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Research Article

A Mixture of Clayton, Gumbel, and Frank Copulas: A Complete Dependence Model

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Knowledge of the dependence between random variables is necessary in the area of risk assessment and evaluation. Some of the existing Archimedean copulas, namely the Clayton and the Gumbel copulas, allow for higher correlations on the extreme left and right, respectively. In this study, we use the idea of convex combinations to build a hybrid Clayton–Gumbel–Frank copula that provides all dependence scenarios from existing Archimedean copulas. The corresponding density and conditional distribution functions of the derived models for two random variables, as well as an estimator for the proportion parameter associated with the proposed model, are also derived. The results show that the proposed model is able to show any case of dependence by providing coefficients for the upper tail and lower tail dependence.

1. Introduction

Dependence measures such as Pearson's correlation coefficient have been and are still being used in several areas of dependence analysis. According to Haugh [1], the Pearson correlation coefficient is not useful when it comes to modeling of increasing and continuous transformation of random variables. Thus, in a case where nonlinear dependence is required, if Pearson's correlation coefficient is used, unreliable estimates may be obtained, leading to an incorrect assessment of risk. Copulas have the ability to cater to the cases of increasing and continuous transformations of random variables. This is because it is built on scale-free measures of dependence.

Copulas have the tendency to identify tail dependencies no matter how extreme. Amongst the set of copulas, the Archimedean copulas most often do not properly cater for dependencies in random variables [2, 3].

There are some existing Archimedean copulas that provide tail dependencies. These dependencies may be upper tail, lower tail, or dependence between the tails. Also, obtaining a generator function for an Archimedean copula is quite difficult [4].

Copulas have had applications in fields such as finance [5], hydrology [6], public health and medicine [7], and actuarial science [8, 9].

The Clayton, Gumbel, and Frank copulas are some of such existing Archimedean copulas. The Clayton copula allows for only lower tail dependence [10], the Frank copula allows for dependence around the mode [11], and the Gumbel copula allows for only upper tail dependence [12]. The difference between the Clayton and Gumbel copulas is that: (i) for the Clayton copula, the correlations on the extreme left sides of distributions are more concentrated (i.e., higher correlations) than those in the extreme right sides of the distributions, and (ii) for the Gumbel copula, the correlations on the extreme right sides of distributions are more concentrated (i.e., higher correlations) than those in the extreme left sides of the distributions.

Mixtures of the Clayton and Gumbel copulas have been discussed in literature [13–17]. Their models provide no explicit formulae for the copula density and conditional distribution function. Also, another limitation is that the proportion of each copula in the mixture is arbitrarily chosen. This means that evaluating the proportions from data will be impossible, given their approach.

To the best of the researcher's knowledge, there have been no mixtures involving the Clayton, Gumbel, and Frank copulas.

Since tail dependence is very vital in modeling dependence in cases of extreme events, this study seeks to introduce a set of mixture Archimedean copulas using the abovementioned copulas that attempt to resolve these problems.

2. Methods

2.1. Archimedean Copulas. The general definition of Archimedean copulas is given and the Clayton, Gumbel, and Frank copulas are also discussed, together with the definition of tail dependence of a bivariate copula. An Archimedean copula with a strict generator has the following form:

$$C\cdot\left(u_{1,}u_{2,\ldots,}u_{d}\right)=\phi^{-1}\left(\phi\left(u_{1}\right)+\cdots+\phi\left(u_{d}\right)\right), \tag{1}$$

where the generator function ϕ satisfies the following conditions [18]:

(i) ϕ is continuous, strictly decreasing and convex function mapping [0, 1] onto $[0,\infty)$

(ii) $\phi\ (0)=\infty$

(iii) $\phi\ (1)=0$

In what follows, the generator functions, inverse generator functions, Kendall's Tau representations, and copula models of the Clayton, Frank, and Gumbel copulas are defined.

