Research Article

n -th Order Sensor Output to Control k -DoF Serial Robot Arms

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Currently, zero-order sensors are commonly used as positioning feedback for the closed-loop control in robotics; thus, in order to expand robots' control alternatives, other paths in sensing should be investigated more deeply. Conditions under which the n -th order sensor output can be used to control k -DoF serial robot arms are formally studied in this work. In obtaining the mentioned control conditions, the Pickard-Lindeloff theorem has been used to prove the existence and uniqueness of the robot's mathematical model solution with n order sensory systems included. To verify that the given conditions and claims guarantee controllability for both continuous-based and variable structure-based systems, two types of control strategies are used in obtaining simulation results: the conventional PID control and a second-order Sliding Mode control.

1. Introduction

Currently, sensors of order zero are the most extensively used positioning detectors in closed-loop control systems in industrial robotics. As a matter of fact, nowadays, robotic systems nearly exclusively use incremental and absolute encoders, resolvers, and high precision potentiometers. This is due to the fact that robotic controllers require delay-free information to achieve the desired control goals: accuracy, precision, repetitiveness, avoidance of tracking error, reduction of speed error, and so on.

An extended paradigm considers that positioning sensors of higher order than zero are not reliable at all. This consideration demotivates engineers for designing sensors of order $n > 0$. Nevertheless, the authors of this work consider that the theme should be sufficiently investigated to eliminate restrictive considerations which could lead in the future, to improve sensor systems in robotics.

In this work, the performance of a k -DoF serial robot arm with dynamical inclusion of linear n -order sensors is investigated, showing that robot's properties with linear n -order sensors inclusion are invariant with respect to robot's theo-

retical dynamics, provided that the solutions of the considered linear n -order sensors exist and are unique.

Interest in studying the use of higher-order sensor devices in the robotic industry has increased due to the need to implement more efficient, faster, and optimal control systems in both types of sensor systems concentrated in a robotic arm or systems with distributed sensors connected in a network and which may also be subject to adverse environmental conditions. However, the use of higher-order sensor systems inevitably introduces time delays in robotic control systems, including those delays from the controller to the actuator and the delays from the sensor to the controller.

On the other hand, some control methodologies have been proposed, ranging from sliding mode and adaptive techniques to manage uncertainties related to the knowledge of external dynamics and disturbances [1, 2]. However, these approaches always work well in lag-free environments.

To try to compensate for these delays, low pass filter-based mechanisms have been implemented to obtain high-quality signals and be able to build control loops for control mechanisms [3].

Mathematical scaffolding to ensure unique solutions in robotic device control algorithms including higher-order sensor systems is not widely studied.

Even more, alternative solutions to deal with nonlinear systems with unmeasurable states have been proposed, in [4] where an adaptive fuzzy output-feedback control scheme is designed and in [5] where a fuzzy logic system-based switched observer is constructed to approximate the unmeasurable states.

1.1. Motivation. Sensors can be designed using a very wide variety of physical principles (e.g., inductive, capacitive); however, many phenomena must be modelled using the n -order differential equations, making them unreliable in the world of robotics.

Motivated by the fact that currently sensory systems of order $n > 0$ are not included either in industrial or in other areas of robotics, the authors are interested in expanding the study of the control of serial mechanisms using sensors that may have the potential to improve in some way the current ones.

As such, the first question is: under which conditions a serial robot mechanism can be controlled by positioning sensors of order different from zero?

If this first question is satisfactorily solved, a second one comes to mind: how can we take advantage of the sensor output for control purposes?

1.2. Contribution of This Work. The main contribution of this work is to study conditions to make possible the control of serial robot arms, taking as the feedback for the closed-loop control the output of the n -th order sensory systems. Also, a technique to obtain the mentioned feedback signal from the sensor's output is proposed.

As formal proofs for claims presented in the text are provided, the methodology of this work is rigorous.

As a proof of concept, an application example is focused on the trajectory tracking control problem (for simplicity); however, using the first time derivative of the position, the robot's joint speed control follows the same rules given herein.

1.3. Problem Statement. Figure 1 shows a block control diagram with a negative feedback summing a reference signal (Ref) to get the tracking error (e) as the input for the control block which is the controller for the system Robot-Sensors. In Figure 1, τ is the control torque, q represents the actual robot's position identified by q^* (i.e., $q^* \rightarrow q$), while $\tilde{q}$ represents the sensor's output. If sensors are of order zero, i.e., $\bar{\beta} \tilde{q} = q$, where matrix $\bar{\beta}^{-1}$ includes the sensor's parameters of sensitiveness, the feedback of states can be taken directly from the sensor's output. Nevertheless, sensors of order n can be designed and/or included to control robotic systems. In such a case, the problem consists of recovering q from the information $\tilde{q}$ provided by sensors, using a processing method for this task. In the following section, this problem is addressed formally.

The remainder of the document is organized as follows: in Section 2, conditions and assumptions to control serial

robots using the output of n order sensory systems are given, and formal proofs for the given claims are presented; in Section 3 an implementation example for a 3-DoF serial robot arm is performed, along with simulation results; and finally, concluding remarks are given in Section 4.

