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Research Article

Bayesian Estimation of the Stress-Strength Reliability Based on Generalized Order Statistics for Pareto Distribution

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The aim of this paper is to obtain a Bayesian estimator of stress-strength reliability based on generalized order statistics for Pareto distribution. The dependence of the Pareto distribution support on the parameter complicates the calculations. Hence, in literature, one of the parameters is assumed to be known. In this paper, for the first time, two parameters of Pareto distribution are considered unknown. In computing the Bayesian confidence interval for reliability based on generalized order statistics, the posterior distribution has a complex form that cannot be sampled by conventional methods. To solve this problem, we propose an acceptance-rejection algorithm to generate a sample of the posterior distribution. We also propose a particular case of this model and obtain the classical and Bayesian estimators for this particular case. In this case, to obtain the Bayesian estimator of stress-strength reliability, we propose a variable change method. Then, these confidence intervals are compared by simulation. Finally, a practical example of this study is provided.

1. Introduction

There are at least two factors in a system, one of which puts stress on the other and the other factor resists. In this case, the stress-strength parameter is raised. In a system where stress is applied to its component and members, it resists that stress. According to this model, the more stress is created on the system, the more likely the system will fail. In other words, the system continues to operate as long as the system's strength is greater than the stress applied to it. The stress-strength parameter is defined as a probability $R = P(X > Y)$, in which Y is the random variable of stress and X is the random variable of strength based on which the survival of a system can be controlled. The term stress-strength model was first coined by [1]. Then, many studies were performed on the stress-strength parameter based on different distributions and different conditions governing random variables. Some of the most recently used distributions include two-parameter bathtub-shaped lifetime distribution [2], POLO distribution [3], finite mixture distributions [4], standard two-sided power distribution [5], Kumaraswamy distribution [6], and Gompertz distribution [7].

The Pareto distribution is one of the most important statistical distributions with heavy and skewed tails that is used as a model for many socioeconomic phenomena. The Pareto distribution is also used to study the lifetime of organisms and the issue of reliability, as well as many statistical issues related to finance, insurance, and hydrology. In recent years, the study of reliability based on the Pareto distribution has become an exciting topic. Reference [8] estimated the reliability of the Pareto distribution in the presence of outliers using maximum likelihood (ML), method of moments, least squares, and modified maximum likelihood. Reference [9] studied the confidence interval estimation and approximate hypothesis testing for the reliability of the Pareto distribution based on progressively type-II-censored data with the generalized variable method. Reference [10] assumed the scale parameter of the Pareto distribution to be known and obtained the Bayesian reliability estimate using conjugate and Jeffrey's priors based on type-II-censored data. Reference [11], with the generalized variable method, investigated the reliability of the generalized Pareto distribution. Reference [12] obtained the ML and Bayesian estimates and also the highest posterior density interval for multicomponent stress-strength reliability by considering different shape parameters and common scale parameters based on upper record values.

The generalized order statistics (GOS) model can be considered as a unified model for studying ordered random variables [13]. The GOS includes a wide range of statistics with a sequential nature, such as ordinary order statistics, progressively type-II, censored order statistics, type-II-censored order statistics, and first n record values as subgroups. The theorems expressed and proved for the GOS are also established in its subgroups. The importance of using these models in terms of reliability cannot be ignored. Recently, the study of Pareto distribution based on the GOS has attracted the attention of many authors. Reference [14] obtained the ratio distribution of the GOS from the Pareto distribution. The recurrence relations of moments for the Pareto distribution's GOS were presented by [15]. Reference [16] estimated the parameters of the generalized Pareto distribution based on GOS using ML, bootstrap, and Bayesian under the LSE and LINEX loss functions. Reference [17] studied the properties, recurrence relations of moments, and ML estimate of the parameters for the generalized Pareto distribution based on the GOS.

Reference [18] studied the analysis of stress-strength reliability model based on the Pareto distribution using records, and [10] studied this model based on censored data. However, an analysis of the stress-strength reliability model for the Pareto distribution based on GOS is not available in the statistical literature. On the other hand, because the support of the Pareto distribution depends on the parameter, due to the difficulty of having two unknown parameters in articles, one parameter is assumed to be fixed and analyses are performed. In this paper, for the first time, we present the estimation of the stress-strength reliability of the Pareto distribution using classical and Bayesian inference based on the GOS, where both parameters are considered unknown. In the Bayesian method, the posterior distribution is not a closed form; so, to produce a sample of it, we propose the acceptance-rejection method. We also introduce a special case of this model. In estimating the reliability of the special case by the Bayesian method, we need to solve an integral that cannot be solved by analytical methods and we propose a method to solve it using variable change and Monte Carlo.

The structure of the article includes seven sections. In Section 2, the generalized, bootstrap percentile, and bootstrap-t confidence intervals of stress-strength reliability (R) for the Pareto distribution are calculated, which we denote these confidence intervals by GCI, BPCI, and BTCI, respectively. In Section 3, R estimation based on GOS is obtained using the ML method. In Section 4, Bayesian inference is provided for this model by using the squared error-loss function. Section 5 obtains ML and Bayesian estimation for the specific model of the Pareto distribution based on GOS. The Monte Carlo simulation for comparing estimators and confidence intervals obtained by ML and Bayesian methods are presented in Section 6. Finally, in Section 7, these methods are applied to real data to demonstrate the application of the proposed methods.

2. The GCI, BPCI, and BTCI of R for Pareto Distribution

The random variable X has a Pareto distribution with the shape parameter δ and scale parameter γ when its cumulative distribution function (CDF) and probability density functions (PDFs) are as follows:

$$\begin{aligned} G(x) &= 1-\left(\frac{\gamma}{x}\right)^{\delta}, x>\gamma, \\ g(x) &= \delta\gamma^{\delta}x^{-(\delta+1)}, x>\gamma. \end{aligned} \tag{1}$$

We denote it by $X \sim \text{Pareto}(\delta,\gamma)$.

To obtain R for Pareto distribution, let $X \sim \text{Pareto}(\delta_1,\gamma_1)$ and $Y \sim \text{Pareto}(\delta_2,\gamma_2)$ be independent. Thus,

$$\begin{aligned} R = P(X>Y) &= \int_{-\infty}^{+\infty} G_Y(\omega) g_X(\omega)d\omega \\ &= \int_{\max(\gamma_1,\gamma_2)}^{+\infty}\left[1-\left(\frac{\gamma_2}{\omega}\right)^{\delta_2}\right]\delta_1\gamma_1^{\delta_1}\omega^{-(\delta_1+1)}d\omega \\ &= \int_{\max(\gamma_1,\gamma_2)}^{+\infty}\delta_1\gamma_1^{\delta_1}\omega^{-(\delta_1+1)}d\omega - \int_{\max(\gamma_1,\gamma_2)}^{+\infty}\delta_1\gamma_1^{\delta_1}\gamma_2^{\delta_2}\omega^{-(\delta_1+\delta_2+1)}d\omega \\ &= \left(\frac{\gamma_1}{\max(\gamma_1,\gamma_2)}\right)^{\delta_1}\left[1-\left(\frac{\delta_1}{\delta_1+\delta_2}\right)\left(\frac{\gamma_2}{\max(\gamma_1,\gamma_2)}\right)^{\delta_2}\right] \\ &= R(\delta_1,\delta_2,\gamma_1,\gamma_2). \end{aligned} \tag{2}$$

The above relation can be restated as follows:

$$R = \left[1 - \left(\frac{\delta_1}{\delta_1 + \delta_2}\right)\left(\frac{\gamma_2}{\gamma_1}\right)^{\delta_2}\right] I(\gamma_1 > \gamma_2) + \left[\left(\frac{\delta_2}{\delta_1 + \delta_2}\right)\left(\frac{\gamma_1}{\gamma_2}\right)^{\delta_1}\right] I(\gamma_1 < \gamma_2) + \left(\frac{\delta_2}{\delta_1 + \delta_2}\right) I(\gamma_1 = \gamma_2). \tag{3}$$

Let $X_1, \ldots, X_{\iota_1} \sim \text{Pareto}(\delta_1, \gamma_1)$ and $Y_1, \ldots, Y_{\iota_2} \sim$ Pareto (δ_2, γ_2). Then, the ML estimator of R is $\widehat{R}_{ml} = R(\widehat{\delta}_{1_{ml}}, \widehat{\delta}_{2_{ml}}, \widehat{\gamma}_{1_{ml}}, \widehat{\gamma}_{2_{ml}})$, where $\widehat{\gamma}_{1_{ml}} = X_{(1)}, \widehat{\gamma}_{2_{ml}} = Y_{(1)}$ and

$$\widehat{\delta}_{1_{ml}} = \left[\frac{1}{\iota_1}\sum_{i=1}^{\iota_1} \log\left(\frac{X_i}{X_{(1)}}\right)\right]^{-1}, \widehat{\delta}_{2_{ml}} = \left[\frac{1}{\iota_2}\sum_{i=1}^{\iota_2} \log\left(\frac{Y_i}{Y_{(1)}}\right)\right]^{-1}. \tag{4}$$

We construct GCI, BPCI, and BTCI for R of the Pareto distribution.

