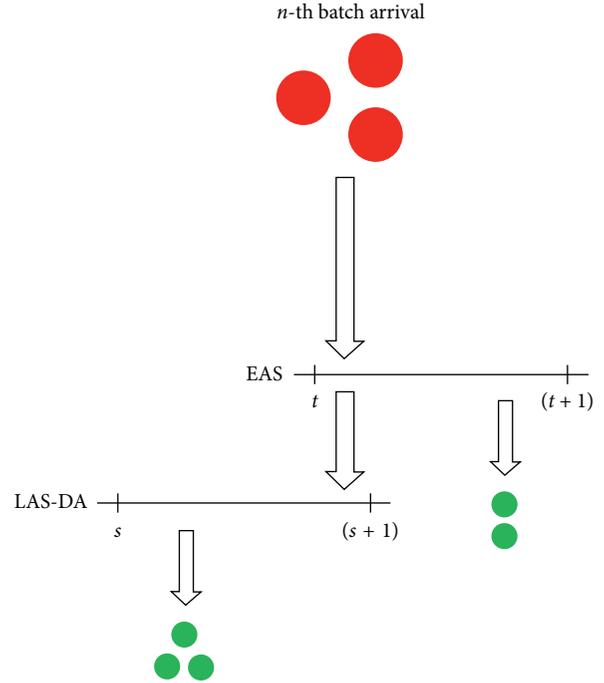


FIGURE 5: A comparison between EAS and LAS-DA at the  $n$ -th batch-arrival.

Since (22) is a  $k$  set of first-order linear difference equations with  $\{P_j^-\}$  being the unknown functions to be determined, we can make a similar assumption as the one made in Sections 3 and 5, and let

$$P_j^- = \sum_{h=1}^r F_h z_h^j, \quad j \geq \max(c, r + 1) \quad (23)$$

be the p.m.f. of the steady-state number of customers in the system for the overloaded servers of  $GI^X/Geo/c$  with LAS-DA at a pre-arrival time epoch. The  $F_h$ 's in (23) are the unknown nonzero constants. As a remark, (23) differs from (9) since we have  $j \geq \max(c, r + 1)$  in (23) but  $j \geq \max(c, r)$  in (9). This difference is due to the property of  $GI^X/Geo/c$  with LAS-DA: assume that there are initially no customers in the system and then a batch of customers arrives after some inter-batch-arrival time. If the next inter-batch-arrival time is one time slot (with probability  $a_1$ ), then no customers access any of the  $c$  servers, whereas if the inter-batch-arrival time is at least two time slots (with probability  $a_m$ ,  $m \geq 2$ ), then customers start accessing the idle servers at the beginning of the second time slot. This property results in the definition of the overloaded servers of  $GI^X/Geo/c$  with LAS-DA to come into effect when  $j \geq \max(c, r + 1)$ . To solve for  $\{P_j^-, 0 \leq j \leq \max(c - 1, r)\}$  and  $F_h$ 's, (22) is alternatively expressed as

$$Q_k^- = \sum_{j=k}^{\max(c-1, r)} P_j^- \omega(j - k | j) + \sum_{j=\max(c, r+1)}^{k+c} \sum_{h=1}^r F_h z_h^j \omega(j - k | j), \quad k \geq 0. \quad (24)$$

We also consider the normalizing condition

$$\sum_{j=0}^{\max(c-1,r)} P_j^- + \sum_{j=\max(c,r+1)}^{\infty} \sum_{h=1}^r F_h z_h^j = 1, \quad (25)$$

such that when  $c \leq r+1$  we let  $k = 1, 2, \dots, 2r$  in (24), and with (25) it leads to  $2r+1$  equations that are required to solve for  $\{P_j^-, 0 \leq j \leq r\}$  and  $\{F_h, 1 \leq h \leq r\}$ . On the contrary, when  $c > r+1$  we let  $k = 1, 2, \dots, c+r-1$  in (24), and with (25) it leads to  $c+r$  equations that are required to solve for  $\{P_j^-, 0 \leq j \leq c-1\}$  and  $\{F_h, 1 \leq h \leq r\}$  (the  $F_h$ 's when  $c > r+1$  are different from those when  $c \leq r+1$ ). Solving these equations leads to  $\{P_j^-, j \geq 0\}$  as

$$P_j^- = \begin{cases} \text{determined above,} & 0 \leq j \leq \max(c-1, r), \\ \sum_{h=1}^r F_h z_h^j, & j \geq \max(c, r+1). \end{cases} \quad (26)$$

## 7. $GI^X/Geo/c$ with LAS-DA at Random and Outside-Observer's Time Epochs

Let  $P_j$  and  $P_j^o$  be the p.m.f.'s of the number of customers in system in  $GI^X/Geo/c$  with LAS-DA at random and

outside observer's time epochs, respectively. In LAS-DA, an outside observer's time epoch falls in a time interval that begins just after a departure and immediately before a batch-arrival. Based on this notion,  $P_j^o$  is the same as  $P_j$ . In addition, we have

$$Q_j^o = P_j^o = P_j, \quad (27)$$

where  $Q_j^o$  is available in (20).

## 8. p.m.f. of Waiting-Time-in-Queue and Performance Measures

Let  $T_q$  be the amount of time an arbitrary customer of an incoming batch spends in queue upon batch-arrival and until entering service. As mentioned by Chaudhry and Templeton [34], if the position of an arbitrary customer of an incoming batch is  $F$ , then its p.m.f. is  $P(F=f) \equiv r_f = \sum_{h=f}^{\infty} (b_h/\bar{b})$  ( $1 \leq f \leq r$ ). Since the system capacity of the  $GI^X/Geo/c$  queues is infinite, the waiting time distribution remains the same in both LAS-DA and EAS (see Hunter [37] and Chaudhry and Gupta [7]). Let the p.m.f. of  $T_q$  be  $w_k = P(T_q = k)$ , ( $k \geq 0$ ). As stated by Chaudhry et al. [31],  $w_k$  is

$$w_k = \begin{cases} \sum_{i=1}^c \hat{Q}_i, & k = 0, \\ \sum_{d=1}^{ck} \sum_{j=\max(0,d-c)}^{d-1} \binom{c(k-1)}{j} \mu^j (1-\mu)^{c(k-1)-j} \sum_{l=d-j}^c \binom{c}{l} \mu^l (1-\mu)^{c-l} \hat{Q}_{c+d}, & k \geq 1, \end{cases} \quad (28)$$

where  $\hat{Q}_i = \sum_{l=0}^{i-1} Q_l^- r_{i-l}$  ( $i \geq 1$ ).

## 9. Performance Measures

The model  $GI^X/Geo/c$  has various performance measures. Since the distributions are known, various moments can be calculated. In particular, denote  $L_s^o$  as the mean number of customers in the system and  $L_q^o$  as the mean number of customers in queue, both at an outside observer's time epoch.  $L_s^o$  and  $L_q^o$  can be found in terms of  $Q_n^o$  such that

$$L_s^o = \sum_{n=1}^{\infty} n Q_n^o, \quad (29)$$

$$L_q^o = \sum_{n=c+1}^{\infty} (n-c) Q_n^o.$$

The mean waiting-time in the system and the mean waiting-time-in-queue can also be found using Little's law:

$$W_s = \frac{\bar{a} L_s^o}{b}, \quad (30)$$

$$W_q = \frac{\bar{a} L_q^o}{b}. \quad (31)$$

In addition,  $W_q$  can be found separately by using the result in Section 8, which is  $W_q = \sum_{k=1}^{\infty} k w_k$ . Furthermore, it can be shown that the average number of idle servers can be found using the expression

$$\sum_{j=0}^{c-1} (c-j) Q_j^o = c(1-\rho). \quad (32)$$

The left-hand side of (32) determines the average number of idle servers using the queue-length-distribution at an outside observer's time epoch. The right-hand side of (32) determines the same number, except that it is independent of  $\{Q_j^-, j \geq 0\}$  (hence independent of roots).