2.2. Clayton Copula. The Clayton copula has a generator function given by

$$\phi(t)=\frac{1}{\theta}\left(t^{-\theta}-1\right),\quad \theta>0,\ t\in[0,1], \tag{2}$$

where θ is the dependence parameter for the Clayton copula.

Using the generator function, we obtain the inverse function as follows:

(i) Let $z=\ 1/\theta\ \left(t^{-\theta}-1\right)$

(ii) $\theta\mathrm{u}+1=t^{-\theta}-1$

(iii) $\theta\mathrm{u}+1=t^{-\theta}$

$$(\theta u+1)^{-(1/\theta)}=t, \tag{3}$$

Thus

$$\phi^{-1}(t)=\left[\left(t^{-\theta}-1\right)+1\right]^{-(1/\theta)}. \tag{4}$$

The Copula obtained from the inverse is

$$C_{C}(u,v)=(u,v)=\left(u^{-\theta}+v^{-\theta}-1\right)^{-(1/\theta)}. \tag{5}$$

Equation (6) gives the relationship between the dependence parameter of the copula and Kendall's Tau. This representation is given as

$$\tau=\frac{\theta}{\theta+2}. \tag{6}$$

2.3. Frank Copula. The generator function of the Frank copula is given as

$$\phi(t)=-\log\left(\frac{\exp(-\omega t)-1}{\exp(-\omega)-1}\right),\quad \omega\in(-\infty,\infty),\ t\in[0,1], \tag{7}$$

where ω is the dependence parameter for the Frank copula.

The inverse generator function of the Frank copula is given as

$$\phi^{-1}(t)=-\frac{1}{\omega}\log[\exp(-t)(\exp(-\omega)-1)+1]. \tag{8}$$

The Frank copula is given as

$$C_{F}(u,v)=-\frac{1}{\omega}\log\left[1+\frac{(\exp(-\omega u)-1)(\exp(-\omega v)-1)}{\exp(-\omega)-1}\right]. \tag{9}$$

Kendall's Tau representation for the Frank copula is

$$\tau=1-\frac{4}{\omega}+\frac{4D_{1(\omega)}}{\omega}, \tag{10}$$

where $D_1(\omega)=\int_0^{\omega}(x/\omega)/(\exp(x)-1)\mathrm{d}x$ (Debye function).

2.4. Gumbel Copula. The generator function of the Gumbel copula is given as

$$\phi(t)=(-\log t)^{\beta},\quad \beta\in[1,\infty],\ t\in[0,1], \tag{11}$$

where β is the dependence parameter for the Gumbel copula.

$$\phi^{-1}(t)=\exp\left(-t\beta^{-1}\right). \tag{12}$$

Kendall's Tau representation for the Gumbel copula is

$$\tau=1-\frac{1}{\beta}C_{G}(u,v)=\exp\left[-\left\{(-\log u)^{\beta}+(-\log v)^{\beta}\right\}^{\beta^{-1}}\right]. \tag{13}$$

2.5. Combination of Copulas. A preliminary study by Boateng et al. [19] on copulas and random variables revealed that, in general, the copula of a pair of random variables is Archimedean. This finding supported those of Adjasi et al. [20] and Trede and Savu [3]. Existing Archimedean copulas need improvement in terms of tail dependence. This part of the study is focused on providing Archimedean copulas with superior tail dependence characteristics. Firstly, a combination based on the product of the selected copulas is explored and their tail dependence assessed. Secondly, a convex combination of all three copulas is discussed.

Table 1: The dependence coefficient (product).