2.1. GCI. The GCI and generalized pivotal quantity (GPQ) were defined by [19]. We propose a GPQ in the following theorem.

Theorem 1. *Let $(\widehat{\delta}, \widehat{\gamma})$ be the ML estimation of (δ, γ) for Pareto(δ, γ). Then,*

(i) $\widehat{\delta}$ and $\widehat{\gamma}$ are independent.

(ii) $W = 2n\delta \log(\widehat{\gamma}/\gamma) \sim \chi^2_{(2)}$.

(iii) $Q = 2n\delta/\widehat{\delta} \sim \chi^2_{(2n-2)}$.

Where $\widehat{\gamma} = X_{(1)}$ and $\widehat{\delta} = [1/n\sum_{i=1}^{n} \log(X_i/X_{(1)})]^{-1}$.

Proof. The proof is similar to [9, 20].

Let $X_1, \ldots, X_n \sim \text{Pareto}(\delta_1, \gamma_1)$ and $Y_1, \ldots, Y_{n'} \sim$ Pareto (δ_2, γ_2) be independent. From Theorem 1, it can be concluded that

$$\begin{aligned} W_1 &= 2n\delta_1 \log\left(\frac{\widehat{\gamma}_1}{\gamma_1}\right) \sim \chi^2_{(2)}, \\ W_2 &= 2n'\delta_2 \log\left(\frac{\widehat{\gamma}_2}{\gamma_2}\right) \sim \chi^2_{(2)}, \\ Q_1 &= 2n\frac{\delta_1}{\widehat{\delta}_1} \sim \chi^2_{(2n-2)}, \\ Q_2 &= 2n'\frac{\delta_2}{\widehat{\delta}_2} \sim \chi^2_{(2n'-2)}. \end{aligned} \tag{5}$$

Here, W_1, W_2, Q_1 and Q_2 are independent. Our proposed GPQ for R is as follows:

$$R_g = R\left(\delta_{1g}, \delta_{2g}, \gamma_{1g}, \gamma_{2g}\right), \tag{6}$$

where

$$\delta_{1g} = \frac{Q_1\widehat{\delta}_{01}}{2n}, \delta_{2g} = \frac{Q_2\widehat{\delta}_{02}}{2n'}, \gamma_{1g} = \frac{\widehat{\gamma}_{01}}{\exp\left(W_1/Q_1\widehat{\delta}_{01}\right)}, \gamma_{2g} = \frac{\widehat{\gamma}_{02}}{\exp\left(W_2/Q_2\widehat{\delta}_{02}\right)}. \tag{7}$$

We use Monte Carlo simulation to find GCI. Algorithm 1 is presented for this purpose. □

2.2. BPCI and BTCI. One of the critical issues in statistical inference is the confidence interval for a parameter, which expresses the status of the parameter at a certain level of confidence. Usually, assuming the population distribution is normal, the z-standard and t-student confidence intervals for the mean population and the mean difference between two populations, the chi-square and Fisher confidence intervals for the variance, and the variance ratio of two populations are used. Nevertheless, the assumption of a normal society is not always established. Statistical studies have shown that when data are selected from a population with a skewed distribution or the sample size is small, the abovementioned confidence intervals do not have the required coverage accuracy. In search of ways to solve these problems, we can point to the bootstrap confidence intervals, which have a high coverage accuracy, and their efficiency is further determined by the size of small samples. BPCI and BTCI are bootstrap confidence intervals [21]. We obtain these two confidence intervals for R of the Pareto distribution with Algorithm 2.

2.3. A Special Case of Pareto Distribution. We consider $X \sim \text{Pareto}(\delta, \gamma_1)$ and $Y \sim \text{Pareto}(\delta, \gamma_2)$. So, we have

(1) Consider $(\widehat{\delta}_{01}, \widehat{\delta}_{02}, \widehat{\gamma}_{01}, \widehat{\gamma}_{02})$, the recorded value of $(\widehat{\delta}_1, \widehat{\delta}_2, \widehat{\gamma}_1, \widehat{\gamma}_2)$;
(2) Given $\widehat{\delta}_{01}, \widehat{\delta}_{02}, \widehat{\gamma}_{01}, \widehat{\gamma}_{02}, n$ and n';
(3) Generate $W_1, W_2 \sim \chi^2_{(2)}, Q_1 \sim \chi^2_{(2n-2)}$, and $Q_2 \sim \chi^2_{(2n'-2)}$;
(4) Calculate $\delta_{1g}, \delta_{2g}, \gamma_{1g}$, and γ_{2g};
(5) Calculate R_g;
(6) Repeat 2–5 steps N times and denote $R_g^1, \ldots, R_g^N$;
(7) Obtain the $100(1-\varepsilon)\%$ GCI by $(R_g^{(\varepsilon/2)}, R_g^{(1-\varepsilon/2)})$, where the ε^{th} quantile of $R_g^1, \ldots, R_g^N$ is the same as $R_g^{(\varepsilon)}$.

Algorithm 1: GCI.

(1) Given $\widehat{\delta}_{01}, \widehat{\delta}_{02}, \widehat{\gamma}_{01}, \widehat{\gamma}_{02}, n$, and n';
(2) Generate $W_1, W_2 \sim \chi^2_{(2)}, Q_1 \sim \chi^2_{(2n-2)}$, and $Q_2 \sim \chi^2_{(2n'-2)}$;
(3) Calculate $\widehat{R}_B = R(\widehat{\delta}_{1B}, \widehat{\delta}_{2B}, \widehat{\gamma}_{1B}, \widehat{\gamma}_{2B})$, where $\begin{matrix} \widehat{\delta}_{1B} = 2n\widehat{\delta}_{01}/Q_1, \widehat{\gamma}_{1B} = \widehat{\gamma}_{01}\exp(W_1/2n\widehat{\delta}_{01}) \text{ and} \\ \widehat{\delta}_{2B} = 2n'\widehat{\delta}_{02}/Q_2, \widehat{\gamma}_{2B} = \widehat{\gamma}_{02}\exp(W_2/2n'\widehat{\delta}_{02}) \end{matrix}$
are the estimations of the bootstrap sample for $\delta_1, \delta_2, \gamma_1$, and γ_2;
(4) Repeat 2 and 3 steps N times and denote $\widehat{R}_B^1, \ldots, \widehat{R}_B^N$;
(5) Obtain the $100(1-\varepsilon)\%$ BPCI by $(\widehat{R}_B^{(N+1)(\varepsilon/2)}, \widehat{R}_B^{(N+1)(1-\varepsilon/2)})$.
(6) Obtain the $100(1-\varepsilon)\%$ BTCI by $[\widehat{R}_{ml} - T_B^{(1-\varepsilon/2)}(\mathrm{Var}(\widehat{R}_B)^{1/2}), \widehat{R}_{ml} + T_B^{(\varepsilon/2)}(\mathrm{Var}(\widehat{R}_B)^{1/2})]$,
where $\widehat{R}_{ml}$ is the ML estimation of R, $\mathrm{Var}(\widehat{R}_B)$ is the variance of $\widehat{R}_B^1, \ldots, \widehat{R}_B^N$, and $T_B^{(\varepsilon)}$ is the ε^{th} quantile of $\Upsilon_B^i = (\widehat{R}_B^i - \widehat{R})(\mathrm{Var}(\widehat{R}_B))^{-1/2}$.

