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Research Article

Hybrid Model for Stock Market Volatility

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Empirical evidence suggests that the traditional GARCH-type models are unable to accurately estimate the volatility of financial markets. To improve on the accuracy of the traditional GARCH-type models, a hybrid model (BSGARCH (1, 1)) that combines the flexibility of B-splines with the GARCH (1, 1) model has been proposed in the study. The lagged residuals from the GARCH (1, 1) model are fitted with a B-spline estimator and added to the results produced from the GARCH (1, 1) model. The proposed BSGARCH (1, 1) model was applied to simulated data and two real financial time series data (NASDAQ 100 and S&P 500). The outcome was then compared to the outcomes of the GARCH (1, 1), EGARCH (1, 1), GJR-GARCH (1, 1), and APARCH (1, 1) with different error distributions (ED) using the mean absolute percentage error (MAPE), the root mean square error (RMSE), Theil's inequality coefficient (TIC) and QLIKE. It was concluded that the proposed BSGARCH (1, 1) model outperforms the traditional GARCH-type models that were considered in the study based on the performance metrics, and thus, it can be used for estimating volatility of stock markets.

1. Introduction

Extensive empirical and theoretical research has been conducted on modelling and forecasting stock market volatility over the past three decades [1–3]. This line of inquiry is motivated by a number of factors. Arguably, volatility is one of the most significant concepts in the whole of finance. Volatility is frequently used as a rough indicator of the total risk of financial assets. The estimation or forecast of a volatility parameter is used in many value-at-risk models for gauging market risk [4–6].

Economies are changing frequently and are susceptible to unpredicted economic shocks or changes in economic policy. It is often difficult to quantify all impacts of the dynamics that bring about change in an economy and because of this; economic forecasting is riddled with problems [7]. It is valuable to be able to consistently identify a forecasting model which is superior in terms of predictability to another model.

One of the most recognised characteristics of financial time series is the presence of nonconstant and time-dependent volatility in returns, and thus, volatility estimation has played a significant role in the fields of statistics, economics, and finance since the seminal work of Engle [8, 9]. Several different techniques attempt to address the issue of estimating the volatility of a financial asset, but the Generalised Autoregressive Conditional Heteroskedastic (GARCH) models have often been used in estimating the volatility of financial time series of stock returns [10, 11]. Especially, the GARCH (1, 1) has been proven to be one of the best predictive models in estimating the volatility of the stock market.

However, according to [12], some researchers have found that there is no unique model of GARCH estimation that consistently gives better results for every stock market. Another, restriction of the GARCH model is that it enforces a symmetric response of volatility to positive and negative shocks [2]. It has been argued that a negative shock to a financial time series is likely to cause volatility to rise by a margin that is more than a positive shock of the same magnitude. This limitation in the GARCH model has been somehow managed by the introduction of volatility treatments that accommodate the asymmetric responses of volatility to positive and negative shocks. Three simple classes of such models that capture the asymmetric nature of stock returns are the GJR-GARCH by [13], the exponential GARCH (EGARCH) by [14], and the asymmetric power ARCH (APARCH) by [15]. Though, the GJR-GARCH, EGARCH, and the APARCH models include the asymmetric response of volatility to positive and negative shocks, they do not capture stock fluctuations with volatility clustering accurately [6]. This fact can lead to errors in volatility estimation.

Several research studies have proposed new approaches to respond to some of these inadequacies of the classical GARCH models. For example, [16] proposed a measurement model that considers the possibility of time-varying interaction of realized volatility and asset returns according to a bivariate model to capture major characteristics such as long-term memory of the volatility process, the heavy-tail of the distribution of returns, and the negative dependence of volatility and daily market returns. Yet another way of dealing with the high persistence in volatility would be to explicitly assume that the volatility process is "smoothly" nonstationary and model it accordingly [17]. In view of this, [18] introduced a time-varying ARCH process for modelling nonstationary volatility [19] also assumed that the variance of a financial time series can be decomposed into stationary and nonstationary components. The stationary component in their work was assumed to follow the GARCH process, while the nonstationary component was described using the exponential quadratic splines. This was achieved through the use of multiplicative decomposition structure [20], though it did not explicitly mention nonstationarity, used multiplicative decomposition structure to correct potential misspecification due to a "rough" parametric GARCH specification by a smooth nonparametric component [17] also introduced two nonstationary GARCH models for situations in which volatility appears to be nonstationary. They proposed an additive time-varying parameter model and multiplicative decomposition of variance into unconditional and conditional components, but focused on the multiplicative decomposition.

The proposal of hybrid models is a potential efficient alternative for modelling and forecasting the volatility of stock markets because such models can account for a number of important features of financial series [2, 5]. Hybrid models combine first principle-based models with data-based models into a joint architecture, supporting enhanced model qualities, such as robustness and explainability [21].

In this paper, an efficient hybrid model that combines the B-spline estimator and the GARCH model are proposed. The piecewise nature of the B-spline estimator allows it to be interpreted as a threshold regime function where the regimes are associated with different regions of the predictor.

2. Materials and Methods Used

2.1. The GARCH Model. Among a number of time series models, the GARCH models proposed by [22] appear to be the most successful and popular form for modelling and forecasting the conditional variance of the return of volatility [14]. The GARCH (p, q) model considers the current conditional variance dependent on the p past conditional variances as well as the q past squared observation of the stochastic process [23]. Let x_t denote a real-valued discrete-time stochastic process and F_t the information set of all information through time t. The GARCH (p, q) process is given by

$$\begin{aligned} x_t \mid F_{t-1} &\sim N(0, h_t), \\ h_t &= \alpha_0 + \sum_{i=1}^{p} \alpha_i x_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j}. \end{aligned} \tag{1}$$

Here, $p > 0, \alpha_0 > 0, \beta_i \geq 0, q > 0, \alpha_i \geq 0$ [22]. In this model, h_t is the conditional variance of x_t. In addition to the nonnegativity of the parameters, there is parameter restriction

$$\sum_{i=1}^{p} \alpha_i + \sum_{j=1}^{q} \beta_j < 1, \tag{2}$$

to ensure the positivity of the conditional variance. The simplicity of the GARCH model and its ability to capture persistence of volatility explains its empirical and theoretical appeal [11].

