Some Improved Classes of Estimators in Stratified Sampling Using Bivariate Auxiliary Information

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1. Introduction

In sampling theory, the appropriate utilization of auxiliary information plays a leading role to improve the efficiency of the estimators. This information may be utilized either at the design phase (sampling design) or at the estimation phase or in both phases. It is very popular when auxiliary information is considered at the estimation phase, the ratio, product, regression, and exponential type estimators are mostly the preferred methods in different dimensions. Shabbir et al. [1] examined the performance of ratio-exponential log type class of estimators using two auxiliary variables. Shahzad et al. [2] introduced a novel family of variance estimators based on L-moments and calibration approach under stratified simple random sampling (SSRS), whereas Shahzad et al. [3] suggested L-Moments and calibration-based variance estimators under double SSRS and discussed an application of COVID-19 pandemic. Shahzad et al. [4] considered the estimation of coefficient of variation using L-moments and calibration approach for nonsensitive and sensitive variables, whereas Shahzad et al. [5] developed variance estimation based on L-moments and auxiliary information. The estimation of population mean is a widely discussed approach in sample surveys and many renowned authors have utilized these auxiliary pieces of information at estimation stage and suggested various modified estimators to date. Especially, under the availability of multi-auxiliary information, the literature contains different kinds of ratio, product, regression, and exponential type estimators. In SSRS, the utilization of auxiliary information at the estimation stage has been discussed by many authors to enhance the efficiency of the estimators. In presence of univariate auxiliary information, Hansen et al. [6]; Kadilar and Cingi [7]; Shabbir and Gupta [8]; Singh and Vishwakarma [9]; Solanki and Singh [10]; Bhushan et al. [11]; etc., suggested various modified estimators of population mean whereas, in presence of multi-auxiliary information, Koyuncu and Kadilar [12] suggested a family of estimators of population mean based on SSRS. Tailor et al. [13] envisaged ratio-cum-product estimator of population mean in SSRS. Tailor and Chouhan [14] addressed ratio-cum-product type exponential estimator of finite population mean. Following Upadhyaya et al. [15]; Singh et al. [16] considered a class of ratio-cum-product estimators using information on two auxiliary...
variables in SSRS. Along the lines of Singh et al. [17]; Lone et al. [18] introduced a generalized ratio-cum-product type exponential estimator in SSRS, whereas Lone et al. [19] suggested efficient separate class of estimators of population mean. Following Singh and Singh [20]; Muneer et al. [21] suggested a class of combined estimators in SSRS. Recently, Muneer et al. [22] introduced a chain ratio exponential family of estimators based on SSRS. In the present study, we propose some improved classes of estimators for the estimation of population mean by employing bivariate auxiliary information under SSRS.

Consider a finite population \( \kappa = (k_1, k_2, \ldots, k_N) \) based on size \( N \) units with study variable \( y \) and two auxiliary variables \( x \) and \( z \), respectively, associated with each unit \( k_i, i = 1, 2, \ldots, N \) of the population. Let the population be divided into \( L \) disjoint strata with the stratum \( h \) comprises of \( N_h \), \( h = 1, 2, \ldots, L \) units. Let a simple random sample of size \( n_h \) be quantified without replacement from the stratum \( h \) such that \( \sum_{h=1}^{L} n_h = n \). Let the observed values of \( y, x \) and \( z \) on the \( i \)th unit of the stratum \( h \) be denoted by \((y_{ih}, x_{ih}, z_{ih}), i = 1, 2, \ldots, N_h \). Let \( \bar{Y}_h = \sum_{i=1}^{N_h} y_{ih}/N_h \), \( \bar{X}_h = \sum_{i=1}^{N_h} x_{ih}/N_h \) and \( \bar{Z}_h = \sum_{i=1}^{N_h} z_{ih}/N_h \) respectively, be the sample means corresponding to the population means \( Y_h = \sum_{i=1}^{N_h} y_{ih}/N_h, X_h = \sum_{i=1}^{N_h} x_{ih}/N_h \) and \( Z_h = \sum_{i=1}^{N_h} z_{ih}/N_h \) of variables \( y, x \) and \( z \), where \( W_h = N_h/N \) is the weight in the stratum \( h \). Let \( \bar{y}_h = \sum_{i=1}^{n_h} y_{ih}/n_h, \bar{x}_h = \sum_{i=1}^{n_h} x_{ih}/n_h \) and \( \bar{z}_h = \sum_{i=1}^{n_h} z_{ih}/n_h \) be the sample means corresponding to the population means \( Y_h = \sum_{i=1}^{N_h} y_{ih}/N_h \), \( X_h = \sum_{i=1}^{N_h} x_{ih}/N_h \) and \( Z_h = \sum_{i=1}^{N_h} z_{ih}/N_h \) of variables \( y, x \) and \( z \) in the stratum \( h \). Let \( s_{y,h}^2 = \sum_{i=1}^{N_h} (y_{ih} - \bar{y}_h)^2/(n_h - 1) \), \( s_{x,h}^2 = \sum_{i=1}^{N_h} (x_{ih} - \bar{x}_h)^2/(n_h - 1) \), \( s_{z,h}^2 = \sum_{i=1}^{N_h} (z_{ih} - \bar{z}_h)^2/(n_h - 1) \), \( s_{xy,h} = \sum_{i=1}^{N_h} ((x_{ih} - \bar{x}_h)(y_{ih} - \bar{y}_h))/(n_h - 1) \), \( s_{xz,h} = \sum_{i=1}^{N_h} ((x_{ih} - \bar{x}_h)(z_{ih} - \bar{z}_h))/(n_h - 1) \) and \( s_{yz,h} = \sum_{i=1}^{N_h} ((y_{ih} - \bar{y}_h)(z_{ih} - \bar{z}_h))/(n_h - 1) \) respectively, be the sample variances and covariances corresponding to the population variances and covariances \( \sigma_{y,h}^2 = \sum_{i=1}^{N_h} (y_{ih} - Y_h)^2/(N_h - 1), \sigma_{x,h}^2 = \sum_{i=1}^{N_h} (x_{ih} - X_h)^2/(N_h - 1), \sigma_{y,z,h} = \sum_{i=1}^{N_h} (y_{ih} - Y_h)(z_{ih} - Z_h)/(N_h - 1), \sigma_{x,z,h} = \sum_{i=1}^{N_h} (x_{ih} - X_h)(z_{ih} - Z_h)/(N_h - 1), \sigma_{x,y,z,h} = \sum_{i=1}^{N_h} ((x_{ih} - X_h)(y_{ih} - Y_h)(z_{ih} - Z_h))/(N_h - 1) \) in the stratum \( h \).

To derive the bias and mean square error (MSE) of different combined estimators, the following notations will be used throughout the paper.

\[
V_{rs} = \sum_{h=1}^{L} W_h^r W_h^{s+e} E \left[ (\bar{Y}_h - \bar{Y}_h)^r (\bar{X}_h - \bar{X}_h)^s (\bar{Z}_h - \bar{Z}_h)^e \right].
\]

Following (1), we can write \( E(\bar{y}_h) = \frac{1}{n_h} \sum_{h=1}^{n_h} w_{ih} y_{ih} = y_{000} \), \( E(\bar{x}_h) = \frac{1}{n_h} \sum_{h=1}^{n_h} w_{ih} x_{ih} = x_{000} \), \( E(\bar{z}_h) = \frac{1}{n_h} \sum_{h=1}^{n_h} w_{ih} z_{ih} = z_{000} \), \( E(\bar{x}_h \bar{y}_h) = \frac{1}{n_h} \sum_{h=1}^{n_h} w_{ih} x_{ih} y_{ih} = x_{001} \), \( E(\bar{x}_h \bar{z}_h) = \frac{1}{n_h} \sum_{h=1}^{n_h} w_{ih} x_{ih} z_{ih} = x_{010} \), \( E(\bar{y}_h \bar{z}_h) = \frac{1}{n_h} \sum_{h=1}^{n_h} w_{ih} y_{ih} z_{ih} = y_{010} \) and \( E(\bar{x}_h \bar{y}_h \bar{z}_h) = \frac{1}{n_h} \sum_{h=1}^{n_h} w_{ih} x_{ih} y_{ih} z_{ih} = xy_{010} \).

2. Existent Estimators

2.1. Combined Estimators. The conventional combined mean estimator under SSRS is given by

\[
y^c_{m} = \bar{Y}_h = \bar{X}_h + \bar{Z}_h. \tag{2}
\]

On the lines of Singh [23]; one may consider the classical combined ratio estimator of population mean \( \bar{Y} \) using bivariate auxiliary information under SSRS as

\[
y^c_{r} = \bar{Y}_h \bar{X}_h \bar{Z}_h. \tag{3}
\]

The classical combined regression estimator of population mean \( \bar{Y} \) based on bivariate auxiliary information under SSRS is given by

\[
y^r_{r} = \bar{Y}_h + \beta_1 (X - \bar{X}_h) + \beta_2 (Z - \bar{Z}_h), \tag{4}
\]

where \( \beta_1 \) and \( \beta_2 \) are the regression coefficients of \( y \) on \( x \) and \( z \), respectively.

Following Olkin [24]; the combined ratio type estimator using bivariate auxiliary information under SSRS is given by

\[
y^c_{a} = \bar{Y}_h \left\{ w \frac{\bar{X}_h}{\bar{z}_h} + (1 - w) \frac{\bar{Z}_h}{\bar{z}_h} \right\}, \tag{5}
\]

where \( w \) is a duly opted scalar.

Along the lines of Abu–Daryeh et al. [25]; some combined classes of ratio type estimators using bivariate auxiliary information under SSRS are given by

\[
y^c_{a} = \bar{Y}_h \left\{ \frac{\bar{X}_h}{\bar{z}_h} \right\}^{\alpha_1} \left\{ \frac{\bar{Z}_h}{\bar{z}_h} \right\}^{\alpha_2}, \tag{6}
\]

\[
y^w_{a} = \bar{Y}_h \left\{ w_1 \left\{ \frac{\bar{X}_h}{\bar{z}_h} \right\}^{\alpha_1} + w_2 \left\{ \frac{\bar{Z}_h}{\bar{z}_h} \right\}^{\alpha_2} \right\}, \tag{7}
\]
where \( a_1, a_2, w_1, \) and \( w_2 \) are duly opted scalars and \( w_1 + w_2 = 1 \).