## 10. Approximation of Loss Probability in $GI^X/Geo/c/S$ Using the Loss Formula

In finite-buffer queues, the loss probability is defined as the probability of a customer being rejected from the system. In the case of finite-buffer queues with bulk-arrivals, an incoming batch of customers may be partially or totally rejected, depending on how close the number of customers in the system is to the finite-buffer immediately prior to a batch-arrival. Gouweleeuw [13] and Gouweleeuw and Tijms

[38] approximate the loss probability of a wide range of queueing models with general input such as  $BMAP/G/1/S$  and  $GI/G/1/S$ . In particular, while discussing the discrete-time model  $GI^X/G/c/S$ , Gouweleew [13] states that “. . . an enormous amount of methods to calculate or approximate the loss probability of a customer in such a queueing system is present in the literature. . . . An overview is presented of those methods which explore the relation between the finite-buffer queue and its infinite-buffer counterpart.” From this statement, it is understandable that Gouweleew’s [13] main effort is in presenting an efficient and accurate approximation of loss probabilities using the steady-state queue-length distribution of the infinite-buffer counterpart.

The asymptotic loss probability is the loss probability as  $S$  becomes a large integer. In discussing the asymptotic loss probability of  $GI^X/M/c/S$ , Kim and Choi [39] express the asymptotic loss probabilities of  $GI^X/M/c/S$  in terms of the roots of the model’s characteristic equation. They prove that there is exactly one positive real root (say  $z_1$ ) that has the largest modulus among the  $r$  roots. Kim and Choi [39] go further to claim that  $z_1$  with its corresponding constant coefficient ( $C_1$ ) can accurately approximate the asymptotic loss probability, even for a moderately sized  $S$ . By this claim, we can also accurately approximate the loss probability of  $GI^X/Geo/c/S$  (both total and partial rejections). Depending on whether the model involves EAS or LAS-DA, we can easily identify  $z_1$  from (8) as well as  $C_1$  and  $F_1$  from Sections 3 and 6, respectively, and approximate the asymptotic loss probabilities of  $GI^X/Geo/c/S$ .

## 11. $GI^X/Geo/c$ Involving Heavy-Tailed Inter-Batch-Arrival Times

The heavy-tailed distributions distinguish themselves from the light-tailed distributions by having a significantly slower rate of decay. In  $GI^X/Geo/c$ , a slow decaying arrival-time distribution renders a very lengthy (or an infinite) mean inter-batch-arrival time that equates to a very small (or a zero) arrival rate ( $\lambda$ ). The arrival rate is directly proportional to  $\rho$ , and as  $\rho \rightarrow 0$ , the roots of the characteristic equation of  $GI^X/Geo/c$  tend towards the origin (see, e.g., Table 1) which was a concern to some researchers.

The heavy-tailed distributions are believed to impose another challenge on the roots method due to the nonclosed form of p.g.f.’s. While the light-tailed distributions have closed form of  $A(z)$  (see Table 2), this is not the case for Weibull, Standard Log-Normal (SLN), and Pareto distributions. Moreover, directly solving (8) with an  $a_m$  that follows the Pareto distribution with  $M \leq 1$  will consume a lengthy computing time (or not compute at all) due to the infinite series of a p.m.f. that decays at an extremely slow rate.

As a remark, in  $Pareto[M]$  the function  $\varphi(\cdot)$  is the Riemann zeta function. The  $k$ -th moment of  $Pareto[M]$  exists as  $E[T^k] = \varphi(M - k + 1)/(\varphi(M + 1))$  if and only if  $1 \leq k < M$  is true. On the contrary, the moments become infinite if  $k \geq M$ .

Solving the  $GI^X/Geo/c$  queueing model involving the heavy-tailed inter-batch-arrival times requires a different numerical approach from the one traditionally used to deal

with the light-tailed inter-batch-arrival time distributions. To accomplish this, we express  $K(z)$  in (8) as

$$K_N(z) = \sum_{m=1}^N a_m (\mu z + 1 - \mu)^{cm}. \quad (33)$$

Modern computing software such as MAPLE can easily determine the roots of (8) when using the above expression of  $K(z)$  with  $N$  being a positive integer. One must choose an adequately sized  $N$  given that the value of  $N$  is indirectly proportional to the rate of decay of  $a_m$  (larger  $N$  is required for  $a_m$  with a slower decay). To offset this balance, we have implemented a simple algorithm in MAPLE that determines  $N$  as well as the roots of (8) (Algorithm 1).

To run the above program,  $\mu$ ,  $a(m)$ ,  $b(h)$ ,  $c$  and  $r$  must be defined. In line (8) of our program the LC of  $10^{-400}$  is used as the default value but it can be scaled depending on the desired degree of accuracy. In addition, the term  $\mu^{c*N}$  in lines (9) and (25) of our program were chosen as the demarcation point to determine  $N$  since it is the constant coefficient of the last term of  $z$  in  $K_N(z)$ ; hence,  $LC$  stands for the Last Coefficient. Line (44) plots all  $r$  roots that are found within the contour of a unit circle  $|z| = 1$ .

In comparing our root-finding method with the one by Harris et al. [17], we first briefly summarize their analytical foundation: Harris et al. [17] solve the continuous-time single-arrival (i.e., single root) models using the transform approximation method (TAM). TAM approximates the L-ST of a heavy-tailed inter-arrival distribution by discrete summation consisting of “ $N$ ” terms such that

$$K_N(z) = \frac{1}{N} \sum_{k=1}^N e^{-zX(k)}, \quad (34)$$

where  $X(k)$ ,  $k = 1, 2, \dots, N$ , are chosen to cover the outcome space of the original inter-arrival random variable with  $K(X(j)) = (j - 0.5)/N$ . Ultimately,  $N$  is manually picked to satisfy  $\lim_{n \rightarrow \infty} K_N(z) = K(z)$ , where  $K(z)$  is the L-ST of a heavy-tailed inter-arrival distribution such that  $K_N(z) = \bar{a}_N (\mu(1 - z))$  for  $GI/M/1$  and  $K_N(z) = \bar{a}_N (c\mu(1 - z))$  for  $GI/M/c$ . Therefore, once  $N$  is known, solving the equation

$$z = K_N(z), \quad (35)$$

results in the root that is needed to solve the single-arrival model. In explaining TAM, Harris et al. [17] place caveat on the fact that TAM results in  $N$  as large as  $10^6$ . Moreover, Harris et al. [17] lay the analytical foundation of TAM based on the single-arrival models such as  $GI/M/1$  and  $GI/M/c$ . Though their method could be extended to bulk-arrival models (i.e., multiple roots), it may not only lead to a laborious analytical foundation, but will require even larger  $N$  to approximate the L-ST of a heavy-tailed inter-batch arrival distribution.

In our numerical results, we have employed  $N = 333$  to determine 1,000 roots of the characteristic equation in  $GI^X/M/c$  when the inter-batch-arrival times follow a heavy-tailed distribution with an infinite mean. In addition, we also compute the roots of the characteristic equation for  $GI^X/M/c$  with heavy-tailed inter-batch-arrival times (see Section 12); hence, a direct comparison between the TAM by Harris et al.

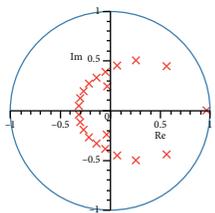
TABLE 1: Discrete probability distributions.