2.2. EGARCH Model. The exponential GARCH (EGARCH) model provides an alternative asymmetric model by considering the leverage effects of a price change on the conditional variance [6]. It was first introduced by [14]. The EGARCH (p, q) model can be represented as

$$\begin{aligned} x_t &= \varepsilon_t \sqrt{h_t}, \\ \ln(h_t) &= \alpha_0 + \sum_{j=1}^{q} \beta_i \ln(h_{t-j}) + \sum_{i=1}^{p} \gamma_i \left(\frac{|x_{t-i}|}{\sqrt{h_{t-i}}} + E\left(\frac{|x_{t-i}|}{\sqrt{h_{t-i}}} \right) \right). \end{aligned} \tag{3}$$

The use of the log form allows the parameters to be negative without the conditional variance becoming negative [2, 5].

2.3. GJR-GARCH Model. [13] also proposed another asymmetric model, the GJR-GARCH model. The model is represented by

$$h_t = \alpha_0 + \sum_{i=1}^{p} \alpha_i x_{t-i}^2 + \sum_{j=1}^{q} \beta_j h_{t-j} + \sum_{k=1}^{p} \gamma_k x_{t-k}^2 I_{t-k}, \tag{4}$$

where

$$I_{t-k} = \begin{cases} 1, & \text{if } x_{t-k} < 0, \\ 0, & \text{if } x_{t-k} > 0. \end{cases} \tag{5}$$

The GJR-GARCH is closely related to the threshold GARCH (TGARCH) model proposed by [24].

2.4. Asymmetric Power ARCH (APARCH) Model. The general structure of the APARCH as introduced by [15] is as follows:

$$x_t = \varepsilon_t \sqrt{h_t}\,\varepsilon_t \sim N(0,1),$$
$$\sigma_t^{\delta} = \alpha_0 + \sum_{i=1}^{p} \alpha_i \left(|x_{t-i}| - \gamma_i x_{t-i} \right)^{\delta} + \sum_{j=1}^{q} \beta_j \sigma_{t-j}^{\delta}, \quad (6)$$

where

$$\alpha_0 > 0, \delta \geq 0, \alpha_i \geq 0, i = 1,\ldots,p, -1 < \gamma_i < 1, \beta_j \geq 0, j = 1,\ldots,q, \quad (7)$$

and $\sigma_t = \sqrt{h_t}$. The model imposes a Box-Cox power transformation of the conditional standard deviation process.

2.5. B-Spline. The term spline is used to refer to a wide class of functions that are used in applications requiring data interpolation and or smoothing. Splines may be used for interpolation and smoothing of either one-dimensional or multidimensional data [25]. Given a partition or a non-decreasing knot sequence t_i, the B-splines of order 1 for this knot sequence are the characteristic functions of this partition, and it is given by

$$B_{i1}(t) \coloneqq X_i(t) \coloneqq \begin{cases} 1, & \text{if } t_i \leq t < t_{i+1}, \\ 0, & \text{otherwise.} \end{cases} \quad (8)$$

The only constraint is that these B-splines should form a partition of unity. That is

$$\sum_i B_{i1}(t) = 1 \forall t. \quad (9)$$

In particular, $t_i = t_{i+1}$ implies $B_{i1} = X_i = 0$. From the first order B-splines, higher order B-splines can be obtained by recurrence:

$$B_{ik} \coloneqq \omega_{ik} B_{i,k-1} + \left(1 - \omega_{i+1,k}\right) B_{i+1,k-1}, \quad (10)$$

where

$$\omega_{ik}(t) \coloneqq \begin{cases} \dfrac{t - t_i}{\left(t_{i+k-1} - t_i\right)}, & \text{if } t_i \neq t_{i+k-1}, \\ 0, & \text{otherwise.} \end{cases} \quad (11)$$

After $k-1$ steps of the recurrence, B_{ik} is obtained in the form

$$B_{ik} = \sum_{j=i}^{i+k-1} b_{jk} X_j. \quad (12)$$

See [26] for details.

2.6. Proposed Model. Let (Ω, F, P) be a probability space and let F_{t-1} be the information set of all information through time $t-1$ which is generated by the stochastic process X_t and assume that under the probability measure P, X_t is given by

$$X_t = \mu_t + \sqrt{h_t} e_t,$$
$$h_t = f\left(X_{t-1}, h_{t-1}\right). \quad (13)$$

Here, e_t is a sequence of independent and identically distributed random variable such that $e_t \sim N(0,1), \mu_t = E(X_t | F_{t-1})$and $\mathrm{h}_t = \mathrm{var}(X_t | F_{t-1})$.

Generally, in financial applications, there is no need to allow for a large degree of flexibility in the dynamics of the conditional means [1]. Thus, it is assumed that the conditional mean is zero. That is

$$\mu_t = 0. \quad (14)$$

The main focus will be devoted to the modelling of the time-varying dynamics of the volatility $h_t = \mathrm{var}(X_t | F_{t-1})$. Estimation and prediction of volatility are a central task in the financial field because of its core importance in many practical applications.

Finding a methodology that yields accurate volatility predictions is one of the main goals in both academic research and practice.