Copula	Lower tail	Upper tail
Clayton	$2^{-(1/\theta)}$	0
Frank	0	0
Gumbel	0	$2-2^{1/\beta}$
Clayton–Gumbel	0	0
Clayton–Gumbel–Frank	0	0

2.6. Multiplication of Copulas (Result 1)

Theorem 1. *Let $C_1(u,v)$ and $C_2(u,v)$ be 2-dimensional Archimedean copulas, respectively. Then $C_3(u,v) = C_1(u,v)\cdot C_2(u,v)$ is also a copula.*

Corollary 1. *Let T_{L_1} and T_{U_1} be the lower and upper tail dependence coefficients of a copula $C_1(u,v)$ and T_{L_2} and T_{U_2} be the lower and upper tail dependence coefficients of a copula $C_2(u,v)$. Then the copula has as lower and upper tail dependence coefficients, respectively.*

Proof. $T_L = \lim_{u\longrightarrow 0}(C_3(u,u)/u)$

Here, we treat $C_3(u,v)$ as $C_1(u,v)$ and $C_2(u,v)$, thus, we find the lower tail dependence as a product of the individual tail dependence of $C_1(u,v)$ and $C_2(u,v)$.

$$T_L = \lim_{u\longrightarrow 0}\frac{(C_1(u,u)C_2(u,u))}{u}\lim_{u\longrightarrow 0}\frac{C_1(u,u)}{u}\lim_{u\longrightarrow 0}\frac{C_2(u,u)}{u} = T_{L_1}T_{L_2}. \tag{14}$$

The lower tail dependence coefficient of the product of two copulas is the product of the individual lower tail dependence coefficients of the copulas being multiplied. This follows from the explanation for the split of the lower tail dependencies.

$$T_U = \lim_{u\longrightarrow 1}\frac{1-2u+C_3(u,u)}{1-u}\lim_{u\longrightarrow 1}\frac{1-2u+C_3(u,u)}{1-u}\lim_{u\longrightarrow 1}\frac{1-2u+C_2(u,u)}{1-u} = T_{U_1}T_{U_2}. \tag{15}$$

Thus, the upper tail dependence coefficient of the product of two copulas is the product of the individual upper tail dependence coefficients of the copulas being multiplied. □

2.7. Tail Dependence Coefficient for Products of Copulas. Applying results from Corollary 1, on the product of the selected copulas (Clayton, Gumbel, and Frank copula), the tail dependence coefficient of the selected copula and their product are given in Table 1.

Remark 1. Using the known tail dependence coefficients of the Clayton, Gumbel, and Frank copulas, the tail coefficient of such a combination (product) is greatly affected. This is especially because none of the copulas in the combination has both lower and upper tail dependence coefficients (this can be seen in Table 1).

2.8. Addition of Copulas (Results 2). Let U_1 and U_2 be two d-dimensional random variables on (Ω, $\mathbb{P}$) distributed according to the copulas C_1 and C_2, respectively. Let Z be a shifted Bernoulli random variable such that $\mathbb{P}(Z=1)=\alpha$ and $\mathbb{P}(Z=2)=1$- α for some $\alpha \in$ I suppose that U_1, U_2, and Z are independent. Now we consider the d-dimension random variable U^*.

$$U^* = \sigma_1(Z)U_1 + \sigma_2(Z)U_2, \tag{16}$$

where for $i \in 1, 2$, $\sigma_i(x) = 1$ if $x=i$, $\sigma_i(x)=0$, otherwise.

Then U^* is distributed according to the copula.

$$\alpha C_1 + (1-\alpha)C_2. \tag{17}$$

Corollary 2. *Suppose $C_C = [u_1^{-\theta} + u_2^{-\theta}, \ldots, +u_d^{-\theta} + 1 - d]^{-(1/\theta)}$ and $C_G = (\exp[-\sum_{j=1}^{d}((-\log u_j)^{\beta})^{\beta-1}])$ are respectively Clayton and Gumbel copulas for the d-dimensional random variable $U=(u_1,\ldots,u_d)$, then for any $\lambda \in [0,1]$,*

$$C_{CG}(u_1,\ldots,u_d) = \lambda\left[u_1^{-\theta} + u_2^{-\theta} + \cdots + u_d^{-\theta} + 1 - d\right]^{-(1/\theta)} + (1-\lambda)\left(\exp\left[-\sum_{j=1}^{d}\left(\left(-\log u_j\right)^{\beta}\right)^{\beta-1}\right]\right). \tag{18}$$

is a copula.