Following Kadilar and Cingi [7], one may suggest some combined ratio-cum-product estimators based on bivariate auxiliary information under SSRS as

\[
\bar{y}_{kc_1} = \bar{y}_a \left[ \frac{aX + b}{\lambda_1 (aX_a + b) + (1 - \lambda_1) (aX + b)} \right]^{g_1} \left( \frac{cZ + d}{\lambda_2 (cZ_a + d) + (1 - \lambda_2) (cZ + d)} \right)^{g_2},
\]

(13)

where \( \lambda_1, \lambda_2, g_1, \) and \( g_2 \) are prescribed scalars, whereas \( a \neq 0, b \) and \( c \neq 0, d \) are either real numbers or functions of the known parameters of auxiliary variables \( x \) and \( z \), respectively.

Along the lines of Singh et al. [17]; Tailor and Chouhan [14] suggested a combined ratio-cum-product type exponential estimator for population mean using bivariate auxiliary information under SSRS as

\[
\bar{y}_c = k \bar{y}_a \exp \left( \frac{X - X_a}{X + X_a} \right) \exp \left( \frac{Z - Z}{Z_a + Z} \right),
\]

(14)

where \( k \) is a duly opted scalar.

Following Singh et al. [17], Lone et al. [18] suggested a combined generalized ratio-cum-product type exponential estimator in SSRS as

\[
\bar{y}_{mu} = [k_1 \bar{y}_a - k_4 (X_a - X)] \left[ \Theta \left\{ 2 - \exp \left( \frac{Z - Z}{Z_a + Z} \right) \right\} + (1 - \Theta) \left\{ \exp \left( \frac{Z - Z}{Z_a + Z} \right) \right\} \right],
\]

(17)

where \( k_3, k_4 \) are duly opted scalars and \( \Theta \) is a scalar assuming values 0 and 1 to design different estimators.

The combined form of Muneer et al. [22] chain ratio exponential family of estimator in SSRS is given by

\[
\bar{y}_{mu} = \bar{y}_a \left( \frac{X}{w_1X_a + (1 - w_1X)} \right)^{a_1} \left( \frac{Z}{w_2Z_a + (1 - w_2Z)} \right)^{a_2} \exp \left( \frac{a_3 (X - X_a)}{X_a + X} \right) \exp \left( \frac{a_4 (Z - Z_a)}{Z_a + Z} \right).
\]

(18)

On the lines of Searls [27], an improved form of the above-combined estimator is given by
\[ \mathcal{Y}_{m_{2i}} = k_1 \bar{Y}_h \left( \frac{X^*}{w_1 X^*_h + (1 - w_1 X^*)} \right)^{a_i} \left( \frac{Z^*}{w_2 Z^*_h + (1 - w_2 Z^*)} \right)^{a_t} \exp \left\{ \frac{\alpha_1 (X - X_h)}{(X + X_h)} \right\} \exp \left\{ \frac{\alpha_t (Z - Z_h)}{(Z + Z_h)} \right\}, \]  

where \( X^* = a X + b, \) \( Z^* = c Z + d, \) \( X^*_h = a X_h + b, \) \( Z^*_h = c Z_h + d, \) \( w_1, \) \( w_2 = (0, 1) \) and \( k_1 \) is a duly opted scalar, \( a_i, j = 1, 2, 3, 4 \) assumes values \(-1, 0, \) and \(+1\) to form different new and existing estimators. Moreover, the authors have shown that more than 65 combined classes of estimators are the members of the estimators \( \mathcal{Y}_{m_{2i}} \) and \( \mathcal{Y}_{m_{2i}} \), respectively, for different values of scalars.

The bias and MSE of the estimators considered in this section are readily discussed in Appendix A.

2.2. Separate Estimators. The conventional separate mean estimator under SSRS is given by

\[ \mathcal{Y}_m = \sum_{h=1}^{L} W_h \bar{Y}_h. \]  

On the lines of Singh [23]; the classical separate ratio estimator of population mean \( \bar{Y} \) using bivariate auxiliary information under SSRS is defined as

\[ \mathcal{Y}_f = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h}{X_h} \right) \left( \frac{Z_h}{Z_h} \right). \]  

The classical separate regression estimator of population mean \( \bar{Y} \) under bivariate auxiliary information using SSRS is

\[ \mathcal{Y}_r = \sum_{h=1}^{L} W_h \left[ \bar{Y}_h + \beta_{1h} (X_h - \bar{X}_h) + \beta_{2h} (Z_h - \bar{Z}_h) \right], \]  

where \( \beta_{1h} \) and \( \beta_{2h} \) are the regression coefficients of \( y \) on \( x \) and \( z \), respectively, in stratum \( h \).

Motivated by Olkin [24]; the separate ratio estimator in SSRS using bivariate auxiliary information is given by

\[ \mathcal{Y}_s = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h}{X_h} \right) + (1 - w_h) \left( \frac{Z_h}{Z_h} \right), \]  

where \( w_h \) is a duly opted scalar in the stratum \( h \) to be determined.

Following Abu-Dayyeh et al. [25]; a separate class of ratio type estimators using bivariate auxiliary information under SSRS is given by

\[ \mathcal{Y}_d = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h}{X_h} \right) \left( \frac{Z_h}{Z_h} \right), \]  

\[ \mathcal{Y}_w = \sum_{h=1}^{L} W_h \bar{Y}_h \left\{ w_1 \left( \frac{X_h}{X_h} \right) + w_2 \left( \frac{Z_h}{Z_h} \right) \right\}, \]  

where \( a_1, a_2, \) \( w_1, \) and \( w_2 \) are duly opted scalars in stratum \( h \) and \( w_1 + w_2 = 1. \)

Following Kadilar and Cingi [7]; one can suggest some separate ratio-cum-product type estimators based on bivariate auxiliary information under SSRS as

\[ \mathcal{Y}_{kc_1} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + C_{x_h}}{X_h + C_{x_h}} \right) \left( \frac{Z_h + C_{z_h}}{Z_h + C_{z_h}} \right), \]  

\[ \mathcal{Y}_{kc_2} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + \beta_1 (x_h)}{X_h + \beta_1 (x_h)} \right) \left( \frac{Z_h + \beta_2 (z_h)}{Z_h + \beta_2 (z_h)} \right), \]  

\[ \mathcal{Y}_{kc_3} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + \beta_1 (x_h)}{X_h + \beta_1 (x_h)} \right) \left( \frac{Z_h + \beta_2 (z_h)}{Z_h + \beta_2 (z_h)} \right), \]  

\[ \mathcal{Y}_{kc_4} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + \beta_1 (x_h)}{X_h + \beta_1 (x_h)} \right) \left( \frac{Z_h + \beta_2 (z_h)}{Z_h + \beta_2 (z_h)} \right), \]  

\[ \mathcal{Y}_{zc_1} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + C_{z_h}}{X_h + C_{z_h}} \right) \left( \frac{Z_h + \beta_2 (z_h)}{Z_h + \beta_2 (z_h)} \right), \]  

\[ \mathcal{Y}_{zc_2} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + \beta_1 (x_h)}{X_h + \beta_1 (x_h)} \right) \left( \frac{Z_h + \beta_2 (z_h)}{Z_h + \beta_2 (z_h)} \right), \]  

\[ \mathcal{Y}_{zc_3} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + \beta_1 (x_h)}{X_h + \beta_1 (x_h)} \right) \left( \frac{Z_h + \beta_2 (z_h)}{Z_h + \beta_2 (z_h)} \right), \]  

\[ \mathcal{Y}_{zc_4} = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h + \beta_1 (x_h)}{X_h + \beta_1 (x_h)} \right) \left( \frac{Z_h + \beta_2 (z_h)}{Z_h + \beta_2 (z_h)} \right), \]  

where \( a_{1h}, a_{2h}, \) \( w_1, \) and \( w_2 \) are duly opted scalars in stratum \( h \) and \( w_1 + w_2 = 1. \)

The separate version of Singh et al. [16] estimator is given by

\[ \mathcal{Y}_s = \sum_{h=1}^{L} W_h \left\{ \eta_{1h} \bar{Y}_h + \eta_{2h} \frac{X_h}{X_h} \left( \frac{Z_h}{Z_h} \right) \right\}, \]  

where \( \eta_{1h}, \eta_{2h} \) are duly opted scalars in stratum \( h \) and \( \alpha_h, \beta_h \) are scalars in stratum \( h \) taking real values.

The separate version of Koyuncu and Kadilar [12] family of estimators is given by

\[ \mathcal{Y}_s^\prime = \sum_{h=1}^{L} W_h \bar{Y}_h \left( \frac{X_h}{X_h} \right) \left( \frac{Z_h}{Z_h} \right) \left\{ \frac{c_{x_h} Z_h + d_h}{\lambda_2 (c_{x_h} Z_h + d_h)} \left( \frac{c_{z_h} Z_h + d_h}{\lambda_1 (c_{z_h} Z_h + d_h)} \right) \right\}, \]  

where \( k_h \) is a duly opted scalar in stratum \( h. \)

On the lines of Lone et al. [18]; a generalized separate ratio-cum-product type exponential estimator in SSRS is defined as
\[
\bar{Y}_i^i = \sum_{h=1}^{l} W_h \bar{Y}_h \exp \left\{ L_{i1} \left( \frac{X_h - \bar{X}_h}{X_h + \bar{X}_h} \right) + L_{i2} \left( \frac{Z_h - \bar{Z}_h}{Z_h + \bar{Z}_h} \right) \right\},
\]

(33)

where \( L_{i1} \) and \( L_{i2} \) are duly opted scalars in the stratum \( h \).