Algorithm 2: BPCI and BTCI.

$$\begin{aligned} R' &= \left(\frac{\gamma_1}{\max(\gamma_1, \gamma_2)}\right)^{\delta}\left[1 - \frac{1}{2}\left(\frac{\gamma_2}{\max(\gamma_1, \gamma_2)}\right)^{\delta}\right] \\ &= \left[1 - \frac{1}{2}\left(\frac{\gamma_2}{\gamma_1}\right)^{\delta}\right] I(\gamma_1 > \gamma_2) + \left[\frac{1}{2}\left(\frac{\gamma_1}{\gamma_2}\right)^{\delta}\right] I(\gamma_1 < \gamma_2) + \frac{1}{2} I(\gamma_1 = \gamma_2). \end{aligned} \tag{8}$$

2.3.1. Confidence Intervals

Theorem 2. *Let $X_1, \ldots, X_n \sim \mathrm{Pareto}(\delta, \gamma_1)$ and $Y_1, \ldots, Y_{n'} \sim \mathrm{Pareto}(\delta, \gamma_2)$ be independent. Consider $\widehat{\gamma}_1, \widehat{\gamma}_2$ and $\widehat{\delta}$ be the ML estimation of γ_1, γ_2 and δ for Pareto(δ, γ_1) and Pareto(δ, γ_2), then*

(i) $W_1' = 2n\delta\log(\widehat{\gamma}_1/\gamma_1) \sim \chi^2_{(2)}$.
(ii) $W_2' = 2n'\delta\log(\widehat{\gamma}_2/\gamma_2) \sim \chi^2_{(2)}$.
(iii) $Q' = 2(n+n')\delta/\widehat{\delta} \sim \chi^2_{(2(n+n')-2)}$.

Where $\widehat{\gamma}_1 = x_{(1)}, \widehat{\gamma}_2 = y_{(1)}$ and $\widehat{\delta} = [1/n + n'(\sum_{i=1}^{n}\log(x_i/x_{(1)}) + \sum_{j=1}^{n'}\log(y_j/y_{(1)}))]^{-1}$.

Proof. The proof is similar to Theorem 1.

W_1', W_2' and Q' are independent. We suggest the following GPQ for R:

$$R'_{g^*} = R'\left(\delta_{g^*}, \gamma_{1g^*}, \gamma_{2g^*}\right), \tag{9}$$

where

$$\delta_{g^*} = \frac{Q'\widehat{\delta}_0}{2\left(n+n'\right)}, \gamma_{1g^*} = \frac{\widehat{\gamma}_{01}}{\exp\left(W_1'/Q'\widehat{\delta}_0\right)}, \gamma_{2g^*} = \frac{\widehat{\gamma}_{02}}{\exp\left(W_2'/Q'\widehat{\delta}_0\right)}. \tag{10}$$

Similar to Algorithm 1, the GCI for R can be obtained for this case. BPCI and BTCI are obtained by Algorithm 3. □

(1) Given $\widehat{\delta}_0, \widehat{\gamma}_{01}, \widehat{\gamma}_{02}, n$, and n';
(2) Generate $W_1', W_2' \sim \chi^2_{(2)}$, and $Q' \sim \chi^2_{(2(n+n')-2)}$;
(3) Calculate $\widehat{R}_B = R(\widehat{\delta}_B, \widehat{\gamma}_{1B}, \widehat{\gamma}_{2B})$, where $\widehat{\delta}_B = 2(n+n')\widehat{\delta}_0/Q', \widehat{\gamma}_{1B} = \widehat{\gamma}_{01} e^{W_1'/2(n+n')\widehat{\delta}_0}$, and $\widehat{\gamma}_{2B} = \widehat{\gamma}_{02} e^{W_2'/2(n+n')\widehat{\delta}_0}$ are the estimations of the bootstrap sample for δ, γ_1, and γ_2;
(4) Perform steps 4, 5, and 6 in Algorithm 3.

Algorithm 3: BPCI and BTCI for the special case $\delta_1 = \delta_2 = \delta$.

3. ML Estimation of R Based on GOS

Let G and g be continuous CDF and PDF, respectively. If the joint PDF of $X_{(1,n,q,h)}, \ldots, X_{(n,n,q,h)}$ are as follows:

$$g_{X_{(1,n,q,h)},\ldots,X_{(n,n,q,h)}}(x_1, \ldots, x_n) = D_n \prod_{i=1}^{n} [1 - G(x_i)]^{\eta_i - \eta_{i+1} - 1} g(x_i), \tag{11}$$

then $X_{(1,n,q,h)}, \ldots, X_{(n,n,q,h)}$ are GOS, where

$$G^{-1}(0) < x_1 < \ldots < x_{n_1} < G^{-1}(1), \quad G^{-1}(v) = \inf\{x\colon G(x) \geq v\}, \tag{12}$$

is the quantile function of G and

$$D_k = \prod_{i=1}^{k} \eta_i, \quad k = 1, 2, \ldots, n, \qquad \eta_i = l + n - i + \sum_{m=i}^{n-1} q_m > 0, \quad i = 1, \ldots, n, \eta_{n+1} = 0, \tag{13}$$

and $(q_1, \ldots, q_{n-1}) \in \mathbb{R}^{n-1}$.

Let $X_{(1,n,q,h)}, \ldots, X_{(n,n,q,h)}$ be GOS from Pareto(δ, γ) and $\mathbf{x}$ be the observation vector. The likelihood function is obtained as follows:

$$L(\delta, \gamma \mid \mathbf{x}) = D_n \delta^n \gamma^{\delta \sum_{i=1}^{n} (\eta_i - \eta_{i+1})} \cdot \prod_{i=1}^{n} x_i^{-\delta(\eta_i - \eta_{i+1}) - 1} u(x_{(1)} - \gamma). \tag{14}$$

The log-likelihood function is

$$\ell(\delta, \gamma \mid \mathbf{x}) \propto \left(n \log\delta + \delta \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log\gamma - \delta \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log x_i \right) u(x_{(1)} - \gamma), \tag{15}$$

where x_i is the recorded value of the GOS sample and

$$u(t) = \begin{cases} 1, & t \geq 0, \\ 0, & t < 0. \end{cases} \tag{16}$$

So, the ML estimator of γ is $\widehat{\gamma} = X_{(1,n,q,h)}$ and taking the derivative of ℓ relative to δ and putting it equal to zero

$$\frac{\partial \ell}{\partial \delta} = \frac{n}{\delta} + \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log\left(\frac{\gamma}{x_i}\right) = 0, \tag{17}$$

where the ML estimator of δ is obtained by

$$\widehat{\delta} = \left[\frac{1}{n} \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log\left(\frac{X_{(i,n,q,h)}}{X_{(1,n,q,h)}}\right)\right]^{-1}. \tag{18}$$

Now, we obtain the ML estimate of R. Let $X_{(i,n,q,h)} \sim$ Pareto(δ_1, γ_1), $i = 1, 2, \ldots, n$ and $Y_{(j,n',q',h')} \sim$ Pareto(δ_2, γ_2) be GOS such that $X_{(i,n,q,h)}$ and $Y_{(j,n',q',h')}$ are independent. According to invariance property of the ML estimator, the R estimate is given by

$$\widehat{R} = R(\widehat{\delta}_1, \widehat{\delta}_2, \widehat{\gamma}_1, \widehat{\gamma}_2) = \left(\frac{\widehat{\gamma}_1}{\max(\widehat{\gamma}_1, \widehat{\gamma}_2)}\right)^{\widehat{\delta}_1} \left[1 - \left(\frac{\widehat{\delta}_1}{\widehat{\delta}_1 + \widehat{\delta}_2}\right) \left(\frac{\widehat{\gamma}_2}{\max(\widehat{\gamma}_1, \widehat{\gamma}_2)}\right)^{\widehat{\delta}_2}\right], \tag{19}$$

where $\widehat{\gamma}_1 = X_{(1,n,q,h)}$ and $\widehat{\gamma}_2 = Y_{(1,n',q',h')}$,

$$\widehat{\delta}_1 = \left[\frac{1}{n} \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log\left(\frac{X_{(i,n,q,h)}}{X_{(1,n,q,h)}}\right)\right]^{-1}, \qquad \widehat{\delta}_2 = \left[\frac{1}{n'} \sum_{j=1}^{n'} (\eta_j' - \eta_{j+1}') \log\left(\frac{Y_{(j,n',q',h')}}{Y_{(1,n',q',h')}}\right)\right]^{-1}. \tag{20}$$

In the above equations,

$$\eta_i = h + n - i + \sum_{s=i}^{n-1} q_s, \quad i = 2, \ldots, n,$$
$$\eta'_j = h' + n' - j + \sum_{t=j}^{n'-1} q_t, \; j = 2, \ldots, n'. \tag{21}$$

In addition, $q = (q_1, \ldots, q_{n-1})$ and $q' = (q'_1, \ldots, q'_{n'-1})$.