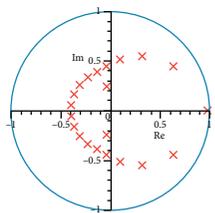
$GI$	$a_m (m \geq 1)$	$A(z) ( z  \leq 1)$	Parameter(s)	Mean
<i>Geometric</i> [ $\lambda$ ]	$\lambda(1-\lambda)^{m-1}$	$\lambda z / (1 - (1-\lambda)z)$	$\lambda > 0$	$1/\lambda$
<i>Poisson</i> [ $\lambda$ ]	$(\lambda^{m-1} / (m-1)!) e^{-\lambda}$	$z e^{-\lambda(1-z)}$	$\lambda > 0$	$1 + \lambda$
<i>Weibull</i> [ $\beta$ ]	$\beta^{-\sqrt{m}} / (\beta^{\sqrt{m}} - 1)$	$(1 / (\beta^{\sqrt{m}} - 1)) \sum_{m=1}^{\infty} \beta^{-\sqrt{m}} z^m$	$\beta' = \sum_{j=0}^{\infty} \beta^{-\sqrt{j}}, \beta > 0$	$\sum_{m=1}^{\infty} (m \beta^{-\sqrt{m}} / (\beta^{\sqrt{m}} - 1))$
<i>SLN</i>	$\varphi e^{-(\ln^2(m)/2)}$	$\varphi \sum_{m=1}^{\infty} e^{-(\ln^2(m)/2)} z^m$	$\varphi = \sum_{j=1}^{\infty} e^{-(\ln^2(j)/2)}$	$\varphi \sum_{m=1}^{\infty} m e^{-(\ln^2(m)/2)}$
<i>Pareto</i> [ $M$ ]	$\delta / (m^{M+1})$	Does not exist	$M > 0,$ $\delta = \varphi (M+1)^{-1}$	$\begin{cases} \infty, & (M \leq k) \\ \varphi (M) / (\varphi (M+1)), & (M > k) \end{cases}$

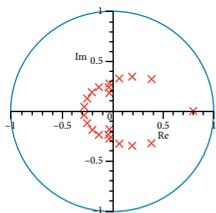
TABLE 2: Binomial batch size distribution with  $b_h = (r/(h-1))p^{h-1}q^{r-h+1}$  ( $p = 0.45, q = 0.55, r = 21$ ) and  $\rho = 0.8$ .

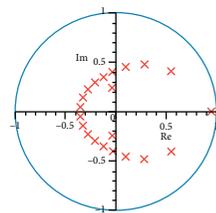
$z_h$	$GI$			
	$GI^X/Geo/5$	$GI$	$GI^X/M/5$	$GI$
	<i>SLN</i>	<i>Weibull</i> [2]	<i>Pareto</i> [4.5, 2]	<i>IG</i> [2, 3]
$z_1$	0.9656	0.9786	0.8031	0.9453
$z_2$	$0.5625 - 0.4433i$	$0.0977 - 0.5157i$	$0.3845 - 0.3232i$	$0.5494 - 0.4076i$
$z_3$	$0.5625 + 0.4433i$	$0.0977 + 0.5157i$	$0.3845 + 0.3232i$	$0.5494 + 0.4076i$
$z_4$	$0.2578 - 0.5013i$	$0.3205 - 0.5482i$	$0.1938 - 0.3462i$	$0.2830 - 0.4770i$
$z_5$	$0.2578 + 0.5013i$	$0.3205 + 0.5482i$	$0.1938 + 0.3462i$	$0.2830 + 0.4770i$
$z_6$	$0.0669 - 0.4518i$	$0.6354 - 0.4463i$	$0.0655 - 0.3256i$	$0.0970 - 0.4537i$
$z_7$	$0.0669 + 0.4518i$	$0.6354 + 0.4463i$	$0.0655 + 0.3256i$	$0.0970 + 0.4537i$
$z_8$	$-0.0364 - 0.2412i$	$-0.3959 - 0.0553i$	$-0.0337 - 0.2760i$	$-0.0364 - 0.2412i$
$z_9$	$-0.0364 + 0.2412i$	$-0.3959 + 0.0553i$	$-0.0337 + 0.2760i$	$-0.0364 + 0.2412i$
$z_{10}$	$-0.0496 - 0.3859i$	$-0.3663 - 0.1622i$	$-0.2536 - 0.1213i$	$-0.3530 - 0.0487i$
$z_{11}$	$-0.0496 + 0.3859i$	$-0.3663 + 0.1622i$	$-0.2536 + 0.1213i$	$-0.3530 + 0.0487i$
$z_{12}$	$-0.1391 - 0.3290i$	$-0.3087 - 0.2578i$	$-0.2789 - 0.0417i$	$-0.3275 - 0.1428i$
$z_{13}$	$-0.1391 + 0.3290i$	$-0.3087 + 0.2578i$	$-0.2789 + 0.0417i$	$-0.3275 + 0.1428i$
$z_{14}$	$-0.2149 - 0.2670i$	$-0.2270 - 0.3348i$	$-0.1367 - 0.2398i$	$-0.2071 - 0.2973i$
$z_{15}$	$-0.2149 + 0.2670i$	$-0.2270 + 0.3348i$	$-0.1367 + 0.2390i$	$-0.2071 + 0.2973i$
$z_{16}$	$-0.3160 - 0.0388i$	$-0.1330 - 0.3877i$	$-0.2052 - 0.1893i$	$-0.2779 - 0.2275i$
$z_{17}$	$-0.3160 + 0.0388i$	$-0.1330 + 0.3877i$	$-0.2052 + 0.1893i$	$-0.2779 + 0.2275i$
$z_{18}$	$-0.3012 - 0.1166i$	$-0.0414 - 0.4447i$	$-0.0332 - 0.1820i$	$-0.1219 - 0.3499i$
$z_{19}$	$-0.3012 + 0.1166i$	$-0.0414 + 0.4447i$	$-0.0332 + 0.1820i$	$-0.1219 + 0.3499i$
$z_{20}$	$-0.2688 - 0.1941i$	$-0.0364 - 0.2412i$	$-0.0406 - 0.2424i$	$-0.0282 - 0.4006i$
$z_{21}$	$-0.2688 + 0.1941i$	$-0.0364 + 0.2412i$	$-0.0406 + 0.2424i$	$-0.0282 + 0.4006i$
$N$	91	144	49	81
$LC$	$10^{-110}$	$10^{-520}$	$10^{-4}$	$10^{-4}$

$|z| = 1$









[17] and our method can be made in terms of the size of  $N$ : finding the roots at a smaller  $N$  equates to higher efficiency and shorter computing time.

## 12. Deriving the Root Equation: From $GI^X/Geo/c$ to $GI^X/M/c$

Here we derive the root equation of  $GI^X/M/c$ . We also explain how to compute the roots of the characteristic equation

of  $GI^X/M/c$  involving heavy-tailed inter-batch-arrival times. Let  $t$  be the inter-batch-arrival time of  $GI^X/M/c$  such that it is a nonzero real-number ( $t \in \mathbb{R}^*$ ). It has a distribution function  $A(t) = P(\hat{T} \leq t) (t > 0)$  with the mean  $E[\hat{T}]$  and arrival rate  $\hat{\lambda}$  such that  $E[\hat{T}] = 1/\hat{\lambda}$ . Similarly, let  $\hat{Y}$  be the exponential service time of a server with the mean  $E[\hat{Y}]$  and exponential service rate  $\hat{\mu}$  ( $0 < \hat{\mu} < 1$ ) such that  $E[\hat{Y}] = 1/\hat{\mu}$ . In  $GI^X/Geo/c$ ,  $T$  is divided into  $m$  time slots, and multiplying  $m$  by a very small, real number (say  $\Delta$ ) equates to a

```

(1) with (plots):
(2) LC := 10-400:
(3) μ :=
(4) a := (m) →
(5) b := (h) →
(6) c :=
(7) r :=
(8) N := 1:
(9) while μc*N * a(N) > LC do
(10)   N := N + 1:
(11) end do:
(12) if μc*N * a(N) ≤ LC then
(13)   N := N - 1:
(14) end if:
(15) y := zr - expand(zr * ∑h=1r b(h) * z-h ∑m=1N (μ * z + 1 - μ)c*m * a(m))
(16) AllRoots := [f solve(y, z, {z = -1 - I..1 + I}, complex)]:
(17) multiple(complexplot, [AllRoots, symbol = diagonalcross, symbolsize = 10,
    style = point, labels = ["Re", "Im"], color = "Re d", [cos + sin I, -π..π, color = "SteelBlue"]])

```

ALGORITHM 1

continuous inter-batch-arrival time  $t$ . Based on this notion, we have  $t = \Delta m$  (similarly,  $E[\hat{T}] = \Delta E[T]$  and  $E[\hat{Y}] = \Delta E[Y]$ ).