Corollary 3. *Let $C_{CG}(u,v)$ be a 2-dimensional Clayton–Gumbel copula, then the conditional distribution function is given by [14, 21]*

$$C_{CG}(v|u) = \frac{\lambda u^{-\theta}\left(u^{-\theta}+v^{-\theta}-1\right)^{-((\theta+1)/\theta)} + \left(\left((\lambda-1)(-\log u)^{\beta} e^{-\left((-\log u)^{\beta}+(-\log v)^{\beta}\right)^{1/\beta}}\left((-\log u)^{\beta}+(-\log v)^{\beta}\right)^{(1/\beta)-1}\right)/\log(u)\right)}{u}. \tag{19}$$

Corollary 4. *Let $C_{CG}(u,v)$ be a 2-dimensional Clayton–Gumbel copula, then the joint density function is given by*

$$\begin{aligned} fu,v(u,v) = \frac{1}{u}\left[-\lambda(-\theta-1)u^{-\theta}v^{-\theta-1}\left(u^{-\theta}+v^{-\theta}-1\right)^{-((\theta-1)/\theta)-1} - \frac{1}{v\log(u)}(\lambda-1)\left(\frac{1}{\beta}-1\right)\beta\right] \times \\ (-\log u)^{\beta}(-\log v)^{\beta-1}(-\log u)^{\beta} e^{-\left((-\log u)^{\beta}+(-\log v)^{\beta}\right)^{1/\beta}}\left((-\log u)^{\beta}+(-\log v)^{\beta}\right)^{(1/\beta)-2} \\ \frac{(\lambda-1)(-\log u)^{\beta}(-\log v)^{\beta-1} e^{-\left((-\log u)^{\beta}+(-\log v)^{\beta}\right)^{1/\beta}}\left((-\log u)^{\beta}+(-\log v)^{\beta}\right)^{(2/\beta)-2}}{v\log(u)} \end{aligned} \tag{20}$$

Corollary 5. $C_{CG}(u_1,\ldots,u_d)$ *and* $C_F(u_1,\ldots,u_d)$ *be the d-dimension Clayton–Gumbel and Frank copula, respectively. Then for any* $\gamma \in [0,1]$, $C_{CGF}(u_1,\ldots,u_d)$ *is a copula given by*

$$\begin{aligned} C_{CGF}(u_1,\ldots,u_d) &= \gamma C_{CG}(u_1,\ldots,u_d) \\ &\quad + (1-\gamma)\left(C_F(u_1,\ldots,u_d)\right). \end{aligned} \tag{21}$$

Corollary 6. *Let $C_{CG}(u|v)$ and $C_F(u|v)$ be the conditional distribution functions of the Clayton–Gumbel and the Frank copulas, respectively. Then the conditional distribution of the Clayton–Gumbel–Frank copula is given by*

$$C_{CGF}(u|v) = \gamma C_{CG}(u|v) + (1-\gamma)F_F(u|v), \quad \gamma \in [0,1]. \tag{22}$$

Corollary 7. *Let $F_{CG}(u,v)$ and $F_F(u,v)$ be the joint distribution function of the Clayton–Gumbel and the Frank copulas, respectively. Then the joint distribution function of the Clayton–Gumbel–Frank copula is given by*

$$F_{CGF}(u,v) = \gamma F_{CG}(u,v) + (1-\gamma)F_F(u,v), \quad \gamma \in [0,1]. \tag{23}$$

Using the fact that the Clayton–Gumbel is also a copula, we combine the Clayton–Gumbel copula with the Frank copula to obtain the Clayton–Gumbel–Frank copula.