4. Bayesian Estimation of R Based on GOS

We consider the prior distributions of parameters δ and γ independently and propose their density as follows:

$$\varphi(\delta_i) = \zeta_i e^{-\zeta_i(\delta_i - \vartheta_i)} u(\delta_i - \vartheta_i), \quad i = 1, 2,$$
$$\varphi(\gamma_i) = \frac{\xi_i \gamma_i^{\xi_i - 1}}{\varrho_i^{\xi_i}} u(\varrho_i - \gamma_i), \quad i = 1, 2. \tag{22}$$

Let $\mathbf{x}$ be the observation vector, then the joint posterior distribution of δ_1 and γ_1 is

$$\Pi(\delta_1, \gamma_1 \mid \mathbf{x}) \propto L(\delta_1, \gamma_1 \mid \mathbf{x}) \varphi(\delta_1) \varphi(\gamma_1)$$
$$\propto \delta_1^n e^{-\delta_1 \tau_1} \gamma_1^{\delta_1 \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) + \xi_1 - 1} u(\delta_1 - \vartheta_1) u(\epsilon_0 - \gamma_1). \tag{23}$$

We obtain the marginal posterior distribution of γ_1 as follows:

$$\varphi_1(\gamma_1 \mid \mathbf{x}) = \int_{\vartheta_1}^{\infty} \delta_1^n e^{-\delta_1 \tau_1} \gamma_1^{\delta_1 \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) + \xi_1 - 1} u(\epsilon_0 - \gamma_1) d\delta_1$$
$$= \frac{\Gamma(n+1, \tau_* \vartheta_1)}{\tau_1^{n+1}} \gamma_1^{\xi_1 - 1} u(\epsilon_0 - \gamma). \tag{24}$$

Also, the conditional posterior distribution of δ_1 is

$$\varphi_1(\delta_1 \mid \gamma_1, \mathbf{x}) = \frac{\Pi(\delta_1, \gamma_1 \mid \mathbf{x})}{\varphi_1(\gamma_1 \mid \mathbf{x})}$$
$$= \frac{\delta_1^n e^{-\delta_1 \tau_1} \gamma_1^{\delta_1 \sum_{i=1}^{n} (\eta_i - \eta_{i+1})} \tau_*^{n+1} u(\delta_1 - \vartheta_1) u(\varrho_1 - \gamma_1)}{\Gamma(n+1, \tau_* \vartheta_1) u(\epsilon_0 - \gamma_1)}, \tag{25}$$

where

$$\epsilon_0 = \min\left(x_{(1)}, \varrho_1\right),$$
$$\tau_1 = \zeta_1 + \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log x_i,$$
$$\tau_* = \zeta_1 + \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log\left(\frac{x_i}{\epsilon_0}\right). \tag{26}$$

Similarly, let $\mathbf{y}$ be the observation vector. We obtain

$$\varphi_2(\gamma_2 \mid \mathbf{y}) = \int_{\vartheta_2}^{\infty} \delta_2^{n'} e^{-\delta_2 \tau_2} \gamma_2^{\delta_2 \sum_{j=1}^{n'} (\eta'_j - \eta'_{j+1}) + \xi_2 - 1} u(\epsilon_{00} - \gamma_2) d\delta_2$$
$$= \frac{\Gamma\left(n' + 1, \tau_{**} \vartheta_2\right)}{\tau_2^{n'+1}} \gamma_2^{\xi_2 - 1} u(\epsilon_{00} - \gamma_2),$$
$$\varphi_2(\delta_2 \mid \gamma_2, \mathbf{y}) = \frac{\Pi(\delta_2, \gamma_2 \mid \mathbf{y})}{\varphi_2(\gamma_2 \mid \mathbf{y})}$$
$$= \frac{\delta_2^{n'} e^{-\delta_2 \tau_2} \gamma_2^{\delta_2 \sum_{j=1}^{n'} (\eta'_j - \eta'_{j+1})} \tau_{**}^{n'+1} u(\delta_2 - \vartheta_2) u(\varrho_2 - \gamma_2)}{\Gamma\left(n' + 1, \tau_{**} \vartheta_2\right) u(\epsilon_{00} - \gamma_2)}, \tag{27}$$

where

$$\epsilon_{00} = \min\left(y_{(1)}, \varrho_2\right),$$
$$\tau_2 = \zeta_2 + \sum_{j=1}^{n'} \left(\eta'_j - \eta'_{j+1}\right) \log y_i,$$
$$\tau_{**} = \zeta_2 + \sum_{j=1}^{n'} \left(\eta'_j - \eta'_{j+1}\right) \log\left(\frac{y_i}{\epsilon_{00}}\right). \tag{28}$$

We propose Algorithm 4 to obtain the Bayesian confidence interval.

5. Special Case $\boldsymbol{\delta_1 = = \delta_2 = = \delta}$

In this section, we obtain the ML and Bayesian estimators for R of Pareto distribution based on GOS for the special case $\delta_1 = \delta_2 = \delta$.

(1) Generate $\gamma_{11} \sim \varphi_1(\gamma_1 \mid \mathbf{x})$ and $\gamma_{22} \sim \varphi_2(\gamma_2 \mid \mathbf{y})$;
(2) Generate $\delta_{11} \sim \varphi_1(\delta_1 \mid \gamma_1, \mathbf{x})$ and $\delta_{22} \sim \varphi_2(\delta_2 \mid \gamma_2, \mathbf{y})$;
(3) Obtain $R_{\text{Bayes}} = R(\delta_{11}, \delta_{11}, \gamma_{11}, \gamma_{22})$;
(4) Repeat the previous three steps N times and denote $R^1_{\text{Bayes}}, \ldots, R^N_{\text{Bayes}}$;
(5) Sort $\{R^i_{\text{Bayes}}, i = 1, \ldots, N\}$ and say $R^{(1)}_{\text{Bayes}}, \ldots, R^{(N)}_{\text{Bayes}}$;
(6) Compute $100(1-\varepsilon)\%$ Bayesian confidence interval by $(R^{N(\varepsilon/2)}_{\text{Bayes}}, R^{N(1-\varepsilon/2)}_{\text{Bayes}})$.

Algorithm 4: Bayesian confidence intervals.

5.1. ML Estimation. We consider $X_{(i,n,q,h)} \sim \text{Pareto}(\delta, \gamma_1), i = 1, 2, \ldots, n$ and $Y_{(j,n',q',h')} \sim \text{Pareto}(\delta, \gamma_2)$ to be GOS. The likelihood function and log-likelihood function are

$$L(\delta, \gamma_1, \gamma_2) \propto \delta^{n+n'} \gamma_1^{\delta \sum_{i=1}^{n} (\eta_i - \eta_{i+1})} \gamma_2^{\delta \sum_{j=1}^{n'} (\eta'_j - \eta'_{j+1})} \times \prod_{i=1}^{n} x_i^{-\delta(\eta_i - \eta_{i+1})} \prod_{j=1}^{n'} y_j^{-\delta(\eta'_j - \eta'_{j+1})} u(x_{(1)} - \gamma_1) u(y_{(1)} - \gamma_2),$$

$$\ell(\delta, \gamma_1, \gamma_2) = (n + n') \log \delta + \delta \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log \gamma_1 + \delta \sum_{j=1}^{n'} (\eta'_j - \eta'_{j+1}) \log \gamma_2 - \delta \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log x_i - \delta \sum_{j=1}^{n'} (\eta'_j - \eta'_{j+1}) \log y_j. \quad (29)$$

Therefore, $\widehat{\gamma}_1 = X_{(1,n,q,h)}, \widehat{\gamma}_2 = Y_{(1,n',q',h')}$, and

$$\widehat{\delta} = \left[\frac{1}{n+n'} \left(\sum_{i=2}^{n} (\eta_i - \eta_{i+1}) \log \left(\frac{X_{(i,n,q,h)}}{X_{(1,n,q,h)}} \right) + \sum_{j=2}^{n'} (\eta'_j - \eta'_{j+1}) \log \left(\frac{Y_{(j,n',q',h')}}{Y_{(1,n',q',h')}} \right) \right) \right]^{-1}, \quad (30)$$

are the ML estimations of the parameters γ_1, γ_2, and δ.