The separate version of Lone et al. [19] estimator for estimating population mean \( \bar{Y} \) as
\[
\bar{Y}_i^s = \sum_{h=1}^{l} W_h \bar{Y}_h \left( \frac{a_h \bar{X}_h + b_h}{a_h \bar{X}_h + b_h} \right) \left( \frac{c_h \bar{Z}_h + d_h}{c_h \bar{Z}_h + d_h} \right).
\]

(34)

\[
\bar{Y}_{mu}^s = \sum_{h=1}^{l} W_h \left\{ k_h \bar{Y}_h - k_h (\bar{X}_h - \bar{X}) \right\} \left[ \Theta_h \left\{ 2 - \exp \left( \frac{\bar{Z}_h - \bar{Z}}{\bar{Z}_h + \bar{Z}} \right) \right\} + (1 - \Theta_h) \left\{ \exp \left( \frac{\bar{Z}_h - \bar{Z}}{\bar{Z}_h + \bar{Z}} \right) \right\} \right],
\]

(35)

where \( k_h \) and \( \Theta_h \) are suitably chosen scalars in stratum \( h \) and \( \Theta_h \) is a real constant in stratum \( h \).

Muneer et al. [22] suggested a separate chain ratio exponential family of estimators in SSRS as
\[
\bar{Y}_{mu}^s = \sum_{h=1}^{l} W_h \left\{ \frac{X^*_h}{\omega_{1h} x^*_h + (1 - \omega_{1h}) \bar{X}_h} \right\}^{a_{ih}} \left\{ \frac{Z^*_h}{\omega_{2h} z^*_h + (1 - \omega_{2h}) \bar{Z}_h} \right\}^{a_{kh}} \times \exp \left\{ \alpha_{jh} (\bar{X}_h - \bar{X}) \right\} \exp \left\{ \alpha_{kh} (\bar{Z}_h - \bar{Z}) \right\} \right].
\]

(36)

On the lines of Searls [27], the modified form of the above separate estimator is given by
\[
\bar{Y}_{mu}^s = \sum_{h=1}^{l} W_h \left\{ \frac{X^*_h}{\omega_{1h} x^*_h + (1 - \omega_{1h}) \bar{X}_h} \right\}^{a_{ih}} \left\{ \frac{Z^*_h}{\omega_{2h} z^*_h + (1 - \omega_{2h}) \bar{Z}_h} \right\}^{a_{kh}} \times \exp \left\{ \alpha_{jh} (\bar{X}_h - \bar{X}) \right\} \exp \left\{ \alpha_{kh} (\bar{Z}_h - \bar{Z}) \right\} \right].
\]

(37)

where \( X^* = a_h \bar{X}_h + b_h, x^* = a_h \bar{x}_h + b_h, Z^* = c_h \bar{Z}_h + d_h, z^* = c_h \bar{z}_h + d_h \) and \( \omega_{1h}, \omega_{2h} = (0, 1) \) and \( k_{ij} \) is a duly opted scalar in stratum \( h \), \( \alpha_{ij} \), \( j = 1, 2, 3, 4 \) assumes values \(-1, 0, \) and \(+1\) in order to form different new and existing separate estimators. Furthermore, one can generate more than 65 separate classes of estimators from \( \bar{Y}_{mu}^s \) and \( \bar{Y}_{mu}^s \) for different values of scalars. The bias and \( MSE \) of the estimators considered in this section are readily discussed in the Appendix B.

3. Proposed Estimators

The objective of this paper is to suggest some improved combined and separate classes of estimators over the existing combined and separate estimators discussed in the previous section. We have extended the work of Bhushan et al. [28] for the estimation of population mean \( \bar{Y} \) by incorporating bivariate auxiliary information under SSRS.

3.1. Combined Estimators. We propose some improved combined classes of estimators based on bivariate auxiliary information under SSRS as
\[
\bar{Y}_{xi}^{c} = \xi_1 \bar{Y}_{xi} \left[ 1 + \log \left( \frac{X_{x_i}}{\bar{X}} \right) \right]^{\theta_1} \left[ 1 + \log \left( \frac{Z_{y_i}}{\bar{Z}} \right) \right]^{\delta_1},
\]

(38)

\[
\bar{Y}_{yi}^{c} = \xi_2 \bar{Y}_{yi} \left[ 1 + \theta_2 \log \left( \frac{X_{y_i}}{\bar{X}} \right) \right] \left[ 1 + \delta_2 \log \left( \frac{Z_{y_i}}{\bar{Z}} \right) \right],
\]

(39)
\begin{align*}
\bar{Y}_n &= \xi_1 \bar{Y}_{\text{at}} + \theta_1 (\bar{X}_{\text{at}} - \bar{X}) + \delta_1 (\bar{Z}_{\text{at}} - \bar{Z}), \\
\bar{Y}_n &= \xi_3 \bar{Y}_{\text{at}} \left( \frac{\bar{X}}{\bar{X}_{\text{at}}} \right) \delta_4 \left( \frac{\bar{Z}}{\bar{Z}_{\text{at}}} \right),
\end{align*}
\tag{40}

where \( \xi_i, \theta_i, \text{and } \delta_i, i = 1, 2, \ldots, 5 \) are duly opted scalars to be determined.

**Theorem 1.** The bias of the proposed combined classes of estimators \( \bar{Y}_n, i = 1, 2, \ldots, 5 \) is given by

\begin{align*}
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_1 \left( \left[ 1 + \left( \frac{\theta_1^2}{2} - \theta_1 \right)V_{020} + \left( \frac{\delta_1^2}{2} - \delta_1 \right)V_{002} + \theta_1 V_{110} \right) \right) - 1 \right\}, \\
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_2 \left[ 1 - \frac{\theta_2^2}{2}V_{020} - \frac{\delta_2^2}{2}V_{002} + \theta_2 V_{110} + \delta_2 V_{011} \right] - 1 \right\}, \\
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_3 \left[ 1 + \frac{\theta_3^2}{2}V_{020} + \frac{\delta_3^2}{2}V_{002} - \theta_3 V_{110} - \delta_3 V_{011} \right] - 1 \right\}, \\
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_4 \left[ 1 + \frac{\theta_4^2}{2}V_{020} + \frac{\delta_4^2}{2}V_{002} - \theta_4 V_{110} - \delta_4 V_{011} \right] - 1 \right\}.
\end{align*}
\tag{43}

\begin{align*}
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_1 \left( \left[ 1 + \left( \frac{\theta_1^2}{2} - \theta_1 \right)V_{020} + \left( \frac{\delta_1^2}{2} - \delta_1 \right)V_{002} + \theta_1 V_{110} + \delta_1 V_{011} \right) \right) - 1 \right\}, \\
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_2 \left[ 1 - \frac{\theta_2^2}{2}V_{020} - \frac{\delta_2^2}{2}V_{002} + \theta_2 V_{110} + \delta_2 V_{011} \right] - 1 \right\}, \\
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_3 \left[ 1 + \frac{\theta_3^2}{2}V_{020} + \frac{\delta_3^2}{2}V_{002} - \theta_3 V_{110} - \delta_3 V_{011} \right] - 1 \right\}, \\
\text{Bias}(\bar{Y}_n) &= Y \left\{ \xi_4 \left[ 1 + \frac{\theta_4^2}{2}V_{020} + \frac{\delta_4^2}{2}V_{002} - \theta_4 V_{110} - \delta_4 V_{011} \right] - 1 \right\}.
\end{align*}
\tag{44}

**Theorem 2.** The MSE of the proposed combined classes of estimators \( \bar{Y}_n, i = 1, 2, \ldots, 5 \) is given by

\begin{align*}
\text{MSE}(\bar{Y}_n) &= Y^2 \left[ 1 + \xi_1 \left[ 1 + V_{200} + 2\theta_1^2 V_{020} + 2\delta_1^2 V_{002} - 2\theta_1 V_{020} - 2\delta_1 V_{002} + 4\theta_1 V_{110} + 4\delta_1 V_{011} \right] \right] \\
&\quad - 2\xi_1 \left[ 1 + \frac{\theta_1^2}{2}V_{020} + \frac{\delta_1^2}{2}V_{002} - \theta_1 V_{020} - \delta_1 V_{002} + \theta_1 V_{110} + \delta_1 V_{011} \right], \\
\text{MSE}(\bar{Y}_n) &= Y^2 \left[ 1 + \xi_2 \left[ 1 + V_{200} + \theta_2^2 V_{020} + \delta_2^2 V_{002} - \theta_2 V_{020} - \delta_2 V_{002} + 4\theta_2 V_{110} + 4\delta_2 V_{011} \right] \right] \\
&\quad - 2\xi_2 \left[ 1 - \frac{\theta_2^2}{2}V_{020} + \theta_2 V_{110} + \delta_2 V_{011} \right], \\
\text{MSE}(\bar{Y}_n) &= Y^2 \left[ 1 + \xi_3 \left[ 1 + V_{200} + \theta_3^2 V_{020} + \delta_3^2 V_{002} - \theta_3 V_{020} - \delta_3 V_{002} + 4\theta_3 V_{110} + 4\delta_3 V_{011} \right] \right] \\
&\quad - 2\xi_3 \left[ 1 + \theta_3^2 V_{020} + \delta_3^2 V_{002} - \theta_3 V_{020} - \delta_3 V_{002} + \theta_3 V_{110} + \delta_3 V_{011} \right], \\
\text{MSE}(\bar{Y}_n) &= Y^2 \left[ (\xi_3 - 1)^2 + \xi_3^2 \left[ 1 + V_{200} + \theta_3^2 V_{020} + \delta_3^2 V_{002} \right] + 2\xi_3 \theta_3 V_{YV} V_{110} + 2\xi_3 \delta_3 V_{ZV} V_{011} \right], \\
\text{MSE}(\bar{Y}_n) &= Y^2 \left[ 1 + \xi_4 \left[ 1 + V_{200} + \theta_4 V_{020} + \delta_4 V_{002} + 2\theta_4 V_{020} + 2\delta_4 V_{002} - 4\theta_4 V_{110} - 4\delta_4 V_{011} \right] \right] \\
&\quad - 2\xi_4 \left[ 1 + \theta_4 V_{020} + \frac{\delta_4^2}{2}V_{002} - \theta_4 V_{020} - \frac{\delta_4^2}{2}V_{002} - \theta_4 V_{110} - \delta_4 V_{011} \right].
\end{align*}
\tag{45}
\[ \text{MSE}(\bar{Y}_{si}) = \sum_{i=1}^{L} W_{si} \left[ X_{hi} \left( 1 + \theta_{hi} \right) U_{002} + \left( \frac{\theta_{hi}}{2} - \delta_{hi} \right) U_{002} \right] + \theta_{hi} U_{110} + \delta_{hi} U_{011} - 1 \]  