2. Robot's Model with Sensor System Inclusion

Let us consider the dynamic differential equation

$$\bar{\beta}_n \frac{d^n}{dt^n} \tilde{q} + \dots + \bar{\beta}_2 \frac{d^2}{dt^2} \tilde{q} + \bar{\beta}_1 \frac{d}{dt} \tilde{q} + \bar{\beta}_0 \tilde{q} = q - F(x, y, z, t), \quad (1)$$

where $\bar{\beta}_i \in \mathbb{R}^{k \times k}$ with $i = 0, 1, \dots, n$ represents the functional diagonal matrices, i.e., each element of $\bar{\beta}$ denoted as $\beta_{\text{line} \times \text{column}} = 0$ for all line $\neq$ column; $\tilde{q} \in \mathbb{R}^k$ is the sensor's response vector; $q \in \mathbb{R}^k$ is the vector for the actual positions of robot joints; $F(x, y, z, t) \in \mathbb{R}^k$ represents a position and/or time dependent possible function; and k is the number of DoF of the robot.

For the case of linear sensors, (1) becomes

$$\sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} = q. \quad (2)$$

Now, let us consider the following nonlinear robot dynamics

$$\dot{x} = f(t, x), \text{ with } x(t_0) = x_0, \quad (3)$$

where vector x represents the states of the electromechanical plant, i.e., $x = [\chi_1, \chi_2]^T$; vector $f(t, x)$ is a time and/or states-possible dependent function; and x_0 is the initial condition value at $t = t_0$.

The inclusion of (2) in (3) is studied in this work, in order to obtain control conditions, according to the following claim:

Claim 1. It is possible the control of serial robots using positioning sensors of the n -th order for the closed-loop feedback of the states if the mathematical structure of the solution $\tilde{q}$ of the differential equations that model sensor's behaviour (2) exists and is unique for all $t \geq t_0$, under the following assumptions:

Assumption 1. Sensors provide the dynamic solution of (2) for all $t \geq t_0$, i.e., it is assumed that the value for the signal $\tilde{q}$ is available to be used at time $t \geq t_0$ by the closed-loop control system.

Assumption 2. It is assumed that the diagonal matrices $\bar{\beta}_i$ with $i = 1, \dots, n$ in (2) are known.

2.1. Existence and Uniqueness. Herein, the existence and uniqueness of the solution for the robot's dynamics are

FIGURE 1: Block diagram of the closed-loop control with sensors inclusion.

verified for the case in which the solution of the sensor's dynamics exists and is unique.

Definition 1 (Lipschitz). The function $f : D \subset \mathbb{R}^{r+1} \mapsto \mathbb{R}^r$ is locally Lipschitz if it is continuous, and for all subset $\kappa \subset D$ compact exists $L \in \mathbb{R}$ such that if (x, t) and $(y, t) \in \kappa$, then,

$$\|f(x, t) - f(y, t)\| \leq L \|x - y\|, \quad (4)$$

where L is called the Lipschitz constant. (See [6] among many others.)

Theorem 1 (Picard-Lindeloff approach). For function $f : D \subset \mathbb{R}^{r+1} \mapsto \mathbb{R}^r$ locally Lipschitz, and also for the set

$$R = \{(t, x) : t_0 \leq t \leq t_0 + a, |x - x_0| \leq b\} \subset D, \quad (5)$$

let H be a bound of $\|f(t, x)\|$ in R , L the Lipschitz constant of $f(t, x)$ in R , and $\alpha = \min \{a, b/H, 1/L\}$. Then, problem (6) with initial conditions has a solution, and it is unique in the interval $[t_0, t_0 + \alpha]$ (in the Picard-Lindeloff approach).

$$\dot{\chi}_1 = f_1(t, \chi_1, \chi_2),$$

$$\dot{\chi}_2 = f_2(t, \chi_1, \chi_2),$$

$$\chi_1(t_0) = (\chi_1)_0, \chi_2(t_0) = (\chi_2)_0, \quad (6)$$

where $f = [f_1, f_2]^T$, $x = [\chi_1, \chi_2]^T$, and $(\chi_1)_0$ and $(\chi_2)_0$ are the initial values for position and velocity, respectively, at $t = t_0$.

Proof. It is assumed that $\tilde{q}$ exists, then, also the set

$$\mathfrak{F}_0 = \left\{ \tilde{q}(t) \in \mathfrak{C}([t_0, t_0 + \alpha], \mathbb{R}^k) : \tilde{q}(t_0) = x_0, \|\tilde{q}(t) - x_0\| \leq b_0 \right\}, \quad (7)$$

exists, where $\mathfrak{C}([t_0, t_0 + \alpha], \mathbb{R}^k)$ is a Banach space with uniform norm

$$\|\tilde{q}\| = \sup_{t_0 \leq t \leq t_0 + \alpha} (\tilde{q}(t)), \quad (8)$$

and $\mathfrak{F}_0$ is closed in $\mathfrak{C}$.

According to (2) and control concept expressed in Figure 1, the identification process of q is

$$q^* = \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} + \varepsilon, \quad (9)$$

where $q^* \in \mathbb{R}^k$ is the result of processing $\tilde{q}$ and $\varepsilon \in \mathbb{R}^k$ is an identification error given by the Euclidian norm $|q^* - q|$. Observe that $q^*(t_0) = \bar{\beta}_0 \tilde{q}(t_0) + \varepsilon$, for simplicity $\bar{\beta}_0 = I$ and $\varepsilon \rightarrow 0$, then, $q^*(t_0) \rightarrow x_0$, and the problem is reduced to the existence of the set

$$\mathfrak{F} = \left\{ q^*(t) \in \mathfrak{C}([t_0, t_0 + \alpha], \mathbb{R}^k) : q^*(t_0) = x_0, \|q^*(t) - x_0\| \leq b \right\}, \quad (10)$$

as the norm $\|q^*(t) - q^*(t_0)\|$ is bounded by b .