5.2. Bayesian Estimation. Consider the following prior distributions for parameters δ, γ_1, and γ_2:

$$\varphi(\delta) = \zeta e^{-\zeta\delta}, \delta > 0,$$
$$\varphi(\gamma_i) = \frac{\xi_i \gamma_i^{\xi_i - 1}}{\varrho_i^{\xi_i}} u(\varrho_i - \gamma_i), \quad i = 1, 2. \quad (31)$$

The joint posterior distribution is

$$\Pi(\delta, \gamma_1, \gamma_2 \mid \mathbf{x}, \mathbf{y}) \propto \delta^{n+n'} \gamma_1^{\delta \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) + \xi_1 - 1} \gamma_2^{\delta \sum_{j=1}^{n'} (\eta'_j - \eta'_{j+1}) + \xi_2 - 1} \times e^{-\delta\tau'} u(\delta) u(\epsilon_0 - \gamma_1) u(\epsilon_{00} - \gamma_2), \quad (32)$$

where $\epsilon_0 = \min\{x_{(1)}, \gamma_1\}, \epsilon_{00} = \min\{y_{(1)}, \gamma_2\}$ and

$$\tau' = \zeta + \sum_{i=1}^{n} (\eta_i - \eta_{i+1}) \log x_i + \sum_{j=1}^{n'} (\eta'_j - \eta'_{j+1}) \log y_j. \quad (33)$$

In this case, the R changes as follows:

$$R' = \left[1 - \frac{1}{2} \left(\frac{\gamma_2}{\gamma_1} \right)^{\delta} \right] I(\gamma_1 > \gamma_2) + \left[\frac{1}{2} \left(\frac{\gamma_1}{\gamma_2} \right)^{\delta} \right] I(\gamma_1 < \gamma_2) + \frac{1}{2} I(\gamma_1 = \gamma_2). \quad (34)$$

Based on the squared error loss function, the Bayesian estimator of R is $\widehat{R'} = E(R' \mid \mathbf{x}, \mathbf{y})$. For this purpose, we must obtain $E[(\gamma_2/\gamma_1)^{\delta} \mid \mathbf{x}, \mathbf{y}]$ and $E[(\gamma_1/\gamma_2)^{\delta} \mid \mathbf{x}, \mathbf{y}]$ under the posterior distribution. By considering $\epsilon_* = \min\{\epsilon_0, \epsilon_{00}\}$, we obtain

$$\begin{aligned} E\left[\left(\frac{\gamma_2}{\gamma_1}\right)^{\delta} \mid \mathbf{x}, \mathbf{y}\right] &= \int_0^{\infty}\int_0^{\epsilon_*}\int_{\gamma_2}^{\epsilon_0}\left(\frac{\gamma_2}{\gamma_1}\right)^{\delta}\delta^{n+n'}\gamma_1^{\delta\sum_{i=1}^{n}(\eta_i-\eta_{i+1})+\xi_1-1} \\ &\quad\times\gamma_2^{\delta\sum_{j=1}^{n'}(\eta'_j-\eta'_{j+1})+\xi_2-1}e^{-\delta\tau'}d\gamma_1 d\gamma_2 d\delta \\ &= \int_0^{\infty}\frac{\delta^{n+n'}e^{-\delta\tau'}}{\delta A+\xi_1}\left[\frac{\epsilon_0^{\delta A+\xi_1}\epsilon_*^{\delta C+\xi_2}}{\delta C+\xi_2}-\frac{\epsilon_*^{\delta D+\xi_1+\xi_2}}{\delta D+\xi_1+\xi_2}\right]d\delta \\ &= \int_0^{\infty}\phi(\delta)d\delta, \end{aligned} \tag{35}$$

where $A = \sum_{i=1}^{n}(\eta_i - \eta_{i+1}) - 1$, $C = \sum_{j=1}^{n'}(\eta'_j - \eta'_{j+1}) + 1$, $D = A + C$, and

$$\phi(\delta) = \frac{\delta^{n+n'}e^{-\delta\tau'}}{\delta A+\xi_1}\left[\frac{\epsilon_0^{\delta A+\xi_1}\epsilon_*^{\delta C+\xi_2}}{\delta C+\xi_2}-\frac{\epsilon_*^{\delta D+\xi_1+\xi_2}}{\delta D+\xi_1+\xi_2}\right]d\delta. \tag{36}$$

Integral 49 cannot be solved by analytical methods. To solve this integral, we propose a variable change method. Let $U = 1/1 + \delta$, then

$$\int_0^{\infty}\phi(\delta)d\delta = \int_0^1\phi\left(\frac{1-u}{u}\right)\left(\frac{1}{u^2}\right)du = E\left[\phi\left(\frac{1-u}{u}\right)\left(\frac{1}{u^2}\right)\right], \tag{37}$$

where $U \sim U(0,1)$ is from uniform distribution. We generate M samples form $U(0,1)$. Thus, under the strong law of large numbers,

$$E\left[\left(\frac{\gamma_2}{\gamma_1}\right)^{\delta} \mid \mathbf{x}, \mathbf{y}\right] \approx \frac{1}{M}\sum_{z=1}^{M}\phi\left(\frac{1-u_z}{u_z}\right)\left(\frac{1}{u_z^2}\right). \tag{38}$$

Similarly, we repeat the above steps for $E[(\gamma_1/\gamma_2)^{\delta} \mid \mathbf{x}, \mathbf{y}]$ as follows:

$$\begin{aligned} E\left[\left(\frac{\gamma_1}{\gamma_2}\right)^{\delta} \mid \mathbf{x}, \mathbf{y}\right] &= \int_0^{\infty}\int_0^{\epsilon_*}\int_{\gamma_1}^{\epsilon_{00}}\left(\frac{\gamma_1}{\gamma_2}\right)^{\delta}\delta^{n+n'}\gamma_1^{\delta\sum_{i=1}^{n}(\eta_i-\eta_{i+1})+\xi_1-1} \\ &\quad\times\gamma_2^{\delta\sum_{j=1}^{n'}(\eta'_j-\eta'_{j+1})+\xi_2-1}e^{-\delta\tau'}d\gamma_2 d\gamma_1 d\delta \\ &= \int_0^{\infty}\frac{\delta^{n+n'}e^{-\delta\tau'}}{\delta A^*+\xi_2}\left[\frac{\epsilon_{00}^{\delta A^*+\xi_2}\epsilon_*^{\delta C^*+\xi_1}}{\delta C^*+\xi_1}-\frac{\epsilon_*^{\delta D^*+\xi_1+\xi_2}}{\delta D^*+\xi_1+\xi_2}\right]d\delta \\ &= \int_0^{\infty}\phi^*(\delta)d\delta \\ &\approx \frac{1}{M}\sum_{z=1}^{M}\phi^*\left(\frac{1-u_z}{u_z}\right)\left(\frac{1}{u_z^2}\right), \end{aligned} \tag{39}$$

where $A^* = \sum_{j=1}^{n'}(\eta'_j - \eta'_{j+1}) - 1$, $C^* = \sum_{i=1}^{n}(\eta_i - \eta_{i+1}) + 1$, $D^* = A^* + C^*$, and

$$\phi^*(\delta) = \frac{\delta^{n+n'}e^{-\delta\tau'}}{\delta A^*+\xi_2}\left[\frac{\epsilon_{00}^{\delta A^*+\xi_2}\epsilon_*^{\delta C^*+\xi_1}}{\delta C^*+\xi_1}-\frac{\epsilon_*^{\delta D^*+\xi_1+\xi_2}}{\delta D^*+\xi_1+\xi_2}\right]. \tag{40}$$

6. Simulation

The Monte Carlo simulation is used to compare GCI, BPCI, and BTCI of R for Pareto distribution and the specific case of Pareto distribution. For this purpose, samples of Pareto distribution with different sample sizes (n, n') and different values of $R = 0.148$ $(\delta_1 = 3, \delta_2 = 3, \gamma_1 = 2, \gamma_2 = 3)$, $R = 0.852$ $(\delta_1 = 3, \delta_2 = 3, \gamma_1 = 3, \gamma_2 = 2)$, and $R = 0.600$ $(\delta_1 = 2, \delta_2 = 3, \gamma_1 = 3, \gamma_2 = 3)$ and also for the special case, samples with different values of $R = 0.222$ $(\delta = 2, \gamma_1 = 2, \gamma_2 = 3)$, $R = 0.987$ $(\delta = 4, \gamma_1 = 5, \gamma_2 = 2)$, and $R= 0.500$ $(\delta_1 = 3, \gamma_1 = 2, \gamma_2 = 2)$ are generated. The length (L) and coverage probability (CP) of these confidence intervals for Pareto distribution and its specific case are summarized in Tables 1 and 2, respectively. Based on these two tables, the CPs of GCI are approximately equal to 0.95, the CPs of BPCI are less than 0.95, and the CPs of BTCI are greater than 0.95. For BPCI and BTCI, in most cases, with the increase of (n, n'), the CPs approach 0.95. We can conclude GCI is better than BPCI and BTCI. Also, with increasing sample size (n, n'), the L of all confidence intervals has decreased.