\[ \text{Bias}(\bar{Y}_{si}) = \sum_{i=1}^{L} W_{si} \left[ \xi_{hi} \left( 1 + \frac{\theta_{hi}}{2} \right) U_{020} + \frac{\delta_{hi}^{2}}{2} U_{002} - \theta_{hi} U_{110} \right] - 1 \]  

\[ \text{Bias}(\bar{Y}_{si}) = \sum_{i=1}^{L} W_{si} \left[ \xi_{hi} \left( 1 + \frac{\theta_{hi}(\theta_{hi} + 1)}{2} U_{020} + \frac{\delta_{hi}^{2}}{2} U_{002} - \theta_{hi} U_{110} \right] - 1 \right]  

\[ \text{Bias}(\bar{Y}_{si}) = \sum_{i=1}^{L} W_{si} \left[ \xi_{hi} \left( 1 + \frac{\theta_{hi}^{2} + \delta_{hi}^{2}}{2} U_{020} + \frac{\theta_{hi}^{2} + \delta_{hi}^{2}}{2} U_{002} - \theta_{hi} U_{110} \right) - 1 \right]  

\[ \text{Proof.} \text{ The precis of the derivations are given in Appendix C.} \]

\[ \text{Corollary 1.} \text{ The minimum MSE of the proposed combined classes of estimators } \bar{Y}_{si}, i=1,2,...,5 \text{ is given by} \]

\[ \text{min MSE}(\bar{Y}_{si}) = \bar{Y}^{2} \left( 1 - \frac{B_{j}^{2}}{A_{j}} \right); \quad i = 1, 2, 4, 5, \]

\[ \text{min MSE}(\bar{Y}_{si}) = \bar{Y}^{2} \left( 1 - \xi_{j}^{2} \right) = \bar{Y}^{2} \left( 1 - \frac{B_{j}^{2}}{A_{j}} \right). \]

\[ \text{Proof.} \text{ The precis of derivations and the definition of} \]

\[ \text{parametric functions } A_{j} \text{ and } B_{j} \text{ are given in Appendix C for quick review.} \]

\[ \text{We note that Theorem 2 and Corollary 1 are important} \]

\[ \text{to derive the efficiency conditions given in Subsection 4.1.} \]

\[ \text{3.2. Separate Estimators. We propose some improved separate classes of estimators based on bivariate auxiliary information under SSRS as} \]

\[ \text{Bias}(\bar{Y}_{si}) = \sum_{i=1}^{L} W_{si} \bar{Y}_{h} \left[ \xi_{hi} \left( 1 + \frac{\theta_{hi}}{2} \right) U_{020} + \left( \frac{\delta_{hi}}{2} \right)^{2} U_{002} - \theta_{hi} U_{110} - \delta_{hi} U_{011} \right] - 1 \right]  

\[ \text{Bias}(\bar{Y}_{si}) = \sum_{i=1}^{L} W_{si} \left[ \xi_{hi} \left( 1 + \frac{\theta_{hi}(\theta_{hi} + 1)}{2} U_{020} + \frac{\delta_{hi}^{2}}{2} U_{002} - \theta_{hi} U_{110} \right] - 1 \right]  

\[ \text{Bias}(\bar{Y}_{si}) = \sum_{i=1}^{L} W_{si} \left[ \xi_{hi} \left( 1 + \frac{\theta_{hi}^{2} + \delta_{hi}^{2}}{2} U_{020} + \frac{\theta_{hi}^{2} + \delta_{hi}^{2}}{2} U_{002} - \theta_{hi} U_{110} \right) + \delta_{hi} U_{011} \right] - 1 \right]  

\[ \text{where } \xi_{hi}, \theta_{hi} \text{ and } \delta_{hi}, i=1,2,...,5 \text{ are duly opted scalars in stratum } h. \]

\[ \text{Theorem 3. The bias of the proposed separate classes of estimators } \bar{Y}_{si}, i=1,2,...,5 \text{ is given by} \]
Proof. The preciscs of the derivations are given in Appendix C for quick review.

\[ \text{Proof.} \]

Theorem 4. The MSE of the proposed separate classes of estimators \( \hat{y}_i \), \( i = 1, 2, \ldots, 5 \) is given by

\[
\begin{align*}
\text{MSE}(\hat{y}_1) &= \sum_{h=1}^{L} W_h^2 h^2 \\
\text{MSE}(\hat{y}_2) &= \sum_{h=1}^{L} W_h^2 h^2 \\
\text{MSE}(\hat{y}_3) &= \sum_{h=1}^{L} W_h^2 h^2 \\
\text{MSE}(\hat{y}_4) &= \sum_{h=1}^{L} W_h^2 h^2 \\
\text{MSE}(\hat{y}_5) &= \sum_{h=1}^{L} W_h^2 h^2 \\
\end{align*}
\]

Proof. The preciscs of the derivations are given in Appendix C for quick review.

Corollary 2. The minimum MSE of the proposed separate classes of estimators \( \hat{y}_i \), \( i = 1, 2, \ldots, 5 \) is given by

\[
\begin{align*}
\min \text{MSE}(\hat{y}_i) &= \sum_{h=1}^{L} W_h^2 h^2 \left( 1 - \frac{B^2_{\delta_h}}{A_{\delta_h}} \right) ; \quad i = 1, 2, 4, 5, \\
\min \text{MSE}(\hat{y}_i) &= \sum_{h=1}^{L} W_h^2 h^2 (1 - \xi_{i, \text{opt}}) = \sum_{h=1}^{L} W_h^2 h^2 \left( 1 - \frac{B^2_{\delta_h}}{A_{\delta_h}} \right).
\end{align*}
\]
Proof. The precis of the derivations and definition of parametric functions $A_i$ and $B_i$ are given in Appendix.

Proof. C for quick review. We again note that Theorem 4 and Corollary 2 are important in order to derive the efficiency conditions given in Subsection 4.2.

4. Efficiency Conditions

In this section, we derive the efficiency conditions under which the proposed combined and separate classes of estimators dominate the existing combined and separate estimators.

4.1. Combined Estimators. On comparing the minimum $MSE$ of the proposed combined estimators $\bar{y}_s, i = 1, 2, \ldots, 5$ from (53) and (54) with the minimum $MSE$ of existing combined estimators from (A.1), (A.3), (A.4), (A.6), (A.8), (A.10), (A.12), (A.14), (A.16), (A.20), (A.22), (A.24), (A.26), and (A.28), we get the following conditions:

\[
MSE(\bar{y}_w) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200}, \quad (72)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} - V_{020} - V_{002} + 2V_{110} + 2V_{101} - 2V_{011}, \quad (73)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \left(\frac{V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right), \quad (74)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} - V_{002} + 2V_{101} - \frac{(V_{101} - V_{002} - V_{110} + V_{011})^2}{(V_{020} + V_{002} - 2V_{011})}, \quad (75)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \left(\frac{V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right), \quad (76)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} - \lambda_2^2V_{020} - \Delta_2^2V_{002} + 2\lambda_1V_{110}, \quad (77)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > \frac{(A_3 - A_1A_4)}{A_1A_2 - A_3^2} - \frac{A_4(A_1A_4 - A_3)}{A_1A_2 - A_3^2}, \quad (78)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \left(\frac{V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right), \quad (79)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \left(\frac{V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right), \quad (80)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \left(\frac{V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right), \quad (81)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \left(\frac{V_{101}V_{020} + V_{110}^2V_{002} - 2V_{011}V_{101}V_{110}}{V_{020}V_{002}(1 - \rho_{x,y}^2)}\right), \quad (82)
\]

\[
MSE(\bar{y}_f) > MSE(\bar{y}_s) \Rightarrow \frac{B_i^2}{A_i} > 1 - V_{200} + \left(\frac{V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011}}{V_{020}V_{002} - V_{011}^2}\right), \quad (83)
\]
On comparing the minimum MSE, better than the other existing combined estimators.

If conditions (72) to (86) hold, then the proposed combined classes of estimators \( \bar{y}_s, i = 1, 2, \ldots, 5 \) perform better than the other existing combined estimators.