The dynamics of the squared volatility is modelled as an additive expansion of a simple univariate B-spline basis function arising from the lagged values of the squared volatility to help uncover hidden trends in the volatility. In detail, the volatility is modelled as

$$h_t = g\left(x_{t-1}, h_{t-1}\right) + \sum_{i=1}^{n} \delta_i \vartheta_i \left(\nu_{t-1}\right)_+. \quad (15)$$

Here, $g(.,.)$ is a simple parametric starting function of the GARCH model in which an attempt is made to improve using the univariate B-spline basis function $\sum_{i=1}^{n} \delta_i \vartheta_i (\nu_{t-1})_+$. Where

$$\vartheta_i \left(\nu_{t-1}\right)_+ = \begin{cases} \vartheta_i \left(\nu_{t-1}\right), & \text{if } \vartheta_i \left(\nu_{t-1}\right) \geq 0, \\ 0, & \text{if } \vartheta_i \left(\nu_{t-1}\right) < 0. \end{cases} \quad (16)$$

Equation (16) is necessary to ensure that equation (15) is always positive. $\vartheta_i(\nu_{t-1})$ is the basis function.

Suppose that all $\delta_i \equiv 0$ which is a possibility, then the classical parametric GARCH (1, 1) model is obtained. ν_{t-1} is the residual at time $t-1$ from the classical GARCH (1, 1) model. In the sequel, equation (15) will be referred to as BSGARCH (1,1).

B-splines allows for a large flexibility in the shape of the conditional variance function depending on how the degree and number of knots of each basis function is chosen. The residual lags are allowed to be a quadratic function and thus

B-spline of degree 3 (that is of order 2) is chosen. The number of knots is a measure for the approximation accuracy. Usually, with a larger number of knots, a better approximation is obtained.

2.7. Algorithm for Parameter Estimation. The parameters in the first term of equation (15) are estimated in the same way the simple parametric GARCH (1, 1) model is estimated. That is using the maximum likelihood estimation method. The focus is on estimating the parameters of the second term. Considering

$$f = \sum_{i=1}^{n} \delta_i \vartheta_i (v_{t-1}), \tag{17}$$

where δ_i represents control points and $\vartheta_i (v_{t-1})$ is the basis function such that $\sum_{i=1}^{n} \vartheta_i (v_{t-1}) = 1$. The estimate is found by minimizing the residual sum of squares. That is

$$S = \arg\min_f \left(\sum_{t=1}^{n} (v_t - f(v_{t-1}))^2 + \lambda \| f''(v_{t-1}) \|_2^2 \right). \tag{18}$$

This is an infinite-dimensional optimization problem over all functions f for which the criterion is finite, which is the optimal solution is a continuous quantity. This criterion trades off the least squares error. Here, $\|\lambda f''(v_{t-1})\|_2^2$ is a roughness penalty which accounts for the fluctuations and controls the roughness of the function. λ is the tuning parameter and is such that $\lambda > 0$. In vector notation form, equation (18) is denoted as

$$\hat{f} = \delta \Psi, \tag{19}$$

with

$$\Psi_{ij} = \vartheta_i (v_{t-1}). \tag{20}$$

Equation (18) can now be written as

$$\left(S = \arg\min_f \left((v_t - \delta\Psi)^T (v_t - \delta\Psi) + \lambda \delta^T \Omega \delta \right) \right). \tag{21}$$

The matrix Ω is a positive semidefinite matrix by construction and it is known as the penalty matrix. It induces a seminorm on $\mathfrak{R}^n$ so that the seminorm $\|f''\|_2$ of f can be expressed in terms of the parameters in the basis expansion using ϑ_j, thus, Ω can be expressed as

$$\Omega_{ij} = \int_a^b \vartheta_i'' (v_{t-1}) \vartheta_j'' (v_{t-1}) dv_{t-1}, \tag{22}$$

where a and b are consecutive knots.

The partial derivative of S with respect to δ is

$$\frac{\partial S}{\partial \delta} = \delta \left(\Psi \Psi^T + \lambda \Omega \right) - \Psi^T v_t. \tag{23}$$

Equating $(\partial S / \partial \delta)$ to 0 and finding δ, we get

$$\hat{\delta} = \left(\Psi \Psi^T + \lambda \Omega \right)^{-1} \Psi^T v_t. \tag{24}$$

Therefore, equation (15) now becomes

$$\hat{f} = \Psi \left(\Psi \Psi^T + \lambda \Omega \right)^{-1} \Psi^T v_t, \tag{25}$$

where Ω is determined as follows:

Let $\vartheta_i'' (v_{t-1}) \vartheta_j'' (v_{t-1}) = q_{ij} (v_{t-1})$ then, equation (22) can be written as

$$\Omega_{ij} = \int_a^b q_{ij} (v_{t-1}) dv_{t-1}. \tag{26}$$

In which $q_{ij} (v_{t-1})$ can be expressed as equation (27) using Newton's divided difference polynomial.

$$q_{ij} (v_{t-1}) = b_0 + b_1 (v_{t-1} - a) + b_2 (v_{t-1} - a) \left(v_{t-1} - \frac{a+b}{2} \right), \tag{27}$$

where

$$\begin{aligned} b_0 &= q_{ij}(a), \\ b_1 &= \frac{q_{ij}(a + b/2) - q_{ij}(a)}{(a + b/2) - a}, \\ b_2 &= \frac{\left(q_{ij}(b) - q_{ij}(a + b/2) \right) - \left(q_{ij}(a + b/2/) - q_{ij}(a)(a + b/) - a \right)/b - (a + b/2)b - a(a + b/2) - a/b - (a + b/2)\,(a + b/2) - a}{b - a}. \end{aligned} \tag{28}$$