Next, we deduce Kendall's tau representation $\in$ for the proposed model for a bivariate case.

Proposition 1. *Let* $\tau = ((\theta/\theta)+2)$ *and* $\tau = 1-(1/\beta)$ *be Kendall's Tau representations for the Clayton and Gumbel copulas, respectively. Then for* $\lambda \in [0,1]$, *Kendell's Tau for the hybrid Clayton–Gumbel copula is given by*

$$\tau = \frac{\lambda\beta\theta + (1-\lambda)(\theta+2)(\beta-1)}{(\theta+2)\beta}. \tag{24}$$

Proposition 2. *Let* $\tau = (\lambda\beta\theta + (1-\lambda)(\theta+2)(\beta-1))/((\theta+2)\beta)$ *for the hybrid Clayton–Gumbel copula and* $\tau = 1-(4/\omega)+(4D_1(\omega)/\omega)$ *for the Frank copula. Then for* $\gamma \in [0,1]$,

$$\begin{aligned} \tau = \gamma\left[\frac{\lambda\beta\theta + (1-\lambda)(\theta+2)(\beta-1)}{(\theta+2)\beta}\right] \\ + (1-\lambda)\left[1-\frac{4}{\omega}+\frac{4D_{1(\omega)}}{\omega}\right]. \end{aligned} \tag{25}$$

From Kendall's Tau representation, the proportion parameters λ and γ are obtained in the proposition as follows:

Proposition 3. *Let $C_{CG}(u,v)$ be a 2-dimensional Clayton–Gumbel copula, with*

$$\tau = \frac{\lambda\beta\theta + (1-\lambda)(\theta+2)(\beta-1)}{(\theta+2)\beta}. \tag{26}$$

The estimate of λ is

$$\hat{\lambda} = \frac{(\theta+2)(\beta\tau-(\beta-1))}{\beta\theta-((\theta+2)(\beta-1))}. \tag{27}$$

Proposition 4. *Let $C_{CGF}(u,v)$ be a 2-dimensional Clayton–Gumbel–Frank copula with*

$$\begin{aligned} \tau = \gamma\left[\frac{\lambda\beta\theta + (1-\lambda)(\theta+2)(\beta-1)}{(\theta+2)\beta}\right] \\ + (1-\lambda)\left[1-\frac{4}{\omega}+\frac{4D_1(\omega)}{\omega}\right], \end{aligned} \tag{28}$$

Then the estimate of γ is

Table 2: Tail dependencies of models.

Copula	Lower tail dependence	Upper tail dependence
Clayton	$2^{-(1/\theta)}$	0
Frank	0	0
Gumbel	0	$2-2^{-(1/\beta)}$
Clayton–Gumbel	$\lambda 2^{-(1/\theta)}$	$(1-\lambda)(2-2^{-(1/\beta)})$
Clayton–Gumbel–Frank	$\gamma(\lambda 2^{-(1/\theta)})$	$(1-\gamma)(1-\lambda)(2-2^{-(1/\beta)})$

Table 3: Descriptive statistics of the returns series for CAL bank and GCB bank.

Index	Mean	Median	Minimum	Maximum	Standard deviation	Skewness	Kurtosis
CAL	0.001445	0.0000	−0.2000	0.6500	0.003936	10.2022	169.7232
GCB	0.009087	0.0000	−0.6500	3.7400	0.19880	15.25557	204.7648

$$\widehat{\gamma}=\frac{\tau-\left[1-(4/\omega)+\left(4D_1(\omega)/\omega\right)\right]}{\left[(\lambda\beta\theta+(1-\lambda)(\theta+2)(\beta-1))/(\theta+2)\beta\right]-\left[1-(4/\omega)+\left(4D_1(\omega)/\omega\right)\right]}. \tag{29}$$

Remark 2. For the Clayton–Gumbel–Frank (C_{CGF}) copula, if the convex combination parameter $\gamma=0$, we obtain the Frank copula. If $\gamma=1$, we obtain the Clayton–Gumbel (C_{CG}) copula. We can derive the proportions in the Clayton–Gumbel copula using λ.