4.2. Separate Estimators. On comparing the minimum MSE of the proposed separate estimators \( \bar{y}_s, i = 1, 2, \ldots, 5 \) from (69) and (70) with the minimum MSE of the existing separate estimators from (B.1), (B.3), (B.4), (B.6), (B.8), (B.10), (B.12), (B.14), (B.16), (B.18), (B.20), (B.22), (B.24), and (B.26), we get the following efficiency conditions:

\[
MSE(\bar{y}_m) > MSE(\bar{y}_s),
\]

\[
\sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( 1 - \frac{B_h^2}{A_h} \right) < \sum_{h=1}^{L} W_h^2 \bar{y}_h^2 U_{200},
\]

\[
MSE(\bar{y}_s) > MSE(\bar{y}_s),
\]

\[
\sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( 1 - \frac{B_h^2}{A_h} \right) < \sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( U_{200} + U_{020} + U_{002} - 2U_{110} - 2U_{101} + 2U_{011} \right),
\]

\[
MSE(\bar{y}_s) > MSE(\bar{y}_s),
\]

\[
\sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( 1 - \frac{B_h^2}{A_h} \right) < \sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( U_{200} + U_{020} + 2U_{101} - \frac{(U_{101} - U_{002} - U_{110} + U_{011})^2}{(U_{020} + U_{002} - 2U_{011})} \right),
\]

\[
MSE(\bar{y}_s) > MSE(\bar{y}_s),
\]

\[
\sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( 1 - \frac{B_h^2}{A_h} \right) < \sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( U_{200} + \frac{(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011})}{U_{020}U_{002} - U_{011}^2} \right),
\]

\[
MSE(\bar{y}_s) > MSE(\bar{y}_s),
\]

\[
\sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( 1 - \frac{B_h^2}{A_h} \right) < \sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( U_{200} + \frac{(U_{002}U_{110}^2 + U_{020}U_{101}^2 - 2U_{110}U_{101}U_{011})}{U_{020}U_{002} - U_{011}^2} \right),
\]

\[
MSE(\bar{y}_s) > MSE(\bar{y}_s),
\]

\[
\sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( 1 - \frac{B_h^2}{A_h} \right) < \sum_{h=1}^{L} W_h^2 \bar{y}_h^2 \left( U_{200} + \frac{\lambda_h^2 U_{020} + \Delta_h^2 U_{002} - 2\lambda_h U_{110}}{-2\Delta_h U_{101} + 2\lambda_h \Delta_h U_{011}} \right),
\]
If the conditions (87) to (101) hold then the proposed separate classes of estimators $\overline{\gamma}_s^i, i = 1, 2, \ldots, 5$ perform better than the other existing separate estimators.

4.3. Comparison of Proposed Combined and Separate Estimators. By comparing the minimum $\text{MSE}$ of the proposed combined and separate classes of estimators $\overline{\gamma}_c^i$ and $\overline{\gamma}_s^i, i = 1, 2, \ldots, 5$, we get

\[
\begin{align*}
\text{MSE}(\overline{\gamma}_c^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}_h^2 \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}_h^2 \left( 1 - \frac{(A_{x_h} - A_{x_h})}{(A_{x_h} + A_{x_h})} \right) \frac{A_{x_h} - A_{x_h} - A_{x_h} - A_{x_h}}{(A_{x_h} + A_{x_h}) - (A_{x_h} + A_{x_h})}, \\
\text{MSE}(\overline{\gamma}_{kk}^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}^i_{kk} \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}^i_{kk} \left( U_{200} - \frac{(U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2U_{110} U_{101} U_{011})}{U_{020} U_{002} - U_{011}^2} \right), \\
\text{MSE}(\overline{\gamma}_{fc}^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}^i_{fc} \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}^i_{fc} \left( k_h^i \left( U_{200} + \frac{1}{4} (U_{020} + U_{002} - 2U_{002}) \right) + U_{101} - U_{110} \right) \\
& \quad + 2k_h^i (k_h^i - 1) \left( \frac{1}{8} (3U_{020} - U_{002} - 2U_{011}) + \frac{1}{2} (U_{101} - U_{110}) \right) + (k_h^i - 1)^2, \\
\text{MSE}(\overline{\gamma}_{i1}^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}^i_{i1} \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}^i_{i1} \left( U_{200} - \frac{(U_{101} U_{020} + U_{110} U_{002} - 2U_{011} U_{101} U_{110})}{U_{020} U_{002} (1 - \rho_{x_h}^2)} \right), \\
\text{MSE}(\overline{\gamma}_{i2}^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}^i_{i2} \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}^i_{i2} \left( U_{200} - \frac{(U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2U_{110} U_{101} U_{011})}{U_{020} U_{002} - U_{011}^2} \right), \\
\text{MSE}(\overline{\gamma}_{mu}^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}^i_{mu} \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}^i_{mu} \left( 1 - \frac{U_{011}^2}{4U_{020} - A_{mu}^2} \right), \\
\text{MSE}(\overline{\gamma}_{mu1}^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}^i_{mu1} \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}^i_{mu1} (1 + P_h - 2O_h), \\
\text{MSE}(\overline{\gamma}_{mu2}^i) & > \text{MSE}(\overline{\gamma}_s^i), \\
\sum_{h=1}^{L} W_h^2 \overline{y}^i_{mu2} \left( 1 - \frac{B_h^2}{A_h} \right) & < \sum_{h=1}^{L} W_h^2 \overline{y}^i_{mu2} \left( 1 - \frac{O_h^2}{P_h} \right). \tag{94}
\end{align*}
\]
The table below shows the ARB and PRE of different combined estimators.

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<th>PRE</th>
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5. Simulation Study

To enhance the credibility of the theoretical development of the proposed combined and separate classes of estimators, we have conducted a simulation study. In the procedure, the following steps are considered:

(i) Generate trivariate random observations of size \( N = 2000 \) units using a trivariate normal distribution in R software with parameters \( \Sigma = 20 \), \( \Xi = 25 \), \( \alpha_y = 5 \), \( \alpha_x = 6 \), \( \alpha_z = 7 \) and different amounts of correlation coefficients \( \rho_{xy}, \rho_{yz}, \) and \( \rho_{xz} \).

(ii) Stratify the above population into 4 equal disjoint strata and quantify a sample of size \( n = 50 \) units from each stratum.

(iii) Tabulate all necessary statistics.

(iv) Using 15,000 iterations to calculate the absolute relative bias (ARB) and percent relative efficiency (PRE) of various combined and separate classes of estimators. The ARB and PRE of different estimators \( T \) are calculated regarding the classical ratio and usual mean estimators and results are reported in Table 1 and Table 2. The ARB and PRE are calculated using the following expressions.

\[
\text{ARB} = \frac{\sum_{i=1}^{15000} (\hat{T}_i - \hat{y})}{\sum_{i=1}^{15000} (T - \hat{y})},
\]

\[
\text{PRE} = \frac{\sum_{i=1}^{15000} (\hat{T}_i - \hat{y})^2}{\sum_{i=1}^{15000} (T - \hat{y})^2} \times 100,
\]
Table 2: ARB and PRE of different separate estimators.

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<td>$\nu_{kk}$</td>
<td>$\nu_{lc}$</td>
<td>$\nu_{tc}$</td>
<td>$\nu_{mc}$</td>
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The simulation findings of the combined and separate classes of estimators are expressed in terms of ARB and PRE in Table 1 and Table 2 for different values of correlation coefficients. The results exhibit the dominance of the proposed combined and separate classes of estimators $\nu_{ma}$ and $\nu_{mc}$, $i = 1, 2, \ldots, 5$, respectively, over the combined and separate usual mean estimators, classical ratio and regression estimators, Olkin [24] type estimator, Abu–Dayyeh et al. [25] type estimators, Kadilar and Cingi [7] type estimator, Singh et al. [16] estimator, Koyuncu and Kadilar [12] estimator, Tailor and Chouhan [14] estimator, Lone et al. [18, 19] estimators and Muneer et al. [21, 22] estimators in terms of PRE. Also, the proposed combined and separate class of estimators $\nu_{ma}$ and $\nu_{mc}$ are found to be most efficient among the proposed combined and separate classes of estimators for each passably chosen values of correlation coefficients.

6. Conclusion

In this article, we propose some improved classes of estimators for population mean by extending the work of Bhushan et al. [28] using bivariate auxiliary information under SSRS. The mathematical expressions of bias and MSE of the proposed classes of estimators are obtained up to the first order of approximation. The efficiency conditions are derived under which the suggested estimators perform better than the other existing estimators. In support of the theoretical results, a simulation study is carried out using an artificially generated population with various amounts of correlation coefficients. From the perusal of the theoretical and simulation results reported in Table 1 and Table 2, we conclude that:

(i) The proposed combined classes of estimators $\nu_{ma}$, $i = 1, 2, \ldots, 5$ perform better than the combined form of usual mean estimator $\nu_{ma}$, classical ratio and regression estimators $\nu_{ma}$, & $\nu_{mc}$, Olkin [24] type estimator $\nu_{ma}$, Abu–Dayyeh et al. [25] type estimators $\nu_{ma}$, & $\nu_{mc}$, Kadilar and Cingi [7] type estimators $\nu_{ma}$, $i = 1, 2, 3, 4$, Koyuncu and Kadilar [12] estimator $\nu_{ma}$, Singh et al. [16] estimator $\nu_{ma}$, Tailor and Chouhan [14] estimator $\nu_{ma}$, Lone et al. [18, 19] estimators $\nu_{ma}$, $i = 1, 2$ and Muneer et al. [21, 22] estimators $\nu_{ma}$ & $\nu_{mc}$, $i = 1, 2$ for different values of correlation coefficients.

(ii) The proposed separate classes of estimators $\nu_{mc}$, $i = 1, 2, \ldots, 5$ dominate the separate form of usual mean estimator $\nu_{mc}$, classical ratio and regression estimators $\nu_{mc}$ & $\nu_{mc}$, Olkin [24] type estimator $\nu_{mc}$, Abu–Dayyeh et al. [25] type estimators $\nu_{mc}$ & $\nu_{mc}$, Kadilar and Cingi [7] type estimators $\nu_{mc}$, $i = 1, 2, 3, 4$, Koyuncu and Kadilar [12] estimator $\nu_{mc}$, Singh et al. [16] estimator $\nu_{mc}$, Tailor and Chouhan [14] estimator $\nu_{mc}$, Lone et al. [18, 19] estimators $\nu_{mc}$, $i = 1, 2$ and Muneer et al. [21, 22] estimators $\nu_{mc}$ & $\nu_{mc}$, $i = 1, 2$ for different values of correlation coefficients.
estimators $\overline{\gamma}_{mu}^i$ & $\overline{\gamma}_{m2}^i$, $i = 1, 2$ for different values of correlation coefficients.

(iii) Since, the proposed combined and separate classes of estimators $\overline{\gamma}_s^i$, $i = 1, 5$ are, respectively, superior to the combined and separate ratio and chain ratio exponential estimators $\overline{\gamma}_{mu}^i$, $\overline{\gamma}_{m2}^i$ envisaged by Muneer et al. [22] consequently the proposed combined and separate classes of estimators $\overline{\gamma}_s^i$, $i = 1, 2, 5$ will also dominate those 65 estimators that are the members of the combined and separate ratio and chain ratio exponential estimators $\overline{\gamma}_{mu}^i$, $\overline{\gamma}_{m2}^i$, respectively.