Let us define $\Gamma : \mathfrak{F} \rightarrow \mathfrak{C}([t_0, t_0 + \alpha], \mathbb{R}^k)$ such that the image of the function $q^*(t)$ is $\Gamma q^*(t)$ defined by

$$\Gamma q^*(t) = x_0 + \int_{t_0}^t f(s, q^*(s)) ds, \quad (11)$$

where we accomplished both

$$\begin{aligned} \Gamma q^*(t_0) &= x_0 + \int_{t_0}^{t_0} f(s, q^*(s)) ds = x_0, \\ \left\| \int_{t_0}^t f(s, q^*(s)) ds \right\| &\leq H|t - t_0| \leq H\alpha \leq b. \end{aligned} \quad (12)$$

Then, Γ is a contraction defined in a closed subset of a complete metric space, which implies that there exists a unique $q^* \in \mathfrak{F}$ such that $\Gamma q^*(t) = q^*(t)$, according to the fixed-point theorem of Banach.

Thus, $q^*(t)$ exists and is a unique solution of (6) for all $t_0 \leq t \leq t_0 + \alpha$.

2.2. Dynamics Invariance. The validity of Claim 1 is conditioned to result in the transformation of the robot's dynamics when the sensor's dynamics are included. To solve this problem, the following theorem is presented:

Theorem 2. *Dynamic properties of*

$$\dot{\chi}_1 = \chi_2,$$

$$\dot{\chi}_2 = -M^{-1} [C(q, \dot{q})\dot{q} + G(q) + F_f(q, \dot{q})] + M^{-1}(q)\tau,$$

$$\chi_1(t_0) = (\chi_1)_0, \chi_2(t_0) = (\chi_2)_0, \chi_1(t_0) = (\chi_1)_0, \chi_2(t_0) = (\chi_2)_0, \quad (13)$$

which are the dynamic robot's model with actual output q , remain invariant with respect to the transformed form

$$\dot{\chi}_1 = \chi_2,$$

$$\dot{\chi}_2 = -M^{-1} \left[C(\tilde{q}, \dot{\tilde{q}}) \frac{d}{dt} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} + G(\tilde{q}) + F_f(\tilde{q}, \dot{\tilde{q}}) \right] + M^{-1}(\tilde{q})\tau,$$

$$\chi_1(t_0) = (\chi_1)_0, \chi_2(t_0) = (\chi_2)_0, \quad (14)$$

which is the robot with the n -th order sensor inclusion, where $\tau \in \mathbb{R}^k$ is the torque vector, $M \in \mathbb{R}^{k \times k}$ is the inertia matrix, matrix $C \in \mathbb{R}^{k \times k}$ represents the Coriolis and centripetal enforces, $G \in \mathbb{R}^k$ is the gravitational vector, and $F_{f1}^T, \dots, F_{fk}^T$ can be obtained by any parametric identification method, and each of them is the sum of viscous, Coulomb, and static friction forces.

The transformation method is given by the sensor inclusion in the robot's joints, represented by

$$\bar{\beta}_i = \begin{bmatrix} \bar{f}_i & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \bar{h}_i \end{bmatrix},$$

$$\chi_1 = q = \left[\sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1, \dots, \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_k \right]^T,$$

$$\chi_2 = \dot{q} = \left[\frac{d}{dt} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1, \dots, \frac{d}{dt} \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_k \right]^T, \quad (15)$$

which are the sensor's gain matrices, the articular positions, and the articular velocities, respectively.

FIGURE 2: Mechanical configuration of a 3-DoF anthropomorphic robot arm: shapes of articular displacements.

Proof. The proof starts with $k = 3$, but the intermediate results are given to make evident it is the extension to $k = 2, 1$, and by induction conclude generalization to (13)–(15), i.e., generalization for all $k \geq 1$.

Let us consider a serial 3-DoF anthropomorphic robot arm as defined in [7] and as illustrated by Figure 2 with positioning sensors of order n .

As dynamics is a consequence of kinematics, let us consider the form of direct and inverse kinematics with n -th order sensor inclusion.

Direct kinematics. The Denavit-Hartenberg (DH) convention (see [8, 9]) is a commonly used method for selecting frames of reference and the relationship between them to obtain the kinematics mathematical model for robotic plants. This method uses arrays as those given by (16) to represent the orientation and the positioning of the end effector (i.e., the tool of the robot) [10, 11].

$$H = \begin{bmatrix} O_{3 \times 3} & X_{3 \times 1} \\ 0 & 1 \end{bmatrix}, \quad (16)$$

where $O_{3 \times 3}$ is a 3×3 matrix for the orientation, $X_{3 \times 1}$ is a 3×1 array for the position of the tool, and H is the so-called homogeneous transformation matrix (see [12]).

The DH convention claims that it is possible to obtain both position and orientation, from the $(j-1)$ -th to the j -th link, through two translation and two rotation movements, as is presented in the following homogeneous transformation matrix,

$${}^{j1}H_i = R_{\tilde{z}_{j-1}, \theta_j} T_{\tilde{z}_{j-1}, d_j} T_{\tilde{x}_j, l_j} R_{\tilde{x}_j, \alpha_j} = \begin{bmatrix} \cos(\theta_j) & -\sin(\theta_j) \cos(\alpha_j) & \sin(\theta_j) \sin(\alpha_j) & \alpha_j \cos(\theta_j) \\ \sin(\theta_j) & \cos(\theta_j) \cos(\alpha_j) & -\cos(\theta_j) \sin(\alpha_j) & \alpha_j \sin(\theta_j) \\ 0 & \sin(\alpha_j) & \cos(\alpha_j) & d_j \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (17)$$

TABLE 1: Denavit-Hartenberg parameters.