We also compare the ML and Bayesian confidence intervals of R based on GOS for Pareto distribution and its specific case. Consider $(n, n') \in \{(10,10), (10,15), (15,10), (20,20)\}$ and the number of repetitions is 10,000. To generate a GOS sample, we perform the algorithm proposed in [22]. The random samples of Pareto distribution with parameters

TABLE 1: L and CP of generalized, bootstrap percentile, and bootstrap-t confidence intervals (95%) of R for Pareto distribution for $R = 0.148\,(\delta_1 = 3, \delta_2 = 3, \gamma_1 = 2, \gamma_2 = 3)$, $R = 0.852\,(\delta_1 = 3, \delta_2 = 3, \gamma_1 = 3, \gamma_2 = 2)$, and $R = 0.6\,(\delta_1 = 2, \delta_2 = 3, \gamma_1 = 3, \gamma_2 = 3)$.

R	n, n'	GCI		BPCI		BTCI	
		L	CP	L	CP	L	CP
	5, 5	0.540	0.957	0.397	0.910	0.233	0.969
	5, 10	0.535	0.959	0.351	0.877	0.219	0.679
	10, 5	0.534	0.951	0.313	0.943	0.216	0.970
	10, 10	0.528	0.950	0.310	0.936	0.201	0.968
0.148	10, 15	0.526	0.952	0.285	0.920	0.200	0.961
	15, 15	0.519	0.952	0.280	0.935	0.162	0.974
	15, 20	0.507	0.951	0.272	0.941	0.162	0.991
	20, 15	0.503	0.950	0.271	0.944	0.159	0.960
	20, 20	0.414	0.952	0.268	0.944	0.158	0.959
	5, 5	0.508	0.950	0.461	0.942	0.211	0.967
	5, 10	0.506	0.951	0.448	0.941	0.251	0.985
	10, 5	0.476	0.952	0.359	0.938	0.304	0.984
	10, 10	0.470	0.955	0.347	0.945	0.258	0.963
0.852	10, 15	0.471	0.952	0.337	0.941	0.264	0.965
	15, 15	0.423	0.957	0.331	0.943	0.314	0.968
	15, 20	0.401	0.954	0.326	0.947	0.410	0.968
	20, 15	0.324	0.952	0.235	0.935	0.271	0.960
	20, 20	0.299	0.956	0.225	0.943	0.221	0.962
	5, 5	0.406	0.950	0.342	0.933	0.333	0.964
	5, 10	0.402	0.953	0.297	0.940	0.312	0.959
	10, 5	0.367	0.954	0.294	0.944	0.312	0.963
	10, 10	0.342	0.954	0.238	0.947	0.312	0.970
0.600	10, 15	0.344	0.952	0.226	0.948	0.297	0.975
	15, 15	0.268	0.958	0.215	0.941	0.288	0.972
	15, 20	0.265	0.953	0.209	0.930	0.282	0.981
	20, 15	0.265	0.957	0.174	0.947	0.263	0.971
	20, 20	0.237	0.954	0.165	0.944	0.261	0.974

TABLE 2: L and CP of generalized, bootstrap percentile, and bootstrap-t confidence intervals (95%) of R in the special case of Pareto distribution for $R = 0.222\,(\delta = 2, \gamma_1 = 2, \gamma_2 = 3)$, $R = 0.987\,(\delta = 4, \gamma_1 = 5, \gamma_2 = 2)$, and $R = 0.5\,(\delta = 3, \gamma_1 = 2, \gamma_2 = 2)$.

R	n, n'	GCI		BPCI		BTCI	
		L	CP	L	CP	L	CP
	5, 5	0.620	0.952	0.322	0.888	0.391	0.958
	5, 10	0.554	0.959	0.315	0.914	0.239	0.964
	10, 5	0.496	0.951	0.275	0.924	0.231	0.961
	10, 10	0.409	0.954	0.273	0.927	0.214	0.967
0.222	10, 15	0.363	0.959	0.273	0.907	0.180	0.968
	15, 15	0.323	0.950	0.260	0.913	0.380	0.962
	15, 20	0.311	0.951	0.260	0.938	0.376	0.973
	20, 15	0.291	0.955	0.260	0.927	0.270	0.965
	20, 20	0.297	0.953	0.252	0.933	0.164	0.976
	5, 5	0.572	0.958	0.551	0.945	0.309	0.971
	5, 10	0.570	0.954	0.551	0.948	0.304	0.961
	10, 5	0.572	0.955	0.551	0.944	0.229	0.969
	10, 10	0.562	0.957	0.550	0.942	0.215	0.969
0.987	10, 15	0.554	0.956	0.549	0.939	0.214	0.976
	15, 15	0.554	0.950	0.450	0.944	0.209	0.972
	15, 20	0.551	0.959	0.449	0.945	0.203	0.961
	20, 15	0.551	0.958	0.449	0.940	0.201	0.961
	20, 20	0.548	0.956	0.449	0.945	0.108	0.961
	5, 5	0.637	0.949	0.733	0.940	0.328	0.963
	5, 10	0.579	0.956	0.713	0.946	0.328	0.970
	10, 5	0.567	0.953	0.601	0.926	0.304	0.974
	10, 10	0.514	0.959	0.595	0.942	0.290	0.978
0.500	10, 15	0.510	0.955	0.596	0.941	0.269	0.972
	15, 15	0.442	0.958	0.556	0.945	0.214	0.969
	15, 20	0.420	0.949	0.549	0.936	0.158	0.961
	20, 15	0.410	0.950	0.539	0.936	0.145	0.976
	20, 20	0.379	0.957	0.530	0.944	0.112	0.959

(1) Generate independently $\varpi_1, \ldots, \varpi_s \sim U(0,1)$;
(2) Set $Z_i = \varpi_i^{1/\eta_i}, i = 1, \ldots, s$, where $\eta_i = l + n - i + \sum_{m=i}^{n-1} q_m$;
(3) Obtain $E_s^* = 1 - \prod_{i=1}^{s} Z_i$, where E_s^* is the s^{th} uniform GOS;
(4) Calculate $X_s' = F^{-1}(E_s^*), s = 1, \ldots, n$, where X_s' is the s^{th} GOS based on F.

Algorithm 5: Generate the GOS sample.

Table 3: Bias, MSE, L, and CP of ML and Bayesian confidence interval based on GOS for Pareto distribution ($\zeta_1 = 2, \zeta_2 = 3, \vartheta_1 = 0, \vartheta_2 = 0, \varrho_1 = 3, \varrho_2 = 5, \xi_1 = 2$, and $\xi_2 = 2$).