Remark 3. In obtaining the proportion parameters λ and γ $\gamma\in[0,1]$, the above estimators may be used. However, the alternative of selecting an arbitrary choice of parameters in [0, 1] and comparing them is not ruled out.

2.9. Tail Dependence Analysis. For the hybrid model, the lower tail coefficient will be the combination of the lower tail dependencies of the copulas combined. Likewise, the upper tail dependence coefficient will be the combination of the upper tail dependencies of the combined copulas.

2.10. Lower Tail Dependence. The lower tail dependence is defined as

$$\lim_{u\longrightarrow 1}\frac{C(u,u)}{u}=T_L=\lim_{u\longrightarrow 0}\frac{\lambda C_1(u,v;\theta_1)+(1-\lambda)C_2(u,v;\theta_2)}{u}=\lambda T_L^{\theta_1}+(1-\lambda)T_L^{\theta_2}. \tag{30}$$

2.11. Upper Tail Dependence. The upper tail dependence is defined as

$$\lim_{u\longrightarrow 1}\frac{1-2uC(u,u)}{1-u}=U_L\lim_{u\longrightarrow 1}\frac{1-2u+\lambda C_1(u,v;\theta_1)+(1-\lambda)C_2(u,v;\theta_2)}{1-u}=\lambda T_U^{\theta_1}+(1-\lambda)T_U^{\theta_2}. \tag{31}$$

Table 2 gives the tail dependence coefficients of the standalone models and that of the proposed Clayton–Gumbel–Frank model.

3. Application of Model to Data

Data obtained from the Ghana Stock Exchange (GSE) for two companies are used in the model validation. These two stocks were chosen because they were the most consistent in terms of trading on the stock exchange. The two stocks had data available for the period under study. The data set was the returns data (https://gse.com.gh/daily-shares-and-etfs-trades/) from the 1st of January 2018 to the 1st of July 2021. In all, there were 2048 data points. The descriptive statistics (Table 3) give a quantitative analysis of the returns series of both CAL and GCB banks.

The returns from the two indices have positive means and more kurtosis than normal. This indicates that both returns series exhibit positive excess kurtosis (fat-tailed distributions), making them leptokurtic. The two returns series are fairly symmetrical, looking at the value of the skewness. Time series models were fitted to the individual stock returns in order to find the appropriate marginal distributions. The returns for CAL had an MA [22] model while that of GCB had an ARMA [1, 1] model. Since stock returns exhibit dynamic volatility in general, the two series were subjected to heteroscedasticity tests. Using the ARCH-LM test, the two series were found to be heteroscedastic.

Table 4: Estimates of proportion.

Copula	Proportion
Clayton–Gumbel (λ)	Clayton = 0.082, Gumbel = 0.918
Clayton–Gumbel–Frank (γ)	Clayton–Gumbel = 0.780, Frank = 0.22

Table 5: Parameter estimates with LTD, UTD, AIC, and BIC.

Copula	Parameter	LTD	UTD	AIC	BIC
C	0.05	9.6×10^{-6}	0	−4.25	−3.51
F	0.19	0	0	−3.29	−3.02
G	1.02	0	0.03	−4.6	−3.41
C-G	0.940	7.87×10^{-7}	0.0275	−4.7824	−3.686
C-F	0.0459	7.872×10^{-7}	0	−3.7604	−2.42
G-F	0.95194	0	0.02754	−4.648	−3.01
F-C	0.0808	7.872×10^{-7}	0	−4.0388	−2.31
G-C	0.94046	7.87×10^{-7}	2.46×10^{-3}	−4.5713	−3.001
F-G	0.8374	0	0.0234	−4.3118	−3.24
CG-F	0.775	6.14×10^{-7}	0.006	−4.8214	−3.9202

C = Clayton, F = Frank, G = Gumbel, C-G = Clayton–Gumbel, C-F = Clayton–Frank, G-F = Gumbel–Frank, F-C = Frank–Clayton, G-C = Gumbel–Clayton, F-G = Frank–Gumbel, and CG-F = Clayton–Gumbel-Frank.