Integrating equation (26) by the Simpson's approximation rule gives equation (28)

$$\frac{b-a}{6} \left(q_{ij}(a) + 4 q_{ij} \left(\frac{a+b}{2} \right) + q_{ij}(b) \right) = \Omega. \tag{29}$$

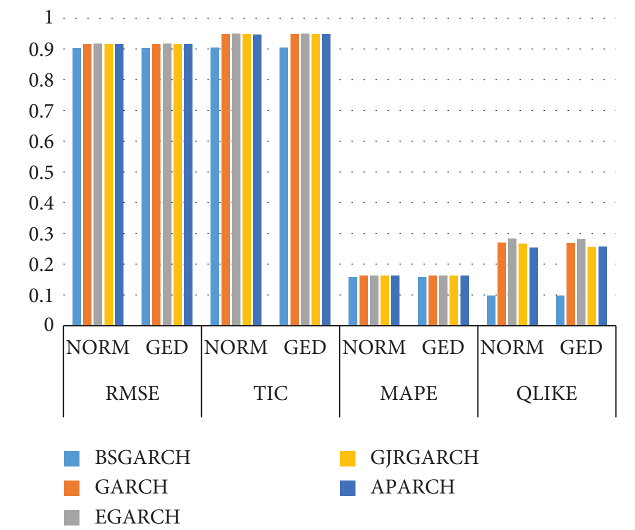

Figure 1: Performance evaluation averaged over 1000 independent simulations for $n = 2000$ observations with parameters $\phi = 0.1, \alpha_0 = 0.1, \alpha_1 = 0.3$.

Now, that the penalty matrix Ω has been established in equation (28), it is also important that the optimal tuning parameter λ be determined. Determination of the optimal tuning parameter λ is exceptionally critical to derive a good curve estimator.

2.8. Determination of Tuning Parameter λ. There has been a number of methods for determining the optimal tuning parameter λ. The most commonly used ones are the ordinary cross-validation (OCV) and the generalized cross-validation (GCV). The leave-$(2l + 1)$-out model of [27] is one of the modifications of the OCV. However, The GCV method found in [28] was adopted for the purpose of this study because it has various advantages over other methods, which includes being asymptotically optimal [29]. It is given as

$$\mathrm{GCV}_\lambda = \frac{\left(\frac{1}{n}\right)\sum_{t=1}^{n}\left(\left(v_t\right) - \widehat{F}\left(\left(v_{t-1}\right)\right)\right)^2}{\left(1 - \left(df/n\right)\right)^2}, \quad (30)$$

where

$$df = \mathrm{trace}\left(\Psi\left(\Psi\Psi^T + \lambda\Omega\right)^{-1}\Psi^T\right). \quad (31)$$

2.9. Performance Metric. Forecasted volatility is usually compared to an ex-post proxy for volatility because true volatility is latent [30]. One such ex-post proxy is realized volatility introduced by [31]. The realized volatility is the square root of the realized variance which is the sum of squared returns [32, 33]. The realized variance is given as

$$\mathrm{RV}_t = \sum_{t=1}^{n} r_t^2. \quad (32)$$

Here,

$$r_t = \ln\left(\frac{P_t}{P_{t-1}}\right), \quad (33)$$

and n is the number of observations.

The ability of the models to accurately forecast realized volatility is assessed using four different loss functions because it is not obvious which loss function is more appropriate for the evaluation of volatility models [34]. The loss functions considered were root mean square error (RMSE), mean absolute percentage error (MAPE), Theil's inequality coefficient (TIC), and QLIKE.

$$\mathrm{RMSE} = \sqrt{\frac{1}{n}\sum_{t=1}^{n}\left(RV_t - h_t\right)^2},$$

$$\mathrm{MAPE} = \frac{1}{n}\sum_{t=1}^{n}\left|\frac{RV_t - h_t}{RV_t}\right|,$$

$$\mathrm{TIC} = \frac{\sqrt{(1/n)\sum_t\left(RV_t - h_t\right)^2}}{\sqrt{(1/n)\sum_t RV_t^2} + \sqrt{(1/n)\sum_t h_t^2}}, \quad (34)$$

$$\mathrm{QLIKE} = \frac{1}{n}\sum_{t=1}^{n}\left(\ln\left(h_t^2\right) + \frac{RV_t^2}{h_t^2}\right).$$

The QLIKE function used is the definition as found in [34].

3. Results and Discussion

3.1. Simulated Data. Similar to the following equation, the data generating process was a time series Y_t that satisfies equation (33)

$$Y_t = \phi Y_{t-1} + \varepsilon_t. \quad (35)$$

First, ε_t was assumed to be normal with mean 0 and conditional variance

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2. \quad (36)$$

The first 70% of the simulated data was used as an in-sample period to estimate the model and the successive 30% as out-of-sample testing period. It should be noted that the conditional mean for all the models were not estimated. It was assumed to be zero. The main focus was on the conditional variance. The in-sample data was used for estimating all the parameters in the model that was then used for the forecasting. The forecasting performance of the GARCH (1, 1), EGARCH (1, 1), GJR-GARCH (1, 1), and the proposed BSGARCH (1, 1) averaged over 1000 repetitions is presented under 6 different scenarios.

3.1.1. Scenario 1. In this scenario, 2000 observations were generated using the parameters $\phi = 0.1, \alpha_0 = 0.1, \alpha_1 = 0.3$. The forecasting performance is presented in Figure 1. From Figure 1, the proposed BSGARCH (1, 1) model had the smallest values in all the performance metric used in this study. This means that, the proposed BSGARCH (1, 1) model outperforms the classical GARCH- type models in forecasting the volatility of financial time series that has low ARCH parameter.