R	n, n'		OOS		FR		PTII	
			MLE	Bayes	MLE	Bayes	MLE	Bayes
0.148	10, 10	Bias	0.077	0.148	0.116	0.049	−0.022	0.158
		MSE	0.013	0.025	0.143	0.007	0.004	0.028
		L	0.352	0.390	0.614	0.348	0.395	0.391
		CP	0.966	0.950	0.964	0.951	0.950	0.950
	10, 15	Bias	0.033	0.117	−0.129	0.046	−0.053	0.116
		MSE	0.006	0.015	0.026	0.006	0.005	0.015
		L	0.290	0.320	0.586	0.344	0.330	0.334
		CP	0.959	0.950	0.954	0.950	0.950	0.950
	15, 10	Bias	0.146	0.166	0.700	0.031	0.037	0.188
		MSE	0.028	0.030	0.579	0.005	0.005	0.039
		L	0.215	0.318	0.525	0.333	0.313	0.324
		CP	0.967	0.951	0.967	0.950	0.954	0.950
	20, 20	Bias	0.104	0.131	−0.091	−0.527	0.002	0.137
		MSE	0.014	0.018	0.134	0.282	0.002	0.020
		L	0.128	0.308	0.205	0.310	0.274	0.314
		CP	0.975	0.950	0.914	0.950	0.975	0.950
0.83	10, 10	Bias	−0.078	−0.421	−0.097	−0.525	0.020	−0.153
		MSE	0.013	0.187	0.143	0.279	0.005	0.028
		L	0.841	0.549	0.577	0.376	0.502	0.726
		CP	0.824	0.953	0.750	0.953	0.951	0.958
	10, 15	Bias	−0.154	−0.482	−0.682	−0.539	−0.045	−0.229
		MSE	0.031	0.237	0.551	0.293	0.006	0.059
		L	0.776	0.458	0.441	0.355	0.501	0.675
		CP	0.952	0.951	0.941	0.952	0.913	0.957
	15, 10	Bias	−0.033	−0.413	0.155	−0.520	0.060	−0.137
		MSE	0.007	0.180	0.027	0.273	0.007	0.022
		L	0.760	0.446	0.424	0.355	0.419	0.632
		CP	0.927	0.955	0.919	0.956	0.969	0.951
	20, 20	Bias	−0.103	−0.499	0.002	−0.208	−0.002	−0.282
		MSE	0.014	0.251	0.171	0.047	0.002	0.085
		L	0.752	0.387	0.424	0.316	0.357	0.631
		CP	0.897	0.951	0.942	0.953	0.953	0.954
0.5	10, 10	Bias	0.003	−0.103	0.017	−0.205	−0.005	−0.008
		MSE	0.015	0.020	0.175	0.046	0.014	0.010
		L	0.644	0.535	0.319	0.379	0.645	0.620
		CP	0.932	0.952	0.817	0.952	0.951	0.954
	10, 15	Bias	−0.091	−0.164	−0.444	−0.222	−0.002	−0.006
		MSE	0.020	0.031	0.232	0.053	0.015	0.012
		L	0.562	0.449	0.205	0.350	0.510	0.539
		CP	0.885	0.958	0.962	0.957	0.785	0.956
	15, 10	Bias	0.087	−0.085	0.446	−0.192	−0.124	−0.098
		MSE	0.019	0.016	0.230	0.040	0.025	0.018
		L	0.514	0.446	0.202	0.349	0.435	0.486
		CP	0.967	0.953	0.948	0.956	0.972	0.951
	20, 20	Bias	−0.002	−0.167	0.454	−0.191	−0.093	−0.078
		MSE	0.006	0.029	0.233	0.040	0.016	0.016
		L	0.509	0.387	0.135	0.319	0.389	0.395
		CP	0.959	0.952	0.933	0.952	0.943	0.951

(1) **Require:** Suppose we want to generate a sample of a PDF f that does not have a simple form,Generate $U^* \sim U(0,1)$;
(2) Generate $y^* \sim g(y^*)$, where $g(y^*)$ is a PDF that is close to f and it is easy to sample;
(3) Obtain $C_m = \max\{f(x)/g(x)\}$;
(4) If $U^* \le f(y^*)/C_m g(y^*)$, then select y^* as the sample, else go to the first step;
(5) Repeat the above steps until reach the desired number of samples.

Algorithm 6: Acceptance-rejection method.

Table 4: Bias, MSE, L, and CP of ML and Bayesian confidence interval based on GOS for Pareto distribution ($\zeta_1 = 2, \zeta_2 = 2, \vartheta_1 = 1, \vartheta_2 = 1, \varrho_1 = 4, \varrho_2 = 4, \xi_1 = 3$, and $\xi_2 = 3$).

R	n, n'		OOS		FR		PTII	
			MLE	Bayes	MLE	Bayes	MLE	Bayes
0.26	10, 10	Bias	0.218	0.217	0.533	0.275	0.067	0.133
		MSE	0.064	0.051	0.392	0.085	0.014	0.021
		L	0.571	0.553	0.472	0.462	0.470	0.475
		CP	0.964	0.950	0.879	0.955	0.968	0.953
	10, 15	Bias	0.157	0.195	0.018	0.274	0.003	0.113
		MSE	0.037	0.041	0.145	0.085	0.007	0.015
		L	0.559	0.527	0.432	0.461	0.376	0.431
		CP	0.972	0.951	0.773	0.953	0.947	0.952
	15, 10	Bias	0.330	0.255	0.728	0.284	0.192	0.197
		MSE	0.120	0.067	0.532	0.091	0.045	0.042
		L	0.535	0.519	0.430	0.376	0.304	0.432
		CP	0.951	0.950	0.871	0.952	0.961	0.950
	20, 20	Bias	0.283	0.237	0.327	−0.053	0.143	0.200
		MSE	0.087	0.058	0.148	0.013	0.025	0.042
		L	0.444	0.436	0.430	0.320	0.293	0.316
		CP	0.950	0.950	0.947	0.947	0.958	0.950
0.60	10, 10	Bias	0.140	−0.082	0.308	−0.051	0.144	−0.063
		MSE	0.031	0.009	0.147	0.013	0.030	0.007
		L	0.453	0.570	0.542	0.665	0.347	0.590
		CP	0.975	0.952	0.835	0.958	0.972	0.951
	10, 15	Bias	0.079	−0.101	−0.101	−0.046	0.046	−0.090
		MSE	0.016	0.012	0.194	0.012	0.013	0.010
		L	0.382	0.543	0.386	0.663	0.324	0.552
		CP	0.967	0.952	0.877	0.955	0.961	0.953
	15, 10	Bias	0.211	−0.059	0.398	−0.038	0.235	−0.041
		MSE	0.051	0.005	0.159	0.011	0.060	0.004
		L	0.373	0.532	0.335	0.620	0.288	0.505
		CP	0.959	0.955	0.880	0.950	0.951	0.954
	20, 20	Bias	0.163	−0.087	0.089	−0.317	0.182	−0.070
		MSE	0.031	0.008	0.025	0.112	0.037	0.006
		L	0.317	0.532	0.326	0.552	0.230	0.456
		CP	0.959	0.955	0.887	0.959	0.953	0.957
0.88	10, 10	Bias	0.001	−0.255	0.083	−0.313	0.053	−0.175
		MSE	0.003	0.069	0.026	0.109	0.005	0.032
		L	0.417	0.481	0.476	0.395	0.239	0.325
		CP	0.947	0.953	0.867	0.950	0.968	0.950
	10, 15	Bias	−0.035	−0.297	−0.239	−0.313	0.019	−0.202
		MSE	0.005	0.091	0.217	0.108	0.002	0.043
		L	0.390	0.433	0.426	0.390	0.235	0.323
		CP	0.940	0.952	0.884	0.951	0.964	0.951
	15, 10	Bias	0.033	−0.237	0.118	−0.300	0.076	−0.159
		MSE	0.003	0.059	0.014	0.101	0.007	0.026
		L	0.327	0.430	0.426	0.391	0.234	0.316
		CP	0.959	0.951	0.898	0.953	0.975	0.950
	20, 20	Bias	−0.001	−0.307	0.118	−0.297	0.054	−0.196
		MSE	0.002	0.096	0.014	0.098	0.004	0.039
		L	0.321	0.417	0.421	0.311	0.225	0.315
		CP	0.958	0.954	0.821	0.950	0.974	0.951

Table 5: Bias, MSE, L, and CP of ML and Bayesian confidence interval based on GOS in the special case of Pareto distribution ($\zeta = 1, \vartheta = 0, \varrho_1 = 5, \varrho_2 = 5, \xi_1 = 1$ and $\xi_2 = 1$).