Autoregressive conditional heteroscedasticity and generalized autoregressive heteroscedasticity models were then applied to the residuals of the MA [22] and ARMA [1, 1] models, respectively. This finally produced an MA [22]-GARCH [1, 1] model as the marginal distribution of the CAL return series and an ARMA [1, 1]-GARCH [1, 1] model as the marginal distribution of the GCB returns series.

3.1. Estimates of λ and γ for the Hybrid Model. From the Kendall's tau representations, the following estimates are obtained and used to calculate the value of λ and γ for the hybrid model: $\theta = 0.05, \beta = 1.02$, $\omega = 0.19, \tau = 0.02$, and $D_1(\omega) = 0.953502$.

Table 4 shows that the proportion (λ) of the Clayton copula in the Clayton–Gumbel mixture is approximately 0.082 with that of Gumbel being approximately 0.918. The proportion in the Clayton–Gumbel–Frank copula (γ) is approximately 0.78 for the Clayton–Gumbel and approximately 0.22 for the Frank copula. This implies that for the returns series under study, they exhibit all scenarios of dependence, i.e., lower tail, upper tail, and dependence in-between the tails. Thus, they are likely to lose and gain together.

3.2. Estimates of Tail Dependencies. Table 5 gives the estimates of the dependence parameters, lower tail dependencies (LTD), upper tail dependencies (UTD), AIC, and BIC of the mixture copulas.

In all, there were 10 alternative models to compare. However, the study objective hinged on the ability of the mixture copulas to provide all possible dependence scenarios. The CG-F copula, while providing for all dependence cases, had the smallest AIC and BIC values, followed by the C-G copula.

Remark 4. In terms of lower and upper tail dependence, the hybrid Clayton–Gumbel–Frank copula behaves like the hybrid Clayton–Gumbel copula. However, the former goes further to show dependencies in-between the tails.

Remark 5. It can be observed that the magnitude of the upper tail dependence for both the Clayton–Gumbel and the Clayton–Gumbel–Frank is greater than that of the lower tail dependence. This means that the two stocks exhibit greater dependence during market upturns than during market downturns.

4. Conclusion

Result 1 showed that if combinations were to be done multiplicatively, the tail dependence coefficients would be zero. Result 2 provided evidence in support of the convex mixture since tail dependence coefficients were obtained for such a mixture. The hybrid Clayton–Gumbel–Frank copula has been constructed together with its corresponding joint density and conditional distribution function for a bivariate case.

The hybrid model possessed all aspects of tail dependence. Specifically, the hybrid Clayton–Gumbel–Frank copula was found to have both lower and upper tail dependence coefficients as well as an inherent dependence between the tails.

The model was applied to the returns series of CAL bank and GCB bank from the Ghana Stock Exchange. Barley et al. [22] examined the effects of perturbations on some selected Archimedean copulas, namely, Clayton, Joe, Frank, and Gumbel copulas. In their study, they found the Gumbel copula to be the most robust. This study looked at the tail dependence strength of the hybrid model built from convex combinations of the Clayton, Gumbel, and Frank copulas. The study has provided a mixture model that gives a full spectrum of possible dependence between random variables. The study also provides explicit formulae for the proportional parameters of the copulas in the mixture. Finally, the results show that the two companies used in the study are likely to lose or gain together based on results for the years under review.

Data Availability

The datasets used in this study were returns data obtained from the Ghana Stock Exchange (https://gse.com.gh/daily-shares-and-etfs-trades/).

Conflicts of Interest

The authors declare that they have no conflicts of interest with this study.

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