To derive the root equation of  $GF^X/M/c$ , we consider the  $k_{i+h-j}$  from Section 2 in a form that is prior to being conditioned on  $a_m$  and  $b_h$  which is

$$\left(\frac{cm}{i+h-j}\right) \mu^{i+h-j} (1-\mu)^{cm-(i+h-j)},$$

$$m \geq 1, 1 \leq h \leq r, i \geq 0, j \geq 1, i+h-j \geq cm. \quad (36)$$

Substituting  $m = t/\Delta$  and  $\mu = \Delta\hat{\mu}$  into the above equation gives

$$\frac{(ct/\Delta)!}{(i+h-j)!((ct/\Delta)-(i+h-j))!} (\Delta\hat{\mu})^{i+h-j} (1-\Delta\hat{\mu})^{(ct/\Delta)-(i+h-j)}. \quad (37)$$

To transform the  $k_{i+h-j}$  of  $GF^X/Geo/c$  into the  $\hat{k}_{i+h-j}$  of  $GF^X/M/c$ , taking the limit of the above result as  $\Delta \rightarrow 0$  (so that the discrete time parameter becomes the continuous time parameter) gives

$$\lim_{\Delta \rightarrow 0} \frac{(ct/\Delta)!}{(i+h-j)!((ct/\Delta)-(i+h-j))!} (\Delta\hat{\mu})^{i+h-j} (1-\Delta\hat{\mu})^{(ct/\Delta)-(i+h-j)}. \quad (38)$$

Multiplying and then dividing the right-hand-side by  $(ct)^{i+h-j}$  ( $t > 0, c \geq 1, i+h \geq j$ ), it becomes

$$\lim_{\Delta \rightarrow 0} \frac{(ct/\Delta)!}{(ct/\Delta)^{i+h-j}((ct/\Delta)-(i+h-j))!} \frac{(ct\hat{\mu})^{i+h-j}}{(i+h-j)!} \cdot \left[ (1-\Delta\hat{\mu})^{1/\Delta} \right]^{ct} (1-\Delta\hat{\mu})^{-(i+h-j)}. \quad (39)$$

Since  $i+h \geq j$  and using the fact that  $\lim_{a \rightarrow \infty} (a/b)(1/a^b) = 1/b!$ , the above simplifies to

$$\frac{(c\hat{\mu}t)^{i+h-j}}{(i+h-j)!} e^{-c\hat{\mu}t}. \quad (40)$$

We now condition this by the p.d.f.  $a(t)$  ( $t > 0$ ) over  $(0, \infty)$  such that

$$\hat{k}_{i+h-j} = \int_0^{\infty} \frac{(c\hat{\mu}t)^{i+h-j}}{(i+h-j)!} e^{-c\hat{\mu}t} a(t) dt, \quad 1 \leq h \leq r, t > 0, \quad (41)$$

or

$$\hat{k}_{i+h-j} = \int_0^{\infty} \frac{(c\hat{\mu}t)^{i+h-j}}{(i+h-j)!} e^{-c\hat{\mu}t} dA(t), \quad 1 \leq h \leq r, t > 0, \quad (42)$$

where  $A(t) = P(\hat{T} \leq t)$  ( $t > 0$ ). The p.m.f.  $\hat{k}_{i+h-j}$  has a p.g.f.  $\hat{K}(z)$

$$\hat{K}(z) = \sum_{i=j-h}^{\infty} \hat{k}_{i+h-j} z^{i+h-j}, \quad 1 \leq h \leq r, |z| \leq 1, \quad (43)$$

which can also be expressed as  $\hat{K}(z) = \int_0^{\infty} e^{-c\hat{\mu}(1-z)t} dA(t) = \tilde{a}(c\hat{\mu}(1-z))$ , where  $\tilde{a}(s)$  is the L-ST  $\tilde{a}(s) = \int_0^{\infty} e^{-st} dA(t)$ .

Now substituting  $\hat{K}(z)$  into (8) leads to

$$1 = \left( \sum_{h=1}^r b_h z^{-h} \right) \left[ \sum_{i=j-h}^{\infty} \int_0^{\infty} e^{-c\hat{\mu}t} \frac{(c\hat{\mu}tz)^{i+h-j}}{(i+h-j)!} dA(t) \right], \quad (44)$$

or

$$1 = B(z^{-1}) \hat{K}(z). \quad (45)$$

The equation (45) coincides with the root equation provided by Chaudhry and Kim [40] who consider the  $GF^X/M/c$  with light-tailed inter-batch-arrival times. To compute the roots of (45) when the inter-batch-arrival times follow a continuous heavy-tailed distribution, we modify  $\hat{K}(z)$

in (45) into  $\widehat{K}_N(z)$ , where  $\widehat{K}_N(z) = \sum_{n=0}^N \int_0^\infty e^{-c\hat{\mu}t} ((c\hat{\mu}tz)^n/n!)dA(t)$ . We replace  $\mu^{c*N}$  in lines (22) and (25) of Algorithm 1 with  $\text{Re}[\int_0^\infty e^{-c\hat{\mu}t} ((c\hat{\mu}t)^N/N!)dA(t)]$ , and replace line (31) entirely with  $y := z^r - \text{expand}(z^r * \sum_{h=1}^r b(h) * z^{-h} \sum_{n=0}^N \int_0^\infty (e^{-c\hat{\mu}t} ((c\hat{\mu}t)^n/n!)a(t) dt)z^n)$  in Section 11. Doing so enables us to easily compute the roots of (45) when the inter-batch-arrival times of  $GI^X/M/c$  follow a distribution such as Pareto (*Pareto* $[\alpha, k]$ ):

$$a(t) = \frac{k^\alpha \alpha}{t^{\alpha+1}}, \quad t \geq k > 0, \alpha > 0, \quad (46)$$

or inverse-Gaussian (*IG* $[\alpha, k]$ ):

$$a(t) = \sqrt{\frac{k}{2\pi t^3}} e^{-(k(t-\alpha)^2/2\alpha^2 t)}, \quad t, \alpha, k > 0. \quad (47)$$

### 13. Numerical Results

The numerical results of  $GI^X/Geo/c$  involving light-tailed inter-batch-arrival times are published by Chaudhry et al. [31]. Though we can recreate their results using the roots method, we instead focus on the numerical results of  $GI^X/Geo/c$  as well as  $GI^X/M/c$  involving heavy-tailed inter-batch-arrival times. Our numerical results are largely divided into three parts.

First, we consider different input parameters (inter-batch-arrival times, maximum batch size, and traffic intensity), and illustrate how the roots behave within the bounds of a unit circle. We then deduce some numerical analysis.