$j[\text{DoF}]$	$\theta_j[\text{rad}]$	$d_j[m]$	$a_j[m]$	$\alpha_j[\text{rad}]$
1	q_1	l_1	0	$\pi/2$
2	$q_2 + \pi/2$	0	l_2	0
3	q_3	0	l_3	0

where θ , d , a , and α are the DH parameters, and their values come from specific aspects of the geometric relationship between the coordinate frames for the mechanical configuration of the robot under study and the way to select each of the coordinate frames; x_j , y_j , and z_j are the axes of the j -th coordinate frame; R_{z_{j-1},θ_j} represents the rotation matrix with respect to z_{j-1} ; T_{z_{j-1},d_j} is a translation matrix with respect to z_{j-1} ; T_{x_j,a_j} is a translation matrix with respect to x_j ; and R_{x_j,α_j} is a rotation matrix with respect to x_j (see [8–13] for a detailed procedure in obtaining (17)).

To feed equation (17) with the values of θ , d , a and α , Table 1 was constructed with the DH parameters of the robot under consideration, taking into account Figure 3 which shows the way in which the author has selected each of the coordinate frames (see [13] for more details).

Table 1 shows the articular plant positions q_1 , q_2 , and q_3 , which represent angular displacements as is shown in Figure 2; $l = [l_1, l_2, l_3]^T$ are the link lengths, which are illustrated in the right side of Figure 3.

A 3-DoF robot arm has $j = 1, 2, 3$, as such, substituting DH parameters in (17), three homogeneous transformation matrices are obtained: 0H_1 , 1H_2 , and 2H_3 . To go from system 0 to system 3, we compute ${}^0H_3 = {}^0H_1 \times {}^1H_2 \times {}^2H_3$, obtaining the final position and orientation of the tool

$${}^0H_3 = \begin{bmatrix} \eta_a & \eta_b & \sin(q_1) & \eta_e \\ \eta_c & \eta_d & -\cos(q_1) & \eta_f \\ \cos(q_2 + q_3) & -\sin(q_2 + q_3) & 0 & \eta_g \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (18)$$

FIGURE 3: Cartesian coordinate frames and home position for the robot. Origins 1, 2, and 3 have z -axes entering the plane of drawing, and origin 0 has a y -axis leaving the plane of the drawing.

with

$$\eta_a = -\cos(q_1) \sin(q_2 + q_3),$$

$$\eta_b = -\cos(q_1) \cos(q_2 + q_3),$$

$$\eta_c = -\sin(q_1) \sin(q_2 + q_3),$$

$$\eta_d = -\sin(q_1) \cos(q_2 + q_3),$$

$$\eta_e = -\cos(q_1)(l_2 \sin(q_2) + l_3 \sin(q_2 + q_3)),$$

$$\eta_f = -\sin(q_1)(l_2 \sin(q_2) + l_3 \sin(q_2 + q_3)),$$

$$\eta_g = l_1 + l_2 \cos(q_2) + l_3 \cos(q_2 + q_3). \quad (19)$$

Thus, from the last column in (18) and including (2), direct kinematics is obtained for the robot with n -order sensor inclusion, as follows

$$X(q(t)) = [x \quad y \quad z]^T = \begin{bmatrix} -\cos\left(\sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1\right) \left(l_2 \sin\left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2\right) + l_3 \sin\left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3\right) \right) \\ -\sin\left(\sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1\right) \left(l_2 \sin\left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2\right) + l_3 \sin\left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3\right) \right) \\ l_1 + l_2 \cos\left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2\right) + l_3 \cos\left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3\right) \end{bmatrix}, \quad (20)$$

where x , y , and z are the spatial Cartesian coordinates of the end effector.

Inverse kinematics. Solving equation (20) for q_1 , q_2 , and q_3 , inverse kinematics (see [14]) is obtained. This can be done involving the Moore-Penrose pseudoinverse Jacobian matrix $J^T(JJ^T)^{-1}$ and taking into account the chain rule $\dot{X} = J\dot{q}$, for the first time derivative of q , given by $\dot{q} = J^T(JJ^T)^{-1}\dot{X}$, where J is the Jacobian matrix $J = (\partial/\partial q)X$ (see [12]). Carrying out this procedure and involving (2), the inverse kinematics for the robot is given by

$$\sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1 = \tan^{-1} \left[\frac{y}{x} \right],$$

$$\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 = \tan^{-1} \left[\frac{(l_2 + (\rho/2l_2))z^* - l_3 \sqrt{1 - (\rho/2l_2l_3)^2} \sqrt{x^2 + y^2}}{(l_2 + (\rho/2l_2)) \sqrt{x^2 + y^2} + l_3 \sqrt{1 - (\rho/2l_2l_3)^2} \cdot z^*} \right],$$

$$\sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 = \tan^{-1} \left[\frac{2l_2l_3}{\rho} \sqrt{1 - \left(\frac{\rho}{2l_2l_3} \right)^2} \right],$$

$$\rho = x^2 + y^2 + z^{*2} - l_2^2 - l_3^2,$$

$$z^* = z - l_1, \quad (21)$$

Attitude in the orientation of the end effector. Three elements of the submatrix $O_{3 \times 3}$ inside (16) are needed to obtain ϕ , γ , and ψ which represent the roll, pitch, and yaw movements of the end effector, respectively (see [13]):

$$\begin{aligned} O_{13} &= \cos(\phi) \sin(\gamma) \cos(\psi) + \sin(\phi) \sin(\psi), \\ O_{31} &= -\sin(\gamma), \\ O_{33} &= \cos(\gamma) \cos(\psi). \end{aligned} \quad (22)$$

Now, taking (18) as the master matrix and solving for ϕ , γ , and ψ , the orientation of the end effector of 3-DoF anthropomorphic robot arms with any order linear positioning sensors is obtained as

$$\begin{bmatrix} \phi \\ \gamma \\ \psi \end{bmatrix} = \begin{bmatrix} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1 \\ \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \\ \pi/2 \end{bmatrix}. \quad (23)$$

The same results presented in (20)–(23) are obtained including directly in (18) the DH parameters with sensor sys-

TABLE 2: Denavit-Hartenberg parameters with sensor system inclusion.