(iv) The proposed combined and separate class of estimators $\overline{\gamma}_s^i$, $i = 1, 2, 5$ perform better among the proposed classes of estimators.

(v) The proposed separate classes of estimators $\overline{\gamma}_s^i$, $i = 1, 2, 5$ dominate the proposed combined classes of estimators $\overline{\gamma}_s^i$, $i = 1, 2, 5$ in terms of greater PRE for various amounts of correlation coefficients.

(vi) The PRE of the proposed combined and separate classes of estimators increases as the values of correlation coefficients increase and vice versa.

Thus, the proposed combined and separate classes of estimators can be preferably used by the survey professionals in practice.

Appendix

A. Bias and MSE of the Existing Combined Estimators

The bias and MSE expressions of the existing combined estimators are expressed as mentioned below:

\[
MSE(\overline{\gamma}_m) = \overline{V}^2 V_{200},
\]
\[
Bias(\overline{\gamma}_m) = \overline{V} [V_{020} + V_{002} + V_{011} - V_{101} + V_{110}],
\]
\[
MSE(\overline{\gamma}_r) = \overline{V}^2 \left[ +V_{020} + V_{002} - 2V_{110} - 2V_{101} + 2V_{011} \right],
\]
\[
\text{min MSE}(\overline{\gamma}_r) = \overline{V}^2 \left[ V_{200} - \frac{(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011})}{(V_{020}V_{002} - V_{011}^2)} \right],
\]
\[
Bias(\overline{\gamma}_0) = \overline{V} [w(V_{110} - V_{101}) + V_{101}],
\]
\[
\text{min MSE}(\overline{\gamma}_0) = \overline{V}^2 \left[ (V_{200} + V_{002} + 2V_{101}) - \frac{(V_{101} - V_{002} - V_{110} + V_{011})^2}{(V_{020} + V_{002} - 2V_{011})} \right],
\]
\[
Bias(\overline{\gamma}_d) = \overline{V} \left[ \frac{a_1(a_1 + 1)}{2} V_{020} + \frac{a_2(a_2 + 1)}{2} V_{002} + a_1a_2 V_{011} - a_1 V_{110} - a_2 V_{101} \right],
\]
\[
\text{min MSE}(\overline{\gamma}_d) = \overline{V}^2 \left[ V_{200} - \frac{(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011})}{(V_{020}V_{002} - V_{011}^2)} \right],
\]
\[
Bias(\overline{\gamma}_w) = \overline{V} \left[ w_1a_1 V_{110} + w_2a_2 V_{101} + w_1 \frac{a_1(a_1 - 1)}{2} V_{020} + w_2 \frac{a_2(a_2 - 1)}{2} V_{002} \right],
\]
\[
\text{min MSE}(\overline{\gamma}_w) = \overline{V}^2 \left[ V_{200} + a_1^2 V_{002} - 2V_{002} V_{101} - \frac{(V_{002}V_{110}^2 + V_{020}V_{101}^2 - 2V_{110}V_{101}V_{011})}{(V_{020}V_{002} - V_{011}^2)} \right],
\]
\[
Bias(\overline{\gamma}_{k_1}, i = 1, 2, 3, 4)
\]
\[
\text{MSE}(\overline{\gamma}_{k_1}, i = 1, 2, 3, 4) = \overline{V}^2 \left[ V_{200} + \frac{\lambda_i^2 V_{020} + \Delta_i^2 V_{002} - 2\lambda_i V_{110} - 2\Delta_i V_{101} + 2\lambda_i \Delta_i V_{011}}{\lambda_i^2 V_{020} + \Delta_i^2 V_{002} - 2\lambda_i V_{110} - 2\Delta_i V_{101} + 2\lambda_i \Delta_i V_{011}} \right],
\]
\[ \text{Bias}(\bar{Y}_i) = \mathbb{Y} \left[ \eta_1 + \eta_2 \left\{ 1 + \frac{\alpha (\alpha + 1)}{2} V_{020} + \frac{\beta (\beta - 1)}{2} V_{002} \right\} - 1 \right], \]  
(A.13)

\[ \min \text{MSE}(\bar{Y}_i) = \mathbb{Y}^2 \left[ 1 - \frac{(A_2 - A_2 A_4)}{A_1 A_2 - A_3^2} - \frac{A_4 (A_1 A_2 - A_3)}{A_1 A_2 - A_3^2} \right], \]  
(A.14)

\[ \text{Bias}(\bar{Y}_{i,k}) = \mathbb{Y} \left[ \frac{-g_1 V_1 V_{101} - g_1 V_{101} - \alpha V_{101}}{V_{020} - \frac{V_{101}^2 V_{110} + V_{020} V_{101} - 2 V_{110} V_{011}}{V_{020} V_{002} - V_{011}}} \right], \]  
(A.15)

\[ \min \text{MSE}(\bar{Y}_{i,k}) = \mathbb{Y}^2 \left[ V_{200} \frac{-V_{002}^2 V_{110} + V_{020} V_{101} - 2 V_{110} V_{011}}{V_{020} V_{002} - V_{011}} \right], \]  
(A.16)

\[ \text{Bias}(\bar{Y}_{i,k}^*) = \mathbb{Y} (k^* - 1) + k^* \left[ \frac{1}{18} (3 V_{020} - V_{002} - 2 V_{011}) + \frac{1}{2} (V_{101} - V_{110}) \right], \]  
(A.17)

\[ \min \text{MSE}(\bar{Y}_{i,k}^*) = \mathbb{Y}^2 \left[ k^* \left\{ V_{200} + \frac{1}{4} (V_{020} + V_{002} - 2 V_{011}) + (k^* - 1)^2 \right\} + \frac{1}{8} (3 V_{020} - V_{002} - 2 V_{011}) + \frac{1}{2} (V_{101} - V_{110}) \right], \]  
(A.18)

\[ \text{Bias}(\bar{Y}_{i,k}^*) = \mathbb{Y} \left[ \frac{\left( L_1 \left( \frac{L_1}{2} - 1 \right) \right) V_{020} + \frac{L_2 \left( L_2 - 1 \right)}{4} \right) V_{002} - \frac{L_1 L_2}{4} V_{011} \right], \]  
(A.19)

\[ \min \text{MSE}(\bar{Y}_{i,k}^*) = \mathbb{Y}^2 \left[ V_{200} - \frac{V_{020} V_{110} V_{020} + V_{101} V_{011} V_{110} - 2 V_{110} V_{011}}{V_{020} V_{002} - V_{011}} \right], \]  
(A.20)

\[ \text{Bias}(\bar{Y}_{i,k}^*) = \mathbb{Y} \left[ V_{020} - V_{002} + V_{101} - V_{110} \right], \]  
(A.21)

\[ \min \text{MSE}(\bar{Y}_{i,k}^*) = \mathbb{Y}^2 \left[ V_{200} - \frac{V_{002}^2 V_{110} + V_{020} V_{101} - 2 V_{110} V_{011}}{V_{020} V_{002} - V_{011}} \right], \]  
(A.22)

\[ \text{Bias}(\bar{Y}_{m,n}) = (k_3 - 1) \mathbb{Y} + k_3 \frac{3}{4} \mathbb{Y} \left[ \frac{\alpha}{2} - \alpha \right] V_{002} - \frac{V_{011}}{2} + \frac{k_3}{2} \mathbb{X} V_{011}, \]  
(A.23)

\[ \min \text{MSE}(\bar{Y}_{m,n}) = \mathbb{Y}^2 \left[ 1 - \frac{V_{011}^2}{4 V_{020} - \frac{A_{m}^2}{B_{m}}} \right], \]  
(A.24)

\[ \text{Bias}(\bar{Y}_{m,n}) = \mathbb{Y} \left[ \left( \frac{\alpha}{2} g_1 \frac{g_1}{2} + \frac{\alpha}{2} \right) V_{110} - \left( \frac{\alpha}{2} g_2 \frac{g_2}{2} + \frac{\alpha}{2} \right) V_{101} \right] + \left( \frac{\alpha}{2} g_2 \frac{g_2}{2} + \frac{\alpha}{2} \right) V_{020} + \left( \frac{\alpha}{2} g_1 \frac{g_1}{2} + \frac{\alpha}{2} \right) V_{002} \]  
(A.25)
\[ \text{MSE}(\hat{y}_{\text{mu}}) = \hat{y}^2 [1 + P - 2O], \] (A.26)

\[ \text{Bias}(\hat{y}_{\text{mu}}) = (k_1 - 1)\hat{y} + k_1\bar{Y}, \] (A.27)

\[
\min \text{MSE}(\hat{y}_{\text{mu}}) = \bar{Y}^2 \left[ 1 - \frac{\sigma^2}{\hat{p}} \right], \tag{A.28}
\]

where \( l_1 = \sum_{h=1}^{L_h} W_h X_h / \sum_{h=1}^{L_h} W_h (X_h + C_x) \); \( \lambda_s = \sum_{h=1}^{L_h} W_h X_h / \sum_{h=1}^{L_h} W_h (X_h + C_x) \); \( \lambda_t = \sum_{h=1}^{L_h} W_h X_h / \sum_{h=1}^{L_h} W_h (X_h + C_x) \); \( \lambda_4 = \sum_{h=1}^{L_h} W_h \bar{X}_h \beta_2 (x_h) / \sum_{h=1}^{L_h} W_h (X_h + C_x) \); \( \gamma_1 = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \); \( \gamma_2 = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \); \( \delta = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \); \( \delta = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \); \( \delta = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \); \( \delta = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \); \( \delta = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \); \( \delta = \sum_{h=1}^{L_h} W_h \bar{Z}_h \beta_2 (z_h) / \sum_{h=1}^{L_h} W_h (Z_h + C_z) \);