We then consider an extreme case where we have plotted 1,000 roots with an infinite mean inter-batch-arrival time. In doing so, we indicate how to adjust the setting in MAPLE that will otherwise not compute those 1,000 roots in default setting.

Lastly, we present the queue-length and the waiting-time-in-queue distributions of  $GI^X/Geo/c$  with heavy-tailed inter-batch-arrival times. Since all queue-length distributions are in terms of roots of the model's characteristic equation, once the roots are found the solution procedure is deemed complete. Hence, this part of our numerical results serves as a proof of concept.

*13.1. Computing the Roots of the Characteristic Equation.* In Tables 1, 3, and 4, we consider the binomial, (1,10), and normalized Poisson batch size distributions, respectively. In the same tables, the traffic intensities are in a descending order, and the maximum batch size includes both even and odd numbers. We find the roots of the characteristic equation for both  $GI^X/Geo/c$  and  $GI^X/M/c$  in Tables 3 and 4, whereas in Table 1 we focus on  $GI^X/Geo/c$  with a larger  $r$ . We have purposely used  $c = 5$  in all tables in order to isolate and study the effect of other input parameters. All computations were performed in MAPLE using the program in Section 11. All roots were found up to the tenth decimal place and rounded to four decimal places. We have also embedded the

figures of plotted roots in each table for visual illustration. At the end of Table 1, we provide our analysis in bullet points.

From the above tables and through additional numerical tests, we have deduced the following points:

Point 1: while  $z_1$  is always a positive real root (as proved by Kim and Choi [39]), when  $r$  is an odd number, there will be  $r - 1$  imaginary roots, and when  $r$  is an even number, there is also a negative real root and  $r - 2$  imaginary roots. In addition,  $z_1$  has the largest modulus out of all  $r$  roots inside  $|z| = 1$ .

Point 2: the imaginary roots of the characteristic equations always exist in complex conjugate pairs. If  $r$  is an odd number, there are  $(r - 1)/2$  pairs of complex conjugates, whereas if  $r$  is an even number, there are  $(r - 2)/2$  pairs of complex conjugates.

Point 3: the  $LC$  for  $GI^X/M/c$  can be significantly larger than the  $LC$  for  $GI^X/Geo/c$  even though the numerical accuracy of roots is preserved. This is due to the magnitude of  $\mu^{c*N}$  being much smaller than that of  $\int_0^\infty e^{-c\hat{\mu}t} ((c\hat{\mu}t)^N/N!)dA(t)$  when  $\mu = \hat{\mu}$ .

Point 4: we can verify the accuracy of roots by back-substituting any of the  $r$  roots into (8) or (45). While substituting into the characteristic equation with a root that is found at a higher  $N$  leads to a value that is closer to 1, a moderately sized  $N$  will find roots that are just as effective. The accuracy of roots can be verified in another way: compute the queue-length distributions with the  $r$  roots found at a moderately sized  $N$  and then use them to compute the left hand side of (32). Since the right hand side of (32) is independent of roots, one can choose to use a higher  $N$  to make the left-hand-side of (32) match the right-hand-side up to the desired decimal place.

Point 5: as seen in Table 1, as  $\bar{a} \rightarrow \infty$ , the roots tend to converge towards the origin. However, the value of  $\mu$ , ( $0 < \mu < 1$ ) also influences the plotting pattern of roots. Given the relation  $\rho = \bar{b}/(\bar{a}c\mu)$ , having  $\bar{a} = \infty$  leads to  $\rho = 0$ . However, making  $\mu \rightarrow 0$  will have a counter effect and may withhold the roots from converging towards the origin (see later in Figure 6, where  $\bar{a} = \infty$  and  $\mu = 10^{-6}$ ). On the other hand, a large  $\mu$  (and  $c$ ) coupled with  $\rho = 0$  would even further the clustering of roots towards the origin. In either case, roots can be found.

Point 6: the root  $z_1$  nears 1 as  $\rho$  approaches 1. This behavior can be seen by observing the decrease in  $z_1$  in the tables as we make  $\rho = 0.8, 0.25,$  and  $0$ . However, similar to Point 5, when  $\rho = 0$ , making  $\mu \rightarrow 0$  will have a counter effect and result in  $z_1 \rightarrow 1$  (see later in Figure 6, where  $\bar{a} = \infty$  and  $\mu = 10^{-6}$ ).

Point 7: the magnitude of  $\mu$  ( $0 < \mu < 1$ ) is directly proportional to the size of  $N$ . While the size of  $N$  is primarily determined by the size of  $LC$ , if  $LC$  is fixed at a certain value, a smaller  $\mu$  will lead to smaller  $N$  while a larger  $\mu$  will lead to a larger  $N$ . This is true since given the same power, an exponent with a smaller base will

TABLE 3: (1,10) batch size distribution with  $b_1 = 0.9, b_{10} = 0.1$ , and  $\rho = 0.25$ .

$z_h$	GI						
	$SLN$	$GI^X/Geo/5$	$Weibull[2]$	$Pareto[4.5, 2]$	$GI^X/M/5$	$IG[2, 3]$	
$z_1$	0.7408		0.7820		0.6065		0.7086
$z_2$	$0.0974 - 0.5902i$		$0.5448 - 0.4547i$		$0.0209 - 0.4062i$		$0.4544 - 0.4278i$
$z_3$	$0.0974 + 0.5902i$		$0.5448 + 0.4547i$		$0.0209 + 0.4062i$		$0.4544 + 0.4278i$
$z_4$	$0.4798 - 0.4518i$		$0.1663 - 0.6423i$		$0.3137 - 0.3790i$		$0.1048 - 0.5568i$
$z_5$	$0.4798 + 0.4518i$		$0.1663 + 0.6423i$		$0.3137 + 0.3790i$		$0.1048 + 0.5568i$
$z_6$	-0.4953		-0.6026		-0.3262		-0.5033
$z_7$	$-0.4344 - 0.2646i$		$-0.5021 - 0.3460i$		$-0.1765 - 0.3123i$		$-0.2121 - 0.4860i$
$z_8$	$-0.4344 + 0.2646i$		$-0.5021 + 0.3460i$		$-0.1765 + 0.3123i$		$-0.2121 + 0.4860i$
$z_9$	$-0.2377 - 0.4915i$		$-0.2221 - 0.5901i$		$-0.2895 - 0.1665i$		$-0.4275 - 0.2780i$
$z_{10}$	$-0.2377 + 0.4915i$		$-0.2221 + 0.5901i$		$-0.2895 + 0.1665i$		$-0.4275 + 0.2780i$
$N$	15		49		22		34
$LC$	$10^{-40}$		$10^{-240}$		$10^{-3}$		$10^{-3}$

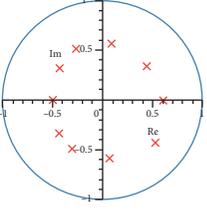
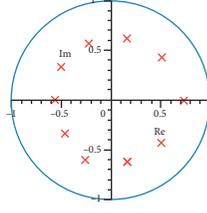
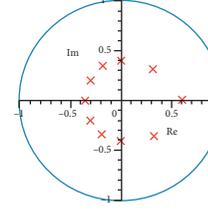
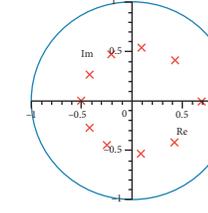
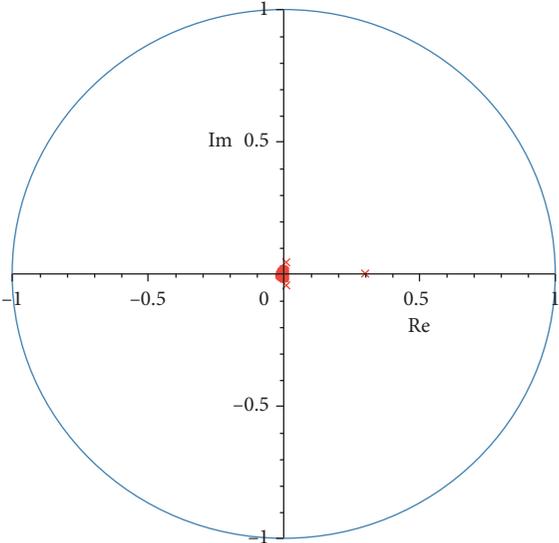
$ z  = 1$				
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TABLE 4: Normalized Poisson batch size distribution with  $b_h = (1/(\sum_{l=1}^r (P^l/l!)))(P^h/h!)$  ( $P = 0.3, r = 50$ ) and  $\rho = 0$ .