$j[\text{DoF}]$	$\theta_j[\text{rad}]$	$d_j[m]$	$a_j[m]$	$\alpha_j[\text{rad}]$
1	$\sum_{i=0}^n \bar{f}_i \left(\frac{d^i}{dt^i} \right) \tilde{q}_1$	l_1	0	$\pi/2$
2	$\sum_{i=0}^n \bar{g}_i \left(\frac{d^i}{dt^i} \right) \tilde{q}_2 + \pi/2$	0	l_2	0
3	$\sum_{i=0}^n \bar{h}_i \left(\frac{d^i}{dt^i} \right) \tilde{q}_3$	0	l_3	0

tem inclusion given by Table 2, concluding that kinematics is invariant to sensor system inclusion.

2.2.1. Robot's Dynamics. Let us represent the total torque vector (τ) of the robot under consideration according to the Euler-Lagrange approach, as

$$\tau = M(q)\ddot{q} + \dot{M}(q)\dot{q} - \frac{\partial}{\partial q} \left[\frac{1}{2} \dot{q}^T M(q) \dot{q} \right] + \frac{\partial U(q)}{\partial q} + F_f^r(q, \dot{q}), \quad (24)$$

where $M \in \mathbb{R}^{3 \times 3}$ is the inertia matrix, $U \in \mathbb{R}^3$ is the potential energy vector, and $F_f^r \in \mathbb{R}^3$ is the friction force vector.

Using the methodology detailed in [15–17] to obtain the torque for the j -th link, (24) is transformed to the following form

$$\tau_i = \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}} L(q, \dot{q}) \right] - \frac{\partial}{\partial q} L(q, \dot{q}) + F_f^r(q, \dot{q}), \quad (25)$$

where $L \in \mathbb{R}^3$ is the Lagrangian given by

$$L(q, \dot{q}) = K(q, \dot{q}) - U(q), \quad (26)$$

with K and U as the kinetic and potential energies, respectively, which according to the referenced methodology are calculated for this work as

$$\begin{aligned} K(q, \dot{q}) &= \frac{1}{2} (I_1 + I_2 + I_3 + m_2 l_{c2}^2 (\sin(q_2))^2 + m_3 l_2^2 (\sin(q_2))^2 \\ &\quad + m_3 l_2 l_{c3} \sin(q_2) \sin(q_2 + q_3) + m_3 l_{c3}^2 (\sin(q_2 + q_3))^2) \dot{q}_1^2 \\ &\quad + \frac{1}{2} (I_2 + I_3 + m_2 l_{c2}^2 + m_3 l_2^2 + m_3 l_2 l_{c3} \cos(q_3)) \dot{q}_2^2 \\ &\quad + \frac{1}{2} (I_3 + m_3 l_{c3}^2) \dot{q}_3^2 + (I_2 + I_3) \dot{q}_1 \dot{q}_2 + I_3 \dot{q}_1 \dot{q}_3 \\ &\quad + (I_3 + m_3 l_2 l_{c3} \cos(q_3) + m_3 l_{c3}^2) \dot{q}_2 \dot{q}_3, \\ U(q) &= m_2 g l_{c2} [1 - \cos(\pi - q_2)] + m_3 g [(l_2 + l_{c3}) \\ &\quad - (l_2 \cos(\pi - q_2) + l_{c3} \cos(\pi - q_2 - q_3))], \end{aligned} \quad (27)$$

$[l_{c1}, l_{c2}, l_{c3}]^T$ are the inertia, mass, and length vectors, respectively, for the j -th centroid, and $g = 9.81 \text{ m/s}^2$.

Separating each element of vector (25) and calculating

$$\begin{aligned}\tau_1 &= \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_1} L(q, \dot{q}) \right] - 0 + F_{f1}^r, \\ \tau_2 &= \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_2} L(q, \dot{q}) \right] - \frac{\partial}{\partial q_2} L(q, \dot{q}) + F_{f2}^r, \\ \tau_3 &= \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_3} L(q, \dot{q}) \right] - \frac{\partial}{\partial q_3} L(q, \dot{q}) + F_{f3}^r, \end{aligned} \quad (28)$$