R	n, n'		OOS		FR		PTII	
			MLE	Bayes	MLE	Bayes	MLE	Bayes
0.75	10, 10	Bias	−0.074	0.250	−0.120	0.250	0.015	0.250
		MSE	0.018	0.062	0.185	0.063	0.008	0.062
		L	0.589	0.550	0.562	0.451	0.439	0.550
		CP	0.896	0.950	0.961	0.951	0.962	0.962
	10, 15	Bias	−0.136	0.250	−0.647	0.248	−0.067	0.250
		MSE	0.032	0.062	0.484	0.062	0.015	0.062
		L	0.543	0.550	0.491	0.451	0.384	0.551
		CP	0.812	0.951	0.665	0.951	0.899	0.950
	15, 10	Bias	−0.033	0.250	0.223	0.247	0.064	0.250
		MSE	0.010	0.062	0.067	0.161	0.009	0.062
		L	0.515	0.520	0.465	0.450	0.358	0.549
		CP	0.929	0.959	0.972	0.950	0.965	0.951
	20, 20	Bias	−0.093	0.250	−0.151	−0.357	−0.007	0.250
		MSE	0.016	0.062	0.219	0.127	0.006	0.062
		L	0.447	0.505	0.464	0.406	0.348	0.548
		CP	0.836	0.950	0.965	0.950	0.948	0.955

Table 6: Rolling contact fatigue data for two steel compositions 10^6 stress cycles.

X	3.19	4.26	4.47	4.53	4.67	4.69	5.78	6.79	9.37	12.75
Y	3.46	5.22	5.69	6.54	9.16	9.40	10.19	10.71	12.58	13.41

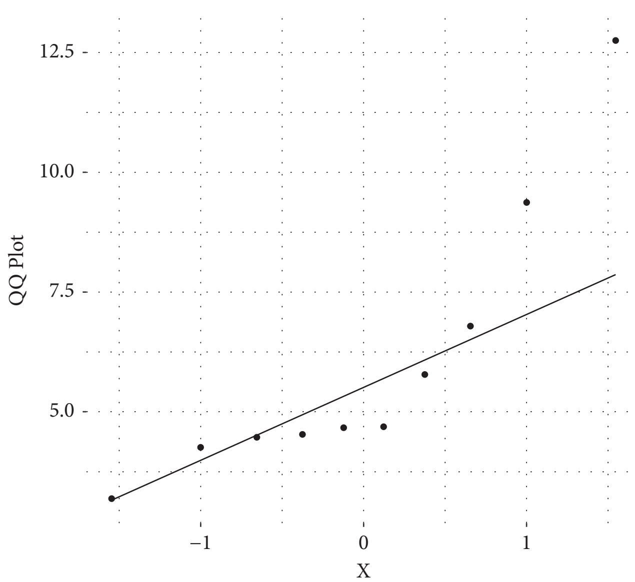

Figure 1: QQ plot for X.

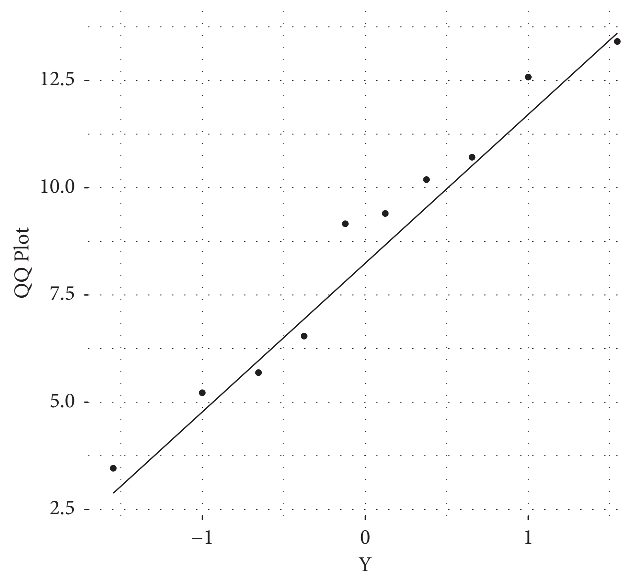

FIGURE 2: QQ plot for Y.

$(\boldsymbol{\delta}, \boldsymbol{\gamma}) \in \{(\delta_1 = 3, \gamma_1 = 2, \delta_2 = 3, \gamma_2 = 3), (\delta_1 = 1, \gamma_1 = 3, \delta_2 = 1, \gamma_2 = 1), (\delta_1 = 1, \gamma_1 = 1, \delta_2 = 1, \gamma_2 = 1)\}$ are generated using Algorithm 5.

The Bayesian confidence interval of R is obtained by Algorithm 4. According to steps 1 and 2 of this algorithm, we need to generate samples from $\varphi_1(\gamma_1 \mid \mathbf{x}), \varphi_2(\gamma_2 \mid \mathbf{y}), \varphi_1(\delta_1 \mid \gamma_1, \mathbf{x})$, and $\varphi_2(\delta_2 \mid \gamma_2, \mathbf{y})$. As can be seen from 31, 32, 36, and 37, these density functions do not have a simple form. To generate samples of these density functions, we propose the Algorithm 6. The values of the hyperparameters are considered $\zeta_1 = 2, \zeta_2 = 3, \vartheta_1 = 0, \vartheta_2 = 0, \varrho_1 = 3, \varrho_2 = 5, \xi_1 = 2$ and $\xi_2 = 2$. Finally, $\widehat{R}$, L, and CP of R by using the ML and Bayesian methods for Pareto distribution are given in Table 3.

The abovementioned steps are repeated with different values of parameters $(\boldsymbol{\delta}, \boldsymbol{\gamma}) \in \{(\delta_1 = 2, \gamma_1 = 2, \delta_2 = 3, \gamma_2 = 3), (\delta_1 = 2, \gamma_1 = 3, \delta_2 = 3, \gamma_2 = 3), (\delta_1 = 2, \gamma_1 = 3, \delta_2 = 3, \gamma_2 = 2)\}$ and different values of hyperparameters $\zeta_1 = 2, \zeta_2 = 2, \vartheta_1 = 1, \vartheta_2 = 1, \varrho_1 = 4, \varrho_2 = 4, \xi_1 = 3$ and $\xi_2 = 3$, and the results are summarized in Table 4. For the special case of Pareto distribution, we produce a sample with parameters $\delta = 1, \gamma_1 = 2, \gamma_2 = 1\,(R = 0.75)$. We consider the hyperparameters $\zeta = 1, \vartheta = 0, \varrho_1 = 5, \varrho_2 = 5, \xi_1 = 1$ and $\xi_2 = 1$ for the Bayesian method and report the results in Table 5. As mentioned earlier, GOS includes many special cases. We consider three cases ordinary order statistics (OOS) with $l = 1, q_i = 0, \eta_i = n - i + 1$, first n record values (FR) with $l = 1, q_i = -1, \eta_i = 1$, and progressively type-II (PTII) with $l = h + 1, q_i = 0, \eta_i = n + h - i + 1$. From Tables 3–5, it can be concluded that for OOS, FR, and PTII, the CP values of the Bayesian method are almost equal to 0.95 but the ML method is far from 0.95, which in most cases approaches 0.95 with the increase of the sample sizes. The L of confidence intervals decreases with increasing (n, n') in both methods for OOS, FR, and PTII.

7. Application

In this section, we use the rolling contact fatigue data for two steel compositions 10^6 stress cycles [10]. These data are reported in Table 6. The Kolmogorov–Smirnov test shows that $X \sim \text{Pareto}(1.79, 3.19)\,(D = 0.3045, p\text{–value} = 0.3125)$ and $Y \sim \text{Pareto}(1.19, 3.46)\,(D = 0.2817, p\text{–value} = 0.4062)$. Also, Figures 1 and 2 show the QQ plots for these data.

For these data, $\widehat{R} = 0.34$ and GCI = $(0.2420, 0.6505)$, BPCI = $(0.1120, 0.5277)$, and BTCI = $(0.1278, 0.6868)$. The estimators of R for OOS, FR, and PTII are 0.343, 0.417, and 0.356, respectively. We can conclude that the estimators for OOS, FR, and PTII are close to the $\widehat{R}$.

8. Conclusion

This paper investigated classical and Bayesian stress-strength reliability estimators based on GOS for Pareto distribution for the first time. It was proposed to calculate generalized confidence intervals and bootstrap Algorithms 1 and 2. Then, the ML estimate of R was obtained based on GOS. To calculate the Bayesian confidence interval, Algorithm 4 was presented due to the complexity of the posterior distribution. In addition, classical and Bayesian inference was performed for a specific case of this model ($\delta_1 = \delta_2 = \delta$). In this case, for Bayesian estimation, we encountered a complex integral that could not be solved analytically and we proposed a change of variable method to solve this integral. In the simulation part, we considered three specific GOS modes including OOS, FR, and PTII and concluded that the CP values of the Bayesian method are approximately equal to 0.95. As the sample size increases, the CP values of the ML method approach 0.95 and the L values decrease in all confidence intervals.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest.

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