$Pareto^X[1]/Geo/5$							
$z_1$	0.3004	$z_{20}$	$0.0072 - 0.0069i$				
$z_2$	$0.0007 - 0.0146i$	$z_{21}$	$0.0072 + 0.0069i$				
$z_3$	$0.0007 + 0.0146i$	$z_{22}$	$0.0073 - 0.0051i$				
$z_4$	$0.0012 - 0.0158i$	$z_{23}$	$0.0073 + 0.0051i$				
$z_5$	$0.0012 + 0.0158i$	$z_{24}$	$0.0101 - 0.0449i$				
$z_6$	$0.0025 - 0.0138i$	$z_{25}$	$0.0101 + 0.0449i$				
$z_7$	$0.0025 + 0.0138i$	$z_{26}$	-0.0146				
$z_8$	$0.0027 - 0.0235i$	$z_{27}$	$-0.0145 - 0.0018i$				
$z_9$	$0.0027 + 0.0235i$	$z_{28}$	$-0.0145 + 0.0018i$				
$z_{10}$	$0.0039 - 0.0127i$	$z_{29}$	$-0.0141 - 0.0036i$				
$z_{11}$	$0.0039 + 0.0127i$	$z_{30}$	$-0.0141 + 0.0036i$				
$z_{12}$	$0.0050 - 0.0115i$	$z_{31}$	$-0.0136 - 0.0053i$				
$z_{13}$	$0.0050 + 0.0115i$	$z_{32}$	$-0.0136 + 0.0053i$				
$z_{14}$	$0.0060 - 0.0101i$	$z_{33}$	$-0.0128 - 0.0069i$	$z_{39}$	$-0.0095 - 0.0111i$	$z_{45}$	$-0.0048 - 0.0138i$
$z_{15}$	$0.0060 + 0.0101i$	$z_{34}$	$-0.0128 + 0.0069i$	$z_{40}$	$-0.0095 + 0.0111i$	$z_{46}$	$-0.0048 + 0.0138i$
$z_{16}$	$0.0067 - 0.0086i$	$z_{35}$	$-0.0119 - 0.0085i$	$z_{41}$	$-0.0080 - 0.0122i$	$z_{47}$	$-0.0030 - 0.0143i$
$z_{17}$	$0.0067 + 0.0086i$	$z_{36}$	$-0.0119 + 0.0085i$	$z_{42}$	$-0.0080 + 0.0122i$	$z_{48}$	$-0.0030 + 0.0143i$
$z_{18}$	$0.0069 - 0.0032i$	$z_{37}$	$-0.0108 - 0.0099i$	$z_{43}$	$-0.0065 - 0.0131i$	$z_{49}$	$-0.0012 - 0.0146i$
$z_{19}$	$0.0069 + 0.0032i$	$z_{38}$	$-0.0108 + 0.0099i$	$z_{44}$	$-0.0065 + 0.0131i$	$z_{50}$	$-0.0012 + 0.0146i$
	$LC = 10^{-55}$					$N = 19$	

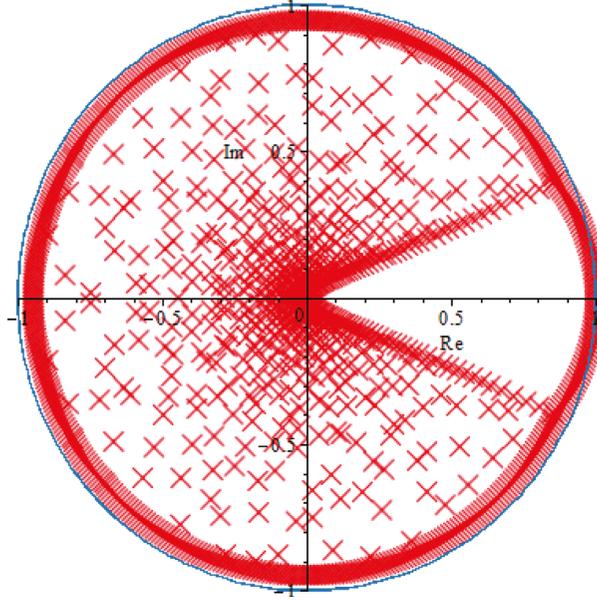


FIGURE 6: Plotting 1,000 roots.

always be smaller than the other exponent with a larger base. In  $GI^X/Geo/c$  and  $GI^X/M/c$  with heavy-tailed inter-batch-arrival times, the  $a_m$  and  $a(t)$  decay at an extremely slow rate as  $m, t \rightarrow \infty$  while  $\mu^{cm}$  and  $e^{-c\hat{\mu}t} ((c\hat{\mu}t)^n/n!)$  decay faster with smaller values of  $\mu$  and  $\hat{\mu}$ , thus requiring less number of iterations ( $N$ ) to reach  $LC$ . We have tested this concept on  $Pareto^X[1]/Geo/5$  by finding the  $N$  at various values of  $\mu$  while letting  $LC = 10^{-100}$ ,  $b_h = (1/(\sum_{l=1}^r (P^l/l!)))(P^h/h!)$  ( $P = 0.1, r = 10$ ) and  $\rho = 0$ . Readers can see Table 5 for results.

Point 8: different batch size distributions lead to different plotting patterns of roots. In Table 4, the (1,10) batch size distribution plotted equal number of roots on each side of the imaginary axis (with the exception of  $Pareto^X[4.5, 2]/M/5$ ), whereas this was not the case in other tables. However, there is always an equal number of roots plotted on each side of the real axis (this is due to the complex conjugate pairing as per Point 2).

Point 9: when  $\bar{a} = \infty$  in  $GI^X/Geo/c$ , it results in  $Q_0 = Q_0^o = P_0^o = P_0 = 1$ . Intuitively, this can be understood as the system being empty (in steady-state) at the outside observer's and random time epochs since there are no arrivals while remaining customers continue to get served. While this concept can be numerically verified, it can be analytically proven as follows: since  $\bar{a} = \infty$ , the relation  $Q_j = \sum_{i=0}^{\infty} Q_i^- P_{i,j}^*$  ( $j \geq 1$ ) becomes  $Q_j = 0$  ( $j \geq 1$ ) which results in  $Q_0 = 1$ . Another approach is by letting  $\rho = 0$  in (32) which leads to  $\sum_{j=0}^{c-1} (c-j)Q_j^o = c$ . This indicates that the mean number of idle servers is  $c$  (i.e., an empty system). Therefore, it can be simplified to  $cQ_0^o = c$  or  $Q_0^o = 1$ . When  $\bar{a} = \infty$  in  $GI^X/M/c$ , the queue-length distribution at a random epoch (say  $p_j, j \geq 0$ ) becomes  $p_0 = 1$ . As a remark, several relations between the performance measures of  $GI^X/M/c$

TABLE 5: Relation between  $\mu$  and  $N$ .