3.1.2. Scenario 2. In this scenario, 255 observations were generated using the parameters $\phi = 0.1, \alpha_0 = 0.1, \alpha_1 = 0.3$. The forecasting performances are shown in Figure 2. Again, the proposed BSGARCH (1, 1) outperforms the classical GARCH-type models in the QLIKE and the RMSE. However, the TIC and MAPE were very close in all the models.

3.1.3. Scenario 3. In this scenario, 2000 observations were generated using the parameters $\phi = 0.6, \alpha_0 = 0.4, \alpha_1 = 0.6$. The forecasting performance is presented in Figure 3. The proposed BSGARCH (1, 1) model had the smallest values in all the performance metric under this scenario also. This means that, the proposed BSGARCH (1, 1) model outperforms the classical GARCH- type models in forecasting the volatility of financial time series that has medium ARCH parameter.

3.1.4. Scenario 4. In this scenario, 255 observations were generated using the parameters $\phi = 0.6, \alpha_0 = 0.4, \alpha_1 = 0.6$. The forecasting performances are shown in Figure 4. In this scenario the forecasting performance of all the models were almost the same. That is the difference was insignificant.

3.1.5. Scenario 5. In this scenario, 2000 observations were generated using the parameters $\phi = 0.8, \alpha_0 = 0.6, \alpha_1 = 0.9$. The forecasting performance is presented in Figure 5. The proposed BSGARCH (1, 1) model had the smallest values in all the performance metric under this scenario also. This means that, the proposed BSGARCH (1, 1) model outperforms the classical GARCH- type models in forecasting the volatility of financial time series that have a high ARCH parameter.

3.1.6. Scenario 6. In this scenario, 255 observations were generated using the parameters $\phi = 0.8, \alpha_0 = 0.6, \alpha_1 = 0.9$. The forecasting performances are shown in Figure 6. In this scenario, the proposed BSGARCH (1, 1) model outperforms the classical GARCH-type models.

After considering all the 6 Scenarios using the conditional variance in equation (34), another simulation was performed where ε_t was assumed to be normal with mean 0 and conditional variance with an additive component given in equation (35)

$$h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + g_t, \quad (37)$$

where g_t is a time-varying persistent deterministic volatility component which was set to be

$$g_t = \sin\left(\frac{2\pi t}{n}\right), \quad (38)$$

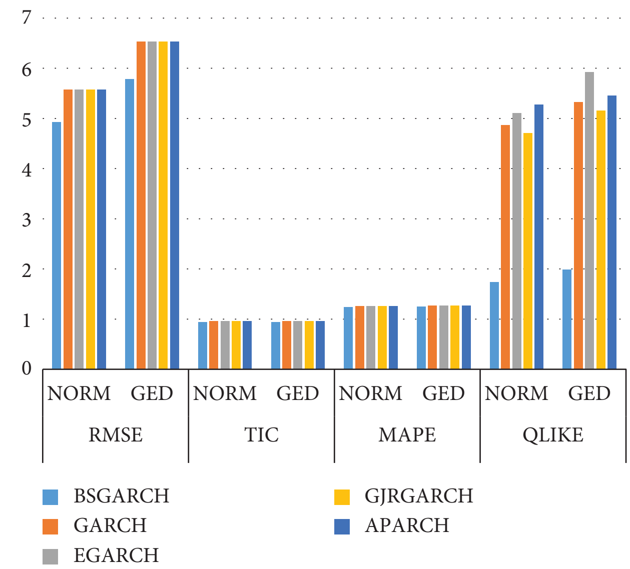

Figure 2: Performance evaluation averaged over 1000 independent simulations for $n = 255$ observations with parameters $\phi = 0.1, \alpha_0 = 0.1, \alpha_1 = 0.3$.

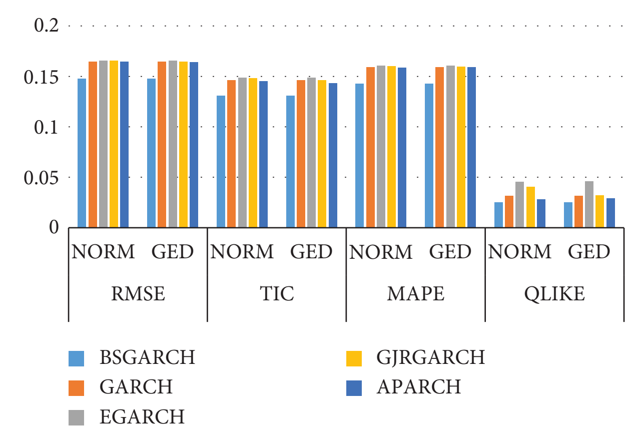

Figure 3: Performance evaluation averaged over 1000 independent simulations for $n = 2000$ observations with parameters $\phi = 0.6, \alpha_0 = 0.4, \alpha_1 = 0.6$.

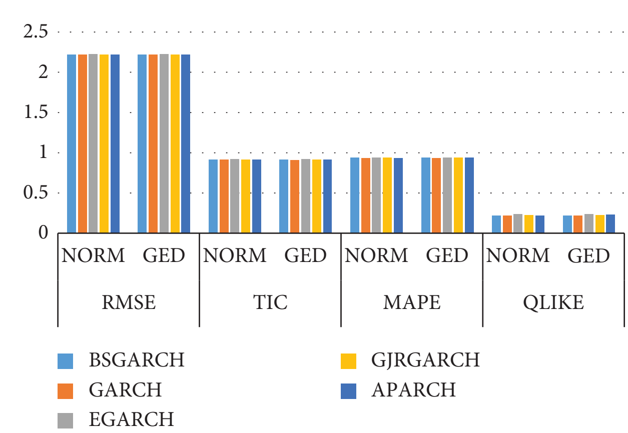

Figure 4: Performance evaluation averaged over 1000 independent simulations for $n = 255$ observations with parameters $\phi = 0.6, \alpha_0 = 0.4, \alpha_1 = 0.6$.