where

$$\begin{aligned} \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_1} L(q, \dot{q}) \right] &= [I_1 + I_2 + I_3 + m_2 l_{c2}^2 (\sin(q_2))^2 \\ &\quad + m_3 l_2^2 (\sin(q_2))^2 + m_3 l_2 l_{c3} \sin(q_2) \sin \\ &\quad \cdot (q_2 + q_3) + m_3 l_{c3}^2 (\sin(q_2 + q_3))^2] \ddot{q}_1 \\ &\quad + [2m_2 l_{c2}^2 \sin(q_2) \cos(q_2) \\ &\quad + 2m_3 l_2^2 \sin(q_2) \cos(q_2) \\ &\quad + m_3 l_2 l_{c3} \cos(q_2) \sin(q_2 + q_3) \\ &\quad + m_3 l_2 l_{c3} \sin(q_2) \cos(q_2 + q_3) \\ &\quad + 2m_3 l_{c3}^2 \sin(q_2 + q_3) \cos(q_2 + q_3)] \dot{q}_1 \dot{q}_2 \\ &\quad + [m_3 l_2 l_{c3} \sin(q_2) \cos(q_2 + q_3) \\ &\quad + 2m_3 l_{c3}^2 \sin(q_2 + q_3) \cos(q_2 + q_3)] \dot{q}_1 \dot{q}_3 \\ &\quad + [I_2 + I_3] \ddot{q}_2 + I_3 \ddot{q}_3, \\ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_2} L(q, \dot{q}) \right] &= [I_2 + I_3] \ddot{q}_1 + [I_2 + I_3 + m_2 l_{c2}^2 \\ &\quad + m_3 l_{c2}^2 + m_3 l_2 l_{c3} \cos(q_3)] \ddot{q}_2 \\ &\quad + [I_3 + m_3 l_{c3}^2 + m_3 l_2 l_{c3} \cos(q_3)] \ddot{q}_3 \\ &\quad - m_3 l_2 l_{c3} \sin(q_3) (\dot{q}_2 \dot{q}_3 + \dot{q}_3^2), \\ \frac{d}{dt} \left[\frac{\partial}{\partial \dot{q}_3} L(q, \dot{q}) \right] &= I_3 \ddot{q}_1 + [I_3 + m_3 l_{c3}^2 + m_3 l_2 l_{c3} \cos(q_3)] \ddot{q}_2 \\ &\quad + [I_3 + m_3 l_{c3}^2] \ddot{q}_3 - m_3 l_2 l_{c3} \sin(q_3) \dot{q}_2 \dot{q}_3, \end{aligned} \quad (29)$$

$$\frac{\partial}{\partial q_1} L(q, \dot{q}) = 0,$$

$$\begin{aligned} \frac{\partial}{\partial q_2} L(q, \dot{q}) &= [m_2 l_{c2}^2 \sin(q_2) \cos(q_2) \\ &\quad + m_3 l_2^2 \sin(q_2) \cos(q_2) \\ &\quad + \frac{1}{2} m_3 l_2 l_{c3} (\sin(q_2) \cos(q_2 + q_3) \\ &\quad + \cos(q_2) \sin(q_2 + q_3)) + m_3 l_{c3}^2 \sin \\ &\quad \cdot (q_2 + q_3) \cos(q_2 + q_3)] \dot{q}_1^2 \\ &\quad + m_2 g l_{c2} \sin(\pi - q_2) + m_3 g l_2 \sin(\pi - q_2) \\ &\quad + m_3 g l_{c3} \sin(\pi - q_2 - q_3), \\ \frac{\partial}{\partial q_3} L(q, \dot{q}) &= \left[\frac{1}{2} m_3 l_2 l_{c3} \sin(q_2) \cos(q_2 + q_3) \right. \\ &\quad \left. + m_3 l_{c3}^2 \sin(q_2 + q_3) \cos(q_2 + q_3) \right] \dot{q}_1^2 \\ &\quad - m_3 l_2 l_{c3} \sin(q_3) \left(\frac{1}{2} \dot{q}_2^2 + \dot{q}_2 \dot{q}_3 \right) \\ &\quad + m_3 g l_{c3} \sin(\pi - q_2 - q_3), \end{aligned} \quad (30)$$

Defining the Coriolis forces and gravitational vectors, respectively, as

$$\begin{aligned} C(q, \dot{q}) \dot{q} &= \dot{M}(q) \dot{q} - \frac{\partial}{\partial q} \left[\frac{1}{2} \dot{q}^T M(q) \dot{q} \right], \\ G(q) &= \frac{\partial U(q)}{\partial q}, \end{aligned} \quad (31)$$

and including (2) in (24), the torque vector takes the form

$$\begin{aligned} \tau &= M(\tilde{q}) \frac{d^2}{dt^2} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} + C(\tilde{q}, \dot{\tilde{q}}) \frac{d}{dt} \\ &\quad \cdot \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} + G(\tilde{q}) + F_f^r(\tilde{q}, \dot{\tilde{q}}). \end{aligned} \quad (32)$$

Rearranging results presented in (29) and (30) and also including (2), the elements of (32), ($M(\tilde{q})$, $C(\tilde{q}, \dot{\tilde{q}})$, and $G(\tilde{q})$), are obtained and given in the following form

$$M(\tilde{q}) = \begin{bmatrix} M_{11} & I_2 + I_3 & I_3 \\ I_2 + I_3 & M_{22} & M_{23} \\ I_3 & M_{32} & I_3 + m_3 l_{c3}^2 \end{bmatrix}, \quad (33)$$

where

$$\begin{aligned} M_{11} &= I_1 + I_2 + I_3 + (m_2 l_{c2}^2 + m_3 l_2^2) \left(\sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right) \right)^2 \\ &\quad + m_3 l_2 l_{c3} \sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right) \sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right) \\ &\quad + m_3 l_{c3}^2 \left(\sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right) \right)^2, \\ M_{22} &= I_2 + I_3 + m_2 l_{c2}^2 + m_3 l_2 l_{c3} \cos \left(\sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right), \\ M_{23} = M_{32} &= I_3 + m_3 l_{c3}^2 + m_3 l_2 l_{c3} \cos \left(\sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right), \end{aligned} \quad (34)$$

$$C(\tilde{q}, \dot{\tilde{q}}) = \begin{bmatrix} 2A \frac{d}{dt} \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_2 & 0 & 2B \frac{d}{dt} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1 \\ -A \frac{d}{dt} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1 & -2D \frac{d}{dt} \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 & -2D \frac{d}{dt} \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \\ -B \frac{d}{dt} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1 & D \frac{d}{dt} \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 & 0 \end{bmatrix}, \quad (35)$$