$Pareto^X[1]/Geo/5$	
$\mu$	$N$
0.1	19
0.2	27
0.3	36
0.4	48
0.5	63
0.6	86
0.7	123
0.8	196
0.9	413
0.99	4239

are derived by Yao et al. [41]. If the inter-batch-arrival times follow the heavy-tailed distributions with an infinite mean, some of their relations that involve  $\rho$  become invalid. As an example, Yao et al. [41] derive the relation

$$W_{q_1}^- = \frac{\rho}{\lambda \bar{b}} \left( L_s^- + 1 - c + \sum_{k=0}^{c-1} (c-k-1)p_k^- \right), \quad (48)$$

where  $W_{q_1}^-$  is the mean waiting-time-in-queue of the first customer within an incoming batch,  $\lambda$  is the arrival rate, and  $p_k^-$  is the p.m.f. of the number of customers in the system at a pre-arrival time epoch with mean  $L_s^-$ . In this relation, when  $\rho = 0$ , the right-hand-side becomes undefined while the left-hand-side may not. The same phenomena can be observed when  $\rho = 0$  in another relation that determines  $W_q^-$  which is the waiting-time-in-queue of a random customer within an incoming batch.

**13.2. Extreme case.** In Figure 6 we have plotted the roots of the characteristic equation for  $Pareto^X[1]/Geo/5$ . Doing so leads to  $\bar{a} = \infty$  which results in  $\rho = 0$ . The batch size distribution is a binomial distribution ( $p = 0.45, q = 0.55$ )

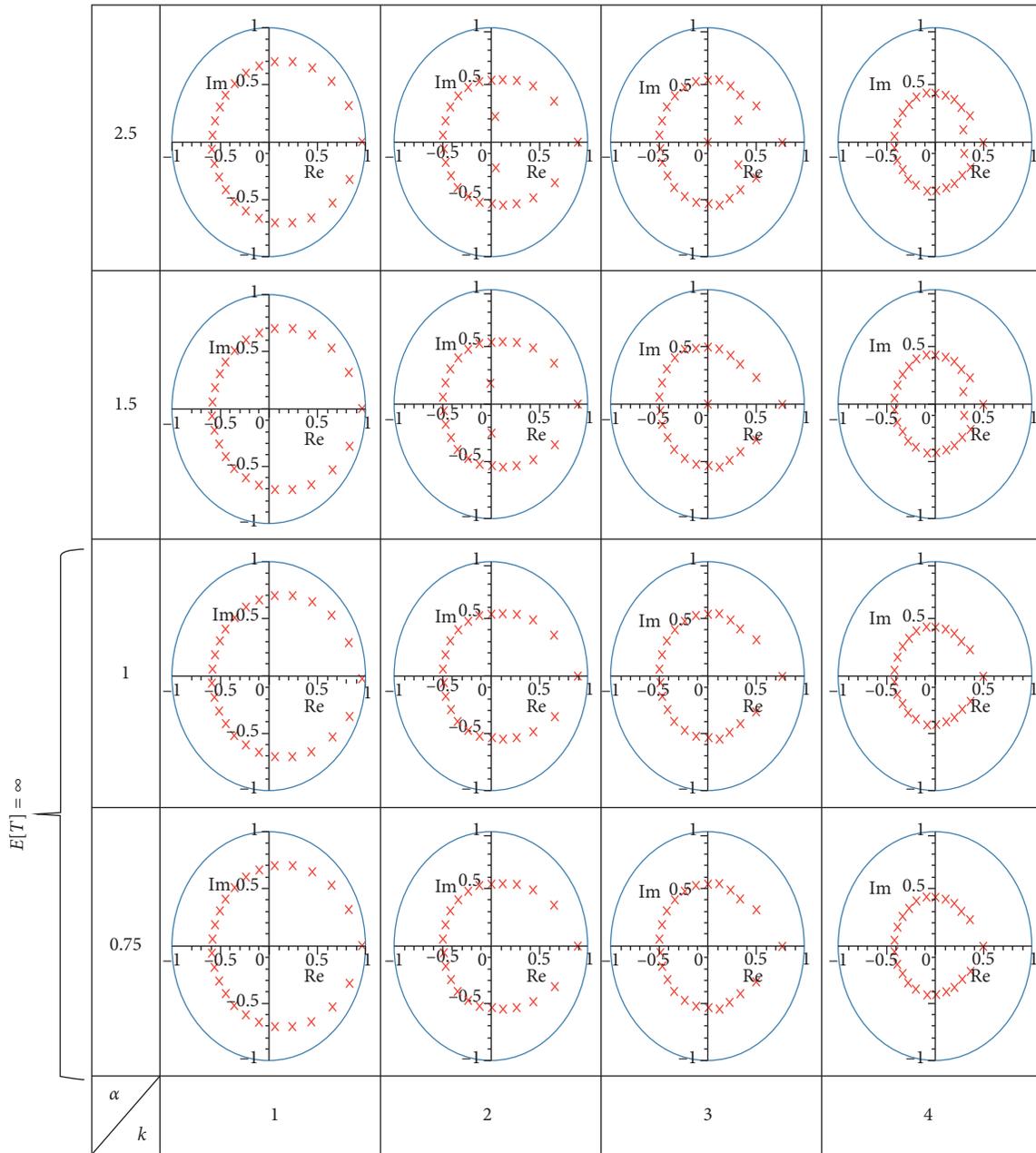


FIGURE 7: Plotting the roots of the characteristic equation for the  $Pareto^X[\alpha, k]/M/7$  queue with  $b_1 = 0.5, b_{25} = 0.5, \mu = 1$ , and  $N = 50$ .

with  $r = 1,000$ . We have used  $\mu = 10^{-6}$  and  $LC = 10^{-10,000}$  which led to  $N = 333$ .

Using our MAPLE program from Section 11, we have successfully plotted all 1,000 roots inside the unit circle with  $z_1 = 0.9999959545$  (a smaller  $LC$  will lead to a  $z_1$  that is closer to 1). As a remark, one may encounter the error message in MAPLE “Length of output exceeds limit of 1000000” as  $z_1$  becomes very close to 1. This can be overcome by changing the default setting in MAPLE in the following manner: go to “Tools,” “Options,” “Precision,” and change the “Limit expression length” from 1,000,000 to 90,000,000 or greater. Doing so prevents MAPLE from

rounding  $z_1$  to 1 which is what is needed to compute the queue-length distributions in terms of roots. Computation of the results for the  $Pareto^X[\alpha, k]/M/c$  queue for various parameters, readers may see Figure 7.

**13.3. Computing the Distributions of  $GI^X/Geo/c$  Involving Heavy-Tailed Distributions.** We have computed the queue-lengths and the waiting-time-in-queue distributions, as well as the performance measures for the queue  $GI^X/Geo/c$ . We have considered the *Weibull* $[\beta]$  and *SLN* distributions as the inter-batch-arrival time distributions. All computations were performed in MAPLE (Tables 6 and 7).

TABLE 6: Inter-batch-arrival pattern is *Weibull*[ $M$ ] with  $a_m = M^{-\sqrt{m}}/(V - 1)$ , where  $V = \sum_{j=0}^{\infty} M^{-\sqrt{j}}$ . Parameters taken are  $M = 2, b_1 = 0.425, b_2 = 0.325, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, \mu = 0.11$ , and  $c = 5$ . This gives  $\bar{b} = 2.125, \bar{a} = 13.7054$ , and  $\rho = 0.2819$ .