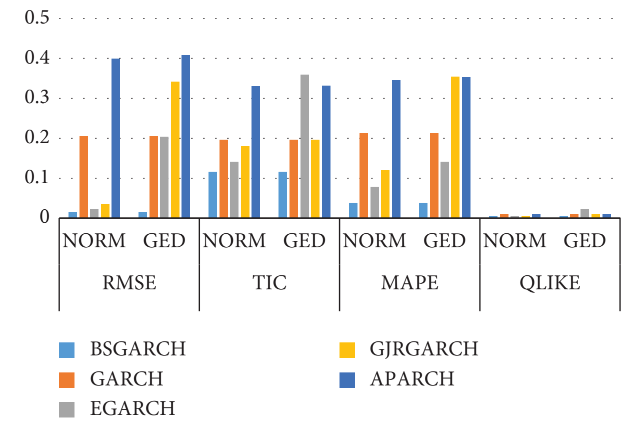

FIGURE 5: Performance evaluation averaged over 1000 independent simulations for $n = 2000$ observations with parameters $\phi = 0.8, \alpha_0 = 0.6, \alpha_1 = 0.9$.

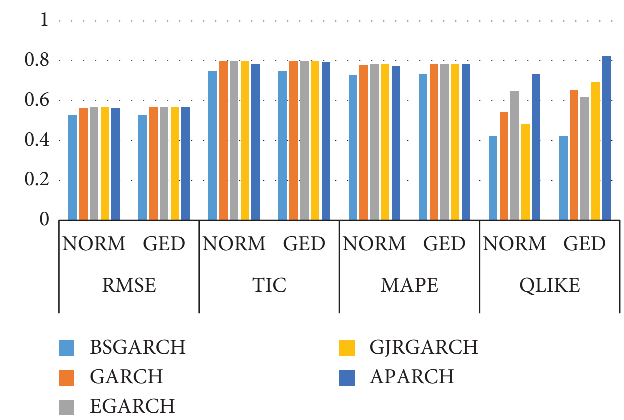

FIGURE 6: Performance evaluation averaged over 1000 independent simulations for $n = 255$ observations with parameters $\phi = 0.8, \alpha_0 = 0.6, \alpha_1 = 0.9$.

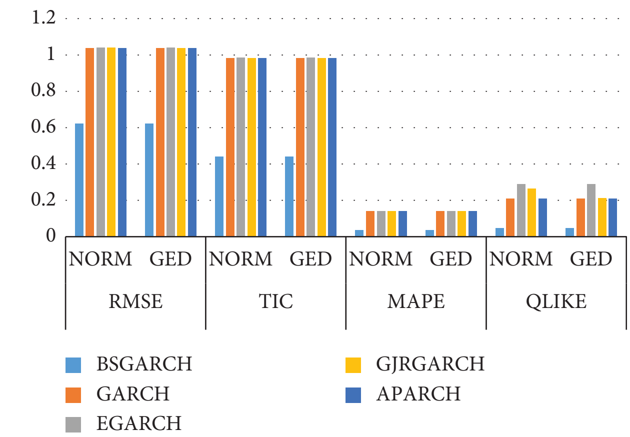

FIGURE 7: Performance evaluation averaged over 1000 independent simulations for $n = 2000$ observations with parameters $\phi = 0.8, \alpha_0 = 0.6, \alpha_1 = 0.9$ and $h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + g_t$.

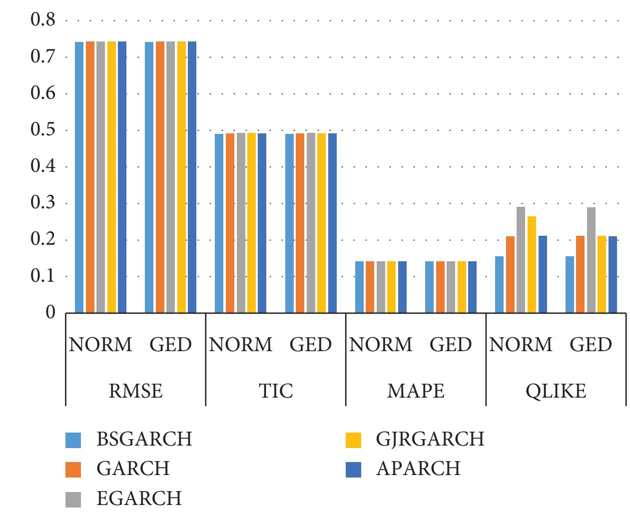

FIGURE 8: Performance evaluation averaged over 1000 independent simulations for $n = 255$ observations with parameters $\phi = 0.8, \alpha_0 = 0.6, \alpha_1 = 0.9$ and $h_t = \alpha_0 + \alpha_1 \varepsilon_{t-1}^2 + g_t$.

and the parameters $\phi = 0.8, \alpha_0 = 0.6, \alpha_1 = 0.9$ was used. The results averaged over 1000 replications with length of observation 2000 and 255 are presented in Figures 7 and 8, respectively.

It can be observed from Figure 7 that the proposed BSGARCH (1, 1) model outperforms the traditional GARCH-type models. However, in Figure 8, it can be observed that the forecasting performance of all the models were close except the QLIKE where the BSGARCH (1, 1) model outperforms the other models.