where

$$A = (m_2 l_{c2}^2 + m_3 l_2^2) \sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right) \cos \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right) + \frac{m_3 l_2 l_{c3}}{2} \cos \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right) \sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right) + B, \quad (36)$$

$$B = \frac{1}{2} m_3 l_2 l_{c3} \sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right) \cos \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right) + m_3 l_{c3}^2 \sin \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right) \cos \left(\sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 + \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right), \quad (37)$$

$$D = \frac{1}{2} m_3 l_2 l_{c3} \sin \left(\sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right), \quad (38)$$

$$G(\tilde{q}) = -g \begin{bmatrix} 0 \\ (m_2 l_{c2} + m_3 l_2) \sin \left(\pi - \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right) + m_3 l_{c3} \sin \left(\pi - \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 - \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right) \\ m_3 l_{c3} \sin \left(\pi - \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 - \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right) \end{bmatrix}. \quad (39)$$

Additionally, the friction vector symbolic form

$$F_f^r(\tilde{q}, \dot{\tilde{q}}) = [F_{f1}^r \quad F_{f2}^r \quad F_{f3}^r]^T, \quad (40)$$

must be included in the dynamic model.

Even more, the same results presented in (33)–(39) are obtained, when (2) is directly included in the Euler-Lagrange (24) and in (31), as follows

$$\begin{aligned} \tau = M(\tilde{q}) & \left[\frac{d^2}{dt^2} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right] + \dot{M}(\tilde{q}) \left[\frac{d}{dt} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right] \\ & - \frac{\partial}{\partial q} \left[\frac{1}{2} \left[\frac{d}{dt} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right]^T M(\tilde{q}) \left[\frac{d}{dt} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right] \right] \\ & + \frac{\partial U(\tilde{q})}{\partial q} + F_f^r(\tilde{q}, \dot{\tilde{q}}), \end{aligned}$$

$$\begin{aligned} C(\tilde{q}, \dot{\tilde{q}}) & \left[\frac{d^2}{dt^2} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right] \\ & = \dot{M}(\tilde{q}) \left[\frac{d}{dt} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right] \\ & - \frac{\partial}{\partial q} \left[\frac{1}{2} \left[\frac{d}{dt} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right]^T M(\tilde{q}) \right. \\ & \left. \cdot \left[\frac{d}{dt} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} \right] \right] G(\tilde{q}) = \frac{\partial U(\tilde{q})}{\partial q}, \end{aligned} \quad (41)$$

where

$$\bar{\beta}_i = \begin{bmatrix} \bar{f}_i & 0 & 0 \\ 0 & \bar{g}_i & 0 \\ 0 & 0 & \bar{h}_i \end{bmatrix},$$

$$\begin{aligned} \chi_1 = q & = \left[\sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1, \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2, \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right]^T, \\ \chi_2 = \dot{q} & = \left[\frac{d}{dt} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1, \frac{d}{dt} \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2, \frac{d}{dt} \sum_{i=0}^n \bar{h}_i \frac{d^i}{dt^i} \tilde{q}_3 \right]^T, \end{aligned} \quad (42)$$

which proves dynamics invariance to sensor inclusion.

2.2.2. Extensive Considerations. Even more, for a serial robot with 2-DoF, from Tables 1 to 2, line $i = 3$ must be removed and (16) becomes

$$H = \begin{bmatrix} O_{2 \times 2} & X_{2 \times 1} \\ 0 & 1 \end{bmatrix}, \quad (43)$$

with

$$\begin{aligned}\bar{\beta}_i &= \begin{bmatrix} \bar{f}_i & 0 \\ 0 & \bar{g}_i \end{bmatrix}, \\ \chi_1 = q &= \left[\sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1, \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right]^T, \\ \chi_2 = \dot{q} &= \left[\frac{d}{dt} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1, \frac{d}{dt} \sum_{i=0}^n \bar{g}_i \frac{d^i}{dt^i} \tilde{q}_2 \right]^T.\end{aligned}\quad (44)$$

Also, when a 1-DoF is considered, from Tables 1 to 2, line $i = 2, 3$ must be removed and (16) becomes

$$H = \begin{bmatrix} O_{1 \times 1} & X_{1 \times 1} \\ 0 & 1 \end{bmatrix}, \quad (45)$$

with

$$\begin{aligned}\bar{\beta}_i &= [\bar{f}_i], \\ \chi_1 = q &= \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1, \\ \chi_2 = \dot{q} &= \frac{d}{dt} \sum_{i=0}^n \bar{f}_i \frac{d^i}{dt^i} \tilde{q}_1.\end{aligned}\quad (46)$$

Since procedures (16)–(40) have intermediated outcomes, it is easy to compute results for a 1-DoF and for a 2-DoF robot, to obtain the same conclusions.

Further, invoking the induction method, these results can be extended to serial robots with k -DoF for all $k \geq 1$.