\[
\beta_{(\text{opt})} = \frac{\bar{Y}}{\bar{Y}^2 \left[ \frac{V_{002} V_{110} - V_{101} V_{011}}{V_{020} V_{002} - V_{011}} \right]}, \tag{A.29}
\]

\[
\theta_{(\text{opt})} = \frac{\bar{Y}}{\bar{Y}^2 \left[ \frac{V_{020} V_{110} - V_{101} V_{011}}{V_{002} V_{002} - V_{011}} \right]}, \tag{A.30}
\]

\[
\omega_{(\text{opt})} = \frac{V_{101} - V_{002} - V_{110} + V_{011}}{V_{020} V_{002} - 2 V_{011}}, \tag{A.31}
\]

\[
a_{1(\text{opt})} = \frac{V_{002} V_{110} - V_{101} V_{011}}{V_{020} V_{002} - V_{011}}, \tag{A.32}
\]

\[
a_{2(\text{opt})} = \frac{V_{020} V_{101} - V_{110} V_{011}}{V_{020} V_{002} - V_{011}}, \tag{A.33}
\]

\[
w_{1(\text{opt})} = \frac{a_2^2 V_{002} + a_1^2 V_{110} - a_1 a_2 V_{011}}{a_1^2 V_{002} + a_2^2 V_{002} - 2 a_1 a_2 V_{011}}, \tag{A.34}
\]

\[
w_{2(\text{opt})} = 1 - w_{1(\text{opt})}, \tag{A.35}
\]

\[
\eta_{1(\text{opt})} = \frac{A_2 - A_3 A_4}{A_1 A_2 - A_3^2}, \tag{A.36}
\]
\[ \eta_{2(\text{opt})} = \frac{A_1 A_4 - A_3}{A_1 A_2 - A_3^2}, \]  
\[ I_{1(\text{opt})} = \left[ \frac{V_{002} V_{110} - V_{101} V_{011}}{IV_{020} V_{002} - V_{011}^2} \right], \]  
\[ I_{2(\text{opt})} = \left[ \frac{V_{020} V_{101} - V_{110} V_{011}}{IV_{020} V_{002} - V_{011}^2} \right], \]  
\[ k_{(\text{opt})} = \left( 1 + \frac{1}{8} (3V_{020} - V_{002} - 2V_{011}) + \frac{1}{2} (V_{101} - V_{110}) \right) + \frac{1}{2} \left( V_{020} + \frac{1}{4} (V_{020} + V_{002} - 2V_{002} + V_{101} - V_{110}) \right) \]  
\[ L_{1(\text{opt})} = \frac{2(V_{002} V_{110} - V_{101} V_{011})}{IV_{020} V_{002} - V_{011}^2}, \]  
\[ L_{2(\text{opt})} = \frac{2(V_{110} V_{011} - V_{020} V_{101})}{IV_{020} V_{002} - V_{011}^2}, \]  
\[ \lambda_{(\text{opt})} = \left( \frac{V_{002} V_{110} - V_{101} V_{011}}{IV_{020} V_{002} - V_{011}^2} \right), \]  
\[ \psi_{(\text{opt})} = \left( \frac{V_{020} V_{101} - V_{110} V_{011}}{IV_{020} V_{002} - V_{011}^2} \right), \]  
\[ k_{3(\text{opt})} = \frac{1 + (3/8 - \alpha/4)V_{002} - (V_{101}/2) - (V_{011} (V_{011} - V_{110})/2V_{020})}{1 + V_{200} + (1 - \alpha/2)V_{002} - 2V_{101} - ((V_{011} - V_{110})^2/V_{020})} = \frac{A_{2m}}{B_{2m}} \]  
\[ k_{4(\text{opt})} = \frac{\psi}{X} \left[ \frac{V_{011}}{2V_{020}} - k_3 \left( \frac{V_{011} - V_{110}}{V_{020}} \right) \right], \]  
\[ k_{1(\text{opt})} = \frac{O}{P}. \]  

### B. Bias and MSE of the Existing Separate Estimators

The bias and MSE expressions of the existing separate estimators are expressed as

\[ \text{MSE} (\bar{\gamma}^l_{m}) = \sum_{h=1}^{L} W^2_{h} V^2_{h} L_{200}, \]  
\[ \text{Bias} (\bar{\gamma}^l_{r}) = \sum_{h=1}^{L} W^2_{h} V^2_{h} [U_{200} + U_{002} + U_{011} - U_{101} + U_{110}], \]  
\[ \text{MSE} (\bar{\gamma}^l_{r}) = \sum_{h=1}^{L} W^2_{h} V^2_{h} [U_{200} + U_{020} + U_{002} - 2U_{110} - 2U_{101} + 2U_{011}]. \]
\[
\min \text{MSE}(\overline{Y}_h) = \sum_{h=1}^{L} W_h^2 \sum_{h} \left[ U_{200} - \frac{U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2 U_{110} U_{101} U_{011}}{(U_{020} U_{002} - U_{011}^2)} \right],
\]
(B.4)

\[
\text{Bias}(\overline{Y}_h) = \sum_{h=1}^{L} W_h \sum_{h} \left[ \omega_h (U_{110} - U_{101}) + U_{101} \right],
\]
(B.5)

\[
\min \text{MSE}(\overline{Y}_h) = \sum_{h=1}^{L} W_h^2 \sum_{h} \left[ (U_{200} + U_{002} + 2 U_{101}) - \frac{(U_{101} - U_{002} - U_{110} + U_{011})^2}{(U_{020} + U_{002} - 2 U_{011})} \right],
\]
(B.6)

\[
\text{Bias}(\overline{Y}_h) = \sum_{h=1}^{L} W_h \sum_{h} \left[ \frac{a_1(a_2 + 1)}{2} U_{020} + \frac{a_2(a_2 + 1)}{2} U_{002} + a_1 a_2 U_{011} \right] - a_1 U_{110} - a_2 U_{101}.
\]
(B.7)

\[
\min \text{MSE}(\overline{Y}_h) = \sum_{h=1}^{L} W_h^2 \sum_{h} \left[ U_{200} - \frac{(U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2 U_{110} U_{101} U_{011})}{(U_{020} U_{002} - U_{011}^2)} \right],
\]
(B.8)

\[
\text{Bias}(\overline{Y}_h) = \sum_{h=1}^{L} W_h \sum_{h} \left[ +w_2 a_2 U_{101} + \frac{a_1(a_1 - 1)}{2} U_{020} \right] + w_2 \frac{a_2(a_1 - 1)}{2} U_{002}.
\]
(B.9)

\[
\min \text{MSE}(\overline{Y}_h) = \sum_{h=1}^{L} W_h^2 \sum_{h} \left[ U_{200} + a_2^2 U_{002} - 2 U_{002} U_{101} \right] - \frac{(U_{002}^2 U_{110}^2 + U_{020} U_{101}^2 - 2 U_{110} U_{101} U_{011})}{(U_{020} U_{002} - U_{011}^2)}.
\]
(B.10)

\[
\text{Bias}(\overline{Y}_h) = \sum_{h=1}^{L} W_h \sum_{h} \left[ \lambda_h U_{002} + \Delta_h U_{002} - \lambda_h U_{110} - \Delta_h U_{101} + \lambda_h \Delta_h U_{011} \right], \quad i = 1, 2, 3, 4.
\]
(B.11)

\[
\text{MSE}(\overline{Y}_h) = \sum_{h=1}^{L} W_h^2 \sum_{h} \left[ U_{200} + \lambda_h^2 U_{020} + \Delta_h^2 U_{002} - 2 \lambda_h U_{110} - 2 \Delta_h U_{101} + 2 \lambda_h \Delta_h U_{011} \right],
\]
(B.12)

\[
\text{Bias}(\overline{Y}_h) = \sum_{h=1}^{L} W_h \sum_{h} \left[ \eta_h + \eta_h \left\{ \frac{1 + \alpha_h(a_2 + 1)}{2} U_{020} + \frac{\beta_h(a_2)}{2} U_{002} \right\} - 1 \right],
\]
(B.13)

\[
\min \text{MSE}(\overline{Y}_h) = \sum_{h=1}^{L} W_h^2 \sum_{h} \left[ 1 - \frac{(A_{A_h} - A_h A_{A_h})}{A_h A_{A_h} - A_{A_h}^2} \right],
\]
(B.14)

\[
\text{Bias}(\overline{Y}_h) = \sum_{h=1}^{L} W_h \sum_{h} \left[ \frac{-g_1 v_2 \lambda_a U_{110} - g_2 v_2 \lambda_a U_{101} + \lambda_a^2 v_2 g_1 v_1 U_{011}}{2} \right] + \frac{g_1 (g_1 + 1)}{2} \lambda_a^2 v_1^2 U_{020} + \frac{g_2 (g_2 + 1)}{2} \lambda_a^2 v_2^2 U_{002}.
\]
(B.15)

\[
\min \text{MSE}(\overline{Y}_h) = \sum_{h=1}^{L} W_h^2 \sum_{h} \left[ U_{200} - \frac{(U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2 U_{110} U_{101} U_{011})}{(U_{020} U_{002} - U_{011}^2)} \right],
\]
(B.16)
\[
\text{Bias}(\bar{y}_{hi}) = \sum_{h=1}^{\ell} W_h \bar{Y}_h \left( k_h^* - 1 \right) + k_h^* \bar{Y}_h \left[ \frac{1}{8} \left( 3U_{020} - U_{002} - 2U_{011} \right) + \frac{1}{2} \left( U_{101} - U_{110} \right) \right], \quad (B.17)
\]

\[
\min \text{MSE}(\bar{y}_{hi}) = \sum_{h=1}^{\ell} W_h^2 \bar{Y}_h^2 \left[ k_h^* \left\{ U_{200} + \frac{1}{4} \left( U_{020} + U_{002} - 2U_{011} \right) + U_{101} - U_{110} \right\} \right], \quad (B.18)
\]