3.2. Real Data. The proposed BSGARCH (1, 1) model was again applied on two different datasets, namely; Nasdaq 100 (NASDAQ100) and Standard and Poor 500 (S&P 500). The data consist of 2403 observations that span from 5th December, 2011 to 30th June, 2021 and were obtained from the Federal Reserve Economic Data (FRED). The first 70% of the data was used as in-sample estimation period and the successive remaining 30% of the data as out-of-sample test data. Since volatility is itself latent, the annualized realized volatility was used as a highly accurate measure for the latent true volatility.

3.3. Residual Analysis. After the proposed model was applied to the two datasets, the McLeod and Li test for autocorrelation and the ACF plot of the standardized residuals were done to check for the validity of the proposed model. Figures 9 and 10 show the ACF plot of standardized residuals from BSGARCH (1, 1) applied to NASDAQ100 and S&P 500, respectively. It was revealed that the BSGARCH (1, 1) method produced forecast that accounted for all available information. The mean of the standardized residual was close to zero and there was no significant correlation in the standardized residual series.

Again, the McLeod-Li test was performed to check for autocorrelations in the standardized residuals. The number of lags considered was 10, 15, 20, 25, and 30. The automatic portmanteau version of [36] can also be used. The null hypothesis for the McLeod-Li test is that, the residuals of the model are independent and identically distributed (i.i.d.)

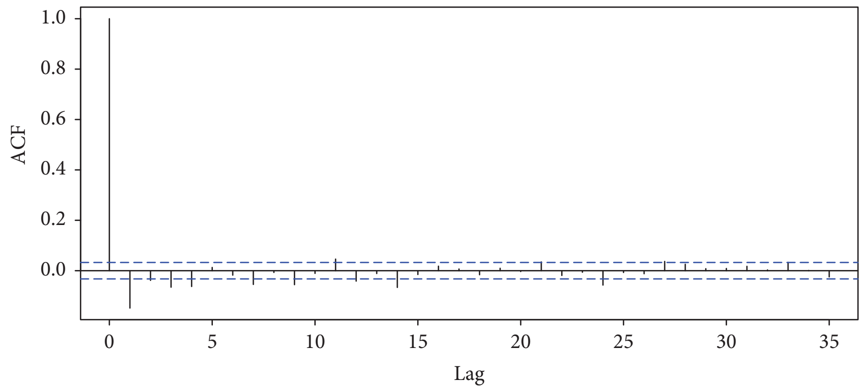

FIGURE 9: ACF of standardized residuals from BSGARCH (1, 1) applied to NASDAQ100.

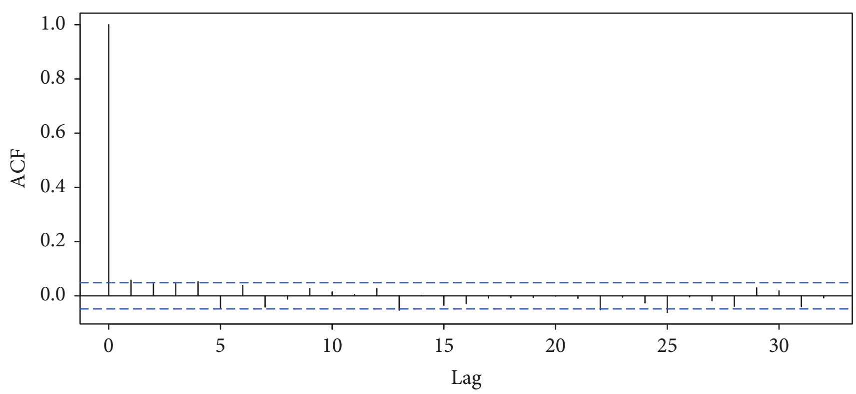

FIGURE 10: ACF of standardized residuals from BSGARCH (1, 1) applied to S&P 500.

TABLE 1: The McLeod-Li test for ARCH effect in standardized residuals.

	Number of lags	Test statistic	P values
S&P 500	10	11.387	0.328
	15	19.975	0.173
	20	22.809	0.298
	25	23.282	0.561
	30	25.494	0.701
NASDAQ100	10	3.981	0.948
	15	15.021	0.450
	20	16.440	0.689
	25	19.325	0.781
	30	21.347	0.877

and the alternative that they are not. This is usually referred to as an ARCH test [37].

From Table 1, the null hypothesis of the McLeod and Li test for both S&P 500 and NASDAQ100 was not rejected at lags 10, 15, 20, 25, and 30 because the p values were greater than the 5% significance level. This implies that there is no sufficient evidence to suggest that the residuals of the BSGARCH (1, 1) model are not independent and identically distributed (i.i.d.). In practical terms, not rejecting the null hypothesis indicates that the BSGARCH (1, 1) model is appropriate for the data, and the estimated coefficients and predicted values are unbiased and efficient.

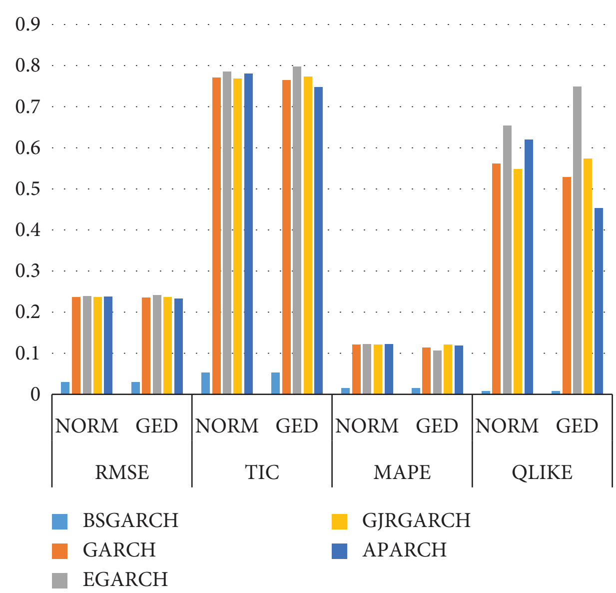

FIGURE 11: Performance evaluation for NASDAQ100.