<i>Weibull</i> <sup>X</sup> [2]/ <i>Geo</i> /5								
$j$	$Q_j^-$	$Q_j$	$Q_j^o$	$P_j^-$	$P_j$	$P_j^o$	$m$	$w_k$
0	0.3300	0.4172	0.4338	0.3041	0.4338	0.4338	0	0.1996
1	0.2279	0.2097	0.2033	0.2161	0.2033	0.2033	1	0.0382
2	0.1606	0.1291	0.1323	0.1623	0.1323	0.1323	2	0.0314
3	0.1002	0.0756	0.0805	0.1053	0.0805	0.0805	3	0.0258
4	0.0635	0.0458	0.0505	0.0690	0.0505	0.0505	4	0.0212
5	0.0411	0.0286	0.0347	0.0497	0.0347	0.0347	5	0.0174
6	0.0270	0.0188	0.0228	0.0327	0.0228	0.0228	6	0.0143
7	0.0176	0.0122	0.0149	0.0215	0.0149	0.0149	7	0.0117
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	Sum	0.9999
			$\sum_{j=0}^{c-1} Q_j^o(c-n) = 3.5905$					
$I_q^o = 0.1847$					$W_q = 1.1912$	$c(1-\rho) = 3.5905$		
$I_s^o = 1.5942$					$W_s = 10.2820$	$(\bar{b}W_q)/\bar{a} = 0.1847$		
						$(\bar{b}W_s)/\bar{a} = 1.5942$		

TABLE 7: Inter-batch-arrival pattern is *SLN* with  $a_m = Ve^{(-\ln(m))^2/2}$  with  $V = 1/\sum_{j=1}^{\infty} e^{(-\ln(j))^2/2}$ . Parameters taken are  $b_1 = 0.425, b_2 = 0.325, b_3 = 0.075, b_4 = 0.05, b_5 = 0.125, \mu = 0.2$ , and  $c = 5$ . This gives  $\bar{b} = 2.125, \bar{a} = 4.6519$ , and  $\rho = 0.4568$ .

<i>SLN</i> <sup>X</sup> / <i>Geo</i> /5								
$j$	$Q_j^-$	$Q_j$	$Q_j^o$	$P_j^-$	$P_j$	$P_j^o$	$m$	$w_k$
0	0.2080	0.2765	0.2339	0.1630	0.2339	0.2339	0	0.4175
1	0.2188	0.2141	0.1860	0.1844	0.1860	0.1860	1	0.0805
2	0.1811	0.1646	0.1602	0.1737	0.1602	0.1602	2	0.0591
3	0.1253	0.1107	0.1190	0.1335	0.1190	0.1190	3	0.0432
4	0.0819	0.0715	0.0837	0.0955	0.0837	0.0837	4	0.0315
5	0.0549	0.0477	0.0630	0.0725	0.0630	0.0630	5	0.0229
6	0.0396	0.0344	0.0464	0.0535	0.0464	0.0464	6	0.0167
7	0.0278	0.0241	0.0331	0.0382	0.0331	0.0331	7	0.0122
⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮	⋮
Sum	0.9999	0.9999	0.9999	0.9999	0.9999	0.9999	Sum	0.9999
			$\sum_{j=0}^{c-1} Q_j^o(c-n) = 2.7160$					
$I_q^o = 0.5059$					$W_q = 1.1075$	$c(1-\rho) = 2.7160$		
$I_s^o = 2.7899$					$W_s = 6.1075$	$(\bar{b}W_q)/\bar{a} = 0.5059$		
						$(\bar{b}W_s)/\bar{a} = 2.7899$		

### 14. Conclusion

Our complete solution procedure to the model  $GI^X/Geo/c$  begins with solving for the queue-length distribution of  $GI^X/Geo/c$  with EAS at a pre-arrival time epoch. In doing so, we express the result in terms of the roots of the characteristic equation. We then develop several relations between the queue-length distributions in order to find the queue-length distributions at other time epochs in  $GI^X/Geo/c$  with EAS and LAS-DA. The significance of our solution procedure is that we have simplified the expression of all queue-length distributions in terms of roots. The roots are quickly and accurately found, even for the heavy-tailed inter-batch-arrival times, and this becomes advantageous for researchers and practitioners who are looking for an analytically simple and computationally efficient way to compute the steady-state queue-length distributions of the model  $GI^X/Geo/c$ .

In our numerical results, we focused on the  $GI^X/Geo/c$  queue as well as the  $GI^X/M/c$  queue involving heavy-tailed inter-batch-arrival times. Because the heavy-tailed distributions do not have the closed form of p.g.f.'s or L-STs and, in certain cases, have an infinite mean (i.e., nonexistent p.g.f.'s or L-STs), applying the roots method was considered

to be not possible and numerically hazardous by some authors. However, we have demonstrated that the roots method is deemed effective and efficient even if the inter-batch-arrival times follow the heavy-tailed distributions.

### Appendix

Linear difference equations have frequently been used in the theory of queues (see, e.g., Chaudhry and Templeton [34]). The application of linear difference equations in solving the  $GI^X/Geo/c$  queues is explained below. First, by rearranging (6), we have

$$Q_j^- = \sum_{h=1}^r b_h \sum_{i=j-h}^{\infty} Q_i^- k_{i+h-j}, \quad j \geq 1. \tag{A.1}$$

Now substituting  $Q_j^- = Cz^j, j \geq \max(c, r)$  into above expression leads to

$$1 = B(z^{-1})K(z), \tag{A.2}$$

which is the underlying characteristic equation of  $GI^X/Geo/c$ . To prove that (8) has  $r$  roots inside the unit circle  $|z| = 1$ , let us rewrite this equation as

$$z^r - \left( \sum_{h=1}^r b_h z^{r-h} \right), \quad K(z) = 0. \quad (\text{A.3})$$

Now let

$$\begin{aligned} f(z) &= z^r, \\ g(z) &= - \left( \sum_{h=1}^r b_h z^{r-h} \right) K(z). \end{aligned} \quad (\text{A.4})$$

Consider absolute values of  $f(z)$  and  $g(z)$  on the circle  $|z| = 1 - \delta$ , where  $\delta$  is positive and sufficiently small. This gives

$$\begin{aligned} |f(z)| &= (1 - \delta)^r = 1 - \delta r + o(\delta), \\ |g(z)| &\leq \sum_{h=1}^r b_h |z|^{r-h} K(|z|) = 1 - \delta(r - \bar{b}) - (\bar{a}c\mu)\delta + o(\delta) \\ &= 1 - \delta r - (\bar{a}c\mu)(1 - \rho)\delta + o(\delta). \end{aligned} \quad (\text{A.5})$$

Finally, we have

$$|g(z)| < 1 - \delta r + o(\delta) = |f(z)|, \quad (\text{A.6})$$

where  $\rho = (\bar{b}/(\bar{a}c\mu)) < 1$ . Thus, for  $\rho < 1$  and  $\delta$  sufficiently small,  $|f(z)| > |g(z)|$  on  $|z| = 1 - \delta$ . Since  $f(z)$  and  $g(z)$  satisfy the conditions of Rouché's theorem, it follows that (8) has  $r$  roots inside the unit circle,  $|z| = 1$ , since  $f(z)$  has  $r$  roots inside  $|z| = 1$ . Extending on the work of root-finding in queueing theory by Chaudhry and Templeton [34] and using complex analysis, each of these  $r$  roots can be individually computed using the following expression:

$$z = \sqrt[r]{\left( \sum_{h=1}^r b_h z^{r-h} \right) K(z) e^{((2\pi i j)/r)}}, \quad j = 0, 1, 2, \dots, r - 1. \quad (\text{A.7})$$

## Data Availability

The data used to support the findings of this study are included within the article.

## Disclosure

This paper is based on the second author's doctoral thesis titled "A Unified Approach to Multi-Server Bulk-Arrival Queues Using Roots" completed on 2 April 2018 at the Royal Military College of Canada (see Kim [42]).

## Conflicts of Interest

The authors declare that they have no conflict of interest.

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