\[
\text{Bias}(\bar{y}_{h}) = \sum_{h=1}^{\ell} W_h \bar{Y}_h \left[ \frac{L_{1k}}{4} \left( \frac{L_{1h}}{2} - 1 \right) U_{020} + \frac{L_{2h}}{4} \left( \frac{L_{2h}}{2} - 1 \right) U_{002} - \frac{L_{1h} L_{2h} U_{011}}{4} \right] + \frac{1}{2} \left( L_{2h} U_{101} - L_{1h} U_{110} \right), \quad (B.19)
\]

\[
\min \text{MSE}(\bar{y}_{h}) = \sum_{h=1}^{\ell} W_h^2 \bar{Y}_h^2 \left[ U_{200} - \frac{\left( \frac{U_{101}^2 U_{020} + U_{110}^2 U_{002} - 2U_{011} U_{101} U_{110}}{U_{020} U_{002} - U_{011}^2} \right)}{U_{020} U_{002} - U_{011}^2} \right], \quad (B.20)
\]

\[
\text{Bias}(\bar{y}_{h}) = \sum_{h=1}^{\ell} W_h \bar{Y}_h \left[ U_{020} - U_{011} + U_{101} - U_{110} \right], \quad (B.21)
\]

\[
\min \text{MSE}(\bar{y}_{h}) = \sum_{h=1}^{\ell} W_h^2 \bar{Y}_h^2 \left[ U_{200} - \frac{\left( \frac{U_{002} U_{110}^2 + U_{020} U_{101}^2 - 2U_{110} U_{101} U_{011}}{U_{020} U_{002} - U_{011}^2} \right)}{U_{020} U_{002} - U_{011}^2} \right], \quad (B.22)
\]

\[
\text{Bias}(\bar{y}_{mu}) = \sum_{h=1}^{\ell} W_h \bar{Y}_h \left[ \left( \frac{3A_{h}}{8} \right) U_{020} - \frac{U_{011}}{2} \right] + k_{h*} \bar{Y}_h U_{011}, \quad (B.23)
\]

\[
\min \text{MSE}(\bar{y}_{mu}) = \sum_{h=1}^{\ell} W_h^2 \bar{Y}_h^2 \left[ 1 - \frac{U_{011}^2}{4U_{020}^2} - \frac{A_{h}^2}{B_{mu}} \right], \quad (B.24)
\]

\[
\text{Bias}(\bar{y}_{mu}) = \sum_{h=1}^{\ell} W_h \bar{Y}_h \left[ \left( \frac{3A_{h}^2}{8} \right) g_{1h}^2 + \frac{3A_{h}}{2} \right] \text{U}_{110} + \left( \frac{3A_{h}}{2} \right) \text{U}_{020} + \left( \frac{3A_{h}}{8} \right) \text{U}_{002} + \left( \frac{3A_{h}}{2} \right) \text{U}_{011}, \quad (B.25)
\]

\[
\text{MSE}(\bar{y}_{mu}) = \sum_{h=1}^{\ell} W_h^2 \bar{Y}_h^2 \left[ 1 + P_h - 2O_h \right], \quad (B.26)
\]
\[ \text{Bias}(\widehat{\varphi}'_{mru}) = \left[ \frac{\lambda_1}{2} \right] \sum_{h=1}^{L} W_h(k_i - 1) \sum_{h=1}^{L} W_h k_i \left( \sum_{h=1}^{L} W_h k_i \right) \left( \sum_{h=1}^{L} W_h k_i \right) \left( \sum_{h=1}^{L} W_h k_i \right) , \right] \]

\[ \min \text{MSE}(\widehat{\varphi}'_{mru}) = \sum_{h=1}^{L} W_h^2 \left[ 1 - \frac{O_h^2}{P_h} \right] , \]

where \( \lambda_1 = \lambda_2 = \lambda_3 = \lambda_4 = \lambda_5 = \lambda_6 = \lambda_7 = \lambda_8 = \lambda_9 = 0 \).
\[ k_{h}(\text{opt}) \]
\[ L_{1h} = \frac{2(U_{002}U_{110} - U_{012}U_{011})}{U_{020}U_{200} - U_{111}} \]
\[ L_{2h} = \frac{2(U_{110}U_{011} - U_{020}U_{101})}{U_{020}U_{002} - U_{011}} \]
\[ \lambda_h(\text{opt}) = \frac{(U_{002}U_{110} - U_{101}U_{011})}{(U_{020}U_{002} - U_{011})} \]
\[ \psi_h(\text{opt}) = \frac{-(U_{020}U_{101} - U_{110}U_{011})}{(U_{020}U_{002} - U_{011})} \]
\[ k_{3h}(\text{opt}) = 1 + ((3/8) - (a_1/4))U_{002} - (U_{101}/2) - (U_{011}U_{110}/2U_{020}) = \frac{A_m}{B_m} \text{ (say),} \]
\[ k_{4h}(\text{opt}) = \frac{\sum_{i}^{} f_i h \left[ \frac{U_{011} - U_{110}}{U_{020}} \right]}{\sum_{i}^{} f_i h^2 \left[ \frac{U_{011} - U_{110}}{U_{020}} \right]} \]
\[ k_{1h}(\text{opt}) = \frac{\sum_{i}^{} f_i h \left[ \frac{U_{011} - U_{110}}{U_{020}} \right]}{\sum_{i}^{} f_i h^2 \left[ \frac{U_{011} - U_{110}}{U_{020}} \right]} \]

\[ (B.30) \]

C. Bias and MSE of the Proposed Combined and Separate Estimators

This section addresses the precis of the proof of Theorem 2 and Corollary 1 of Subsection 3.1.

Consider the estimator
\[ \bar{y}_s = \xi_s y_{st} + \theta_s (x_{st} - \bar{x}) + \delta_s (z_{st} - \bar{z}). \]  

(MSE) of the estimator as
\[ MSE(\bar{y}_s) = \bar{y}^2 \left[ (\xi_s - 1)^2 \bar{y}^2 + \xi_s^2 \bar{y}^2 V_{200} + \theta_s \bar{x}^2 V_{020} + \delta_s^2 \bar{z}^2 V_{002} \right] + 2\bar{y} \xi_s \theta_s \bar{x} V_{110} + 2\xi_s \delta_s \bar{z} V_{101} + 2\theta_s \bar{x} \delta_s \bar{z} V_{011} \]  

(C.3)

The optimum values of \( \xi_s, \theta_s, \) and \( \delta_s \) can be obtained by minimizing (C.3) w.r.t \( \xi_s, \theta_s, \) and \( \delta_s \) as
\[ \xi_s(\text{opt}) = \frac{1}{1 + V_{200} \left( V_{020} V_{101} + V_{002} V_{110} - 2V_{110} V_{011} \right) \left( V_{020} V_{002} - V_{011} \right)} \]
\[ \theta_s(\text{opt}) = \xi_s(\text{opt}) \left( \frac{\bar{y}}{\bar{x}} \right) \left( \frac{V_{101} V_{011} - V_{002} V_{110}}{V_{020} V_{002} - V_{011}} \right) \]
\[ \delta_s(\text{opt}) = \xi_s(\text{opt}) \left( \frac{\bar{y}}{\bar{z}} \right) \left( \frac{V_{110} V_{011} - V_{020} V_{101}}{V_{020} V_{002} - V_{011}} \right) \]

Putting \( \xi_s(\text{opt}), \theta_s(\text{opt}) \) and \( \delta_s(\text{opt}) \) in (C.3), we get the minimum MSE as
\[ \min MSE(\bar{y}_s) = \bar{y}^2 (1 - \xi_s(\text{opt})) = \bar{y}^2 \left( 1 - \frac{\theta_s^2}{\theta_s} \right) \] 

(C.7)

In similar way, we can tabulate the MSE of other estimators \( \bar{y}_s, i = 1, 2, 4, 5 \) as
\[ MSE(\bar{y}_s) = \bar{y}^2 \left[ 1 + \xi_i^2 A_i - 2\xi_i B_i \right] \] 

(C.8)
The minimum MSE of the estimators $\bar{y}_i$, $i = 1, 2, 4, 5$ is given by

$$\min MSE(\bar{y}_i) = Y^2 \left(1 - \frac{B_i^2}{A_i}\right). \quad (C.9)$$

The optimum values of the scalars involved are given hereunder

$$\xi_{i(\text{opt})} = \frac{B_i}{A_i} \quad (C.10)$$

$$\theta_{1(\text{opt})} = \frac{(V_{101}V_{011} - V_{002}V_{110})}{(V_{020}V_{002} - V_{011})} = \theta_{2(\text{opt})}, \quad (C.11)$$

$$\delta_{1(\text{opt})} = \frac{(V_{110}V_{011} - V_{020}V_{101})}{(V_{020}V_{002} - V_{011})} = \delta_{2(\text{opt})}, \quad (C.12)$$

$$\theta_{4(\text{opt})} = \frac{(V_{002}V_{110} - V_{010}V_{101})}{(V_{020}V_{002} - V_{011})} = \theta_{5(\text{opt})}, \quad (C.13)$$

$$\delta_{4(\text{opt})} = \frac{(V_{020}V_{101} - V_{011}V_{010})}{(V_{020}V_{002} - V_{011})} = \delta_{5(\text{opt})}, \quad (C.14)$$

where $A_1 = \left[1 + V_{200} - 2((V_{020}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2) - 2((V_{020}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2)) + (V_{020}V_{002} - V_{011}^2)^2/B_1 = [1 - (V_{020}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2)]B_1 = [1 + (V_{020}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2) - 2(V_{020}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2) \right] \cdot B_2 = [1 - (V_{020}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2)]B_2 = [1 - (V_{020}V_{101}V_{011})/(V_{020}V_{002} - V_{011}^2)]$.

$$\theta_{1(\text{opt})} = \theta_{2(\text{opt})}, \quad \delta_{1(\text{opt})} = \delta_{2(\text{opt})}, \quad \theta_{4(\text{opt})} = \theta_{5(\text{opt})}, \quad \delta_{4(\text{opt})} = \delta_{5(\text{opt})}.$$