3.4. Performance Evaluation. Figures 11 and 12 present the performance of the classical GARCH (1, 1), EGARCH (1, 1), GJR-GARCH (1, 1), and APARCH with different error distributions and the BSGARCH (1, 1) model in terms of volatility estimates and forecasts for NASDAQ100 and S&P

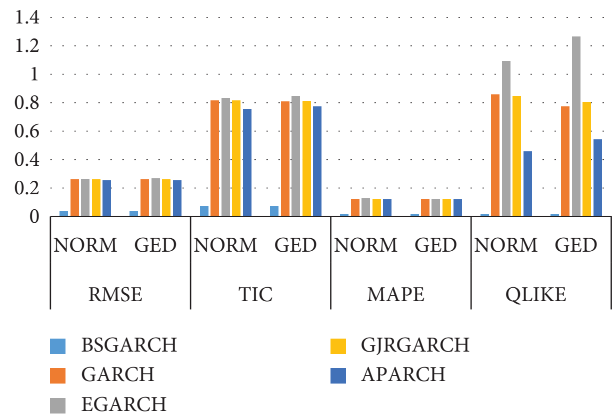

Figure 12: Performance evaluation for S&P 500.

Table 2: Test for superior predictive ability.

Models	ED	Test values	P values
NASDAQ100			
BSGARCH vs. GARCH	NORM	203.97	<0.001
	GED	175.63	<0.001
BSGARCH vs. EGARCH	NORM	263.94	<0.001
	GED	276.30	<0.001
BSGARCH vs. GJR-GARCH	NORM	197.29	<0.001
	GED	208.64	<0.001
BSGARCH vs. APARCH	NORM	225.05	<0.001
	GED	152.52	<0.001
S&P 500			
BSGARCH vs. GARCH	NORM	678.79	<0.001
	GED	646.22	<0.001
BSGARCH vs. EGARCH	NORM	706.79	<0.001
	GED	721.72	<0.001
BSGARCH vs. GJR-GARCH	NORM	661.22	<0.001
	GED	647.70	<0.001
BSGARCH vs. APARCH	NORM	312.95	<0.001
	GED	392.86	<0.001

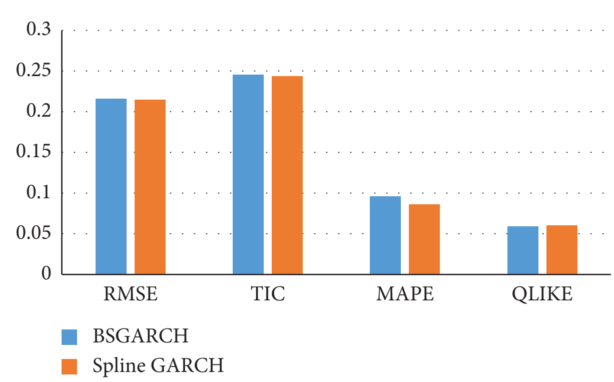

Figure 13: Comparison of BSGARCH (1,1) and Spline-GARCH (1,1) models.

500, respectively. For both datasets, the BSGARCH (1, 1) model consistently beat all the classical methods that were considered in the study. This was because the BSGARCH (1, 1) model had the least RMSE, MAPE, TIC, and QLIKE values both. This indicated that the proposed model is a major improvement over the classical methods considered in estimating and forecasting volatility. This means that the proposed model has a higher forecasting ability especially when the sample size is large.

3.5. Average Superior Predictive Ability (aSPA) Test. The average Superior Predictive Ability (aSPA) test introduced by [38] was used to assess the predictive ability of the models. Positive test values are in favour of the BSGARCH (1, 1) model. From Table 2, the aSPA test indicated that indeed the BSGARCH (1, 1) has a superior prediction ability for the dataset under consideration.

3.6. Comparison of BSGARCH (1, 1) with the Spline-GARCH Model. After comparison with the classical GARCH-type models, the forecasting ability of the proposed BSGARCH (1, 1) model was again compared with the spline-GARCH model of [19] which used the exponential quadratic spline to model the nonstationary part of volatility. The data used for this comparison is the data found in [19]. The performance metric used for this comparison were, RMSE, MAPE, TIC, and QLIKE and the results shown in Figure 13. In the out-of-sample data prediction which represents the ability of the models to predict unknown values, the spline-GARCH model of [19] slightly outperformed the proposed BSGARCH (1, 1) model. In fact, the percentage gains of the spline-GARCH model of [19] over the proposed model in RMSE are approximately 0.6% which is negligible. This means that the proposed BSGARCH (1, 1) model can be a good alternative to the spline-GARCH model of [19].

4. Conclusions

A flexible BSGARCH (1, 1) hybrid model has been proposed to estimate the volatility of a financial time series. This model combines the GARCH (1, 1) model and the B-spline method. The proposed hybrid model is able to fit well in both simulated and real data. The results showed that the proposed BSGARCH (1, 1) model outperforms the traditional GARCH-type models that were considered in the study especially when the long-term volatility is of interest. However, within a short period, the performance of the proposed model is no different from the classical GARCH-type models. Again, the proposed BSGARCH (1, 1) can be a good alternative to the spline-GARCH model of [19], and thus, it can be used for estimating the volatility of stock markets.

Data Availability

The data used for the study can be obtained from FRED using the R code "getSymbols ("SP500," src = "FRED," auto.assign = FALSE)" and getSymbols ("NASDAQ100," src = "FRED," auto.assign = FALSE).

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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