In order to deal with sensor inclusion mathematical complexity, the following remarks are useful for practical implementations:

$$\frac{d^p}{dt^p} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} = \sum_{i=0}^n \bar{\beta}_i \frac{d^{i+p}}{dt^{i+p}} \tilde{q}. \quad (47)$$

Remark 4. For the case to consider matrix β_i as a constant, the following identity should be used, in order to simplify the algebraic complications,

And for the case to consider matrix $\beta_i = I$,

$$\frac{d^p}{dt^p} \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q} = \sum_{i=p}^{n+p} \frac{d^i}{dt^i} \tilde{q}. \quad (48)$$

Remark 5. For initial conditions at the origin $\tilde{q}(0) = \dot{\tilde{q}}(0) = \ddot{\tilde{q}}(0) = \dots = 0$, the Laplace transform for sensors of the k -th order with $\beta_i = I$,

$$\mathfrak{L} \left[\sum_{i=0}^n \frac{d^i}{dt^i} \tilde{q} \right] = \tilde{Q}(s) \sum_{i=0}^n s^i = Q(s), \quad (49)$$

such that

$$f(q) = \mathfrak{L}^{-1} \left[\frac{Q(s)}{\sum_{i=0}^n s^i} \right]. \quad (50)$$

2.3. Control Using n -th Order Sensors' Output. Now, in order to take advantage of Claim 1, the following control conclusions are given:

Assumption 3. It is assumed that sensors are previously well characterized, *i.e.*, both n (the order of every sensor) and sensor's dynamics are known; this implies that $\varepsilon \rightarrow 0$.

Claim 2. Supported by Theorems 2 and 3, and if $q^* \rightarrow q$, it can be claimed that equation (2) becomes the method to identify q from $\tilde{q}$ to control (14), which represents the robot with n -th order sensor inclusion. Thus,

$$e = X^{\text{ref}} - \sum_{i=0}^n \bar{\beta}_i \frac{d^i}{dt^i} \tilde{q}, \quad (51)$$

where $e \in \mathbb{R}^k$ is the tracking error, and $X_{\text{ref}} \in \mathbb{R}^k$ represents the reference signal.

The method is depicted in Figure 4, where the dashed squared block of the negative feedback represents (2).

Note 1. Solution $\tilde{q}$ of (2) cannot be directly used to control (6).

Extension of Claim 2. In [18], the Lasalle-invariance principle (generally used to show asymptotic stability in the Lyapunov approach) is extended to nonautonomous switching systems. This implies that Claim 2 is valid using both continuous-based and discontinuous-based controllers, *e.g.*, Proportional Integral Derivative (PID) control and Sliding Mode (SM) controllers.

Even more, in [19], an extensive study to show the control of MIMO systems by the twisting-algorithm (a second-order SM control) is presented, showing asymptotic convergence of the system trajectories to the selected sliding manifold.

Further, in order to review the stability for biorder SM (twisting-algorithm included) control with variable PID gains, [20] can be consulted.

3. Implementation Example

To verify the validity of Claims 1 and 2, this example is designed to solve the trajectory tracking control of the anthropomorphic robot with 3-DoF (see Figures 2, 3, and 5) when n -th order sensors are included on it and when conditions (including assumptions) are accomplished.

FIGURE 4: Block diagram of the closed-loop control with sensor inclusion.

Selecting $n = 2$, and

$$\begin{aligned}\bar{\beta}_0 &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0.9 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \\ \bar{\beta}_1 &= \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.3 \end{bmatrix}, \\ \bar{\beta}_2 &= \begin{bmatrix} 0.1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix},\end{aligned}\quad (52)$$

linear sensors, with constant coefficients of order two, zero, and one, are included in joints one, two, and three, respectively, as

$$\begin{aligned}q_1^* &= \tilde{q}_1 + \frac{1}{2} \dot{\tilde{q}}_1 + \frac{1}{10} \ddot{\tilde{q}}_1, \\ q_2^* &= \frac{9}{10} \tilde{q}_2, \\ q_3^* &= \tilde{q}_3 + \frac{3}{10} \dot{\tilde{q}}_3.\end{aligned}\quad (53)$$

Reference signal to track for the robot is selected as

$$\begin{aligned}X_1^{\text{ref}} &= \begin{cases} \frac{1}{2} & \text{if } 0 \leq t \leq 5, \\ -\frac{1}{2} & \text{if } 5 < t \leq 10, \end{cases} \\ X_2^{\text{ref}} &= \frac{2}{5} \sin\left(\frac{2}{5}\pi t\right), \\ X_3^{\text{ref}} &= \frac{2}{50}t + \frac{1}{10}.\end{aligned}\quad (54)$$

FIGURE 5: Actual 3-DoF anthropomorphic robot arm parameter model.

TABLE 3: Parameters of the actual 3-DoF anthropomorphic robot arm.

Link number i (DoF)	Total length l [m]	Centroid length l_c [m]	Total mass m [kg]	Inertia average I [kg · m ²]
1	0.218	0.162	2.8	0.88
2	0.16	0.058	1.2	0.96
3	0.255	0.12	0.52	1.08

Thus, the ideal form of sensor systems is

$$\begin{aligned}\tilde{q}_1 + \frac{1}{2} \dot{\tilde{q}}_1 + \frac{1}{10} \ddot{\tilde{q}}_1 &= \pm \frac{1}{2}, \\ \frac{9}{10} \tilde{q}_2 &= \frac{2}{5} \sin\left(\frac{2}{5}\pi t\right), \\ \tilde{q}_3 + \frac{3}{10} \dot{\tilde{q}}_3 &= \frac{2}{50}t + \frac{1}{10}.\end{aligned}\quad (55)$$

Solving for $\tilde{q}$, for the open-loop system, the solution has the structure

$$\begin{aligned}\tilde{q}_1 &= \frac{1}{2} \mp \frac{1}{2} \exp\left(-\frac{5}{2}t\right) \cos\left(\frac{|\sqrt{15}|}{2}t\right) \mp \frac{5}{2|\sqrt{15}|} \exp \\ &\cdot \left(-\frac{5}{2}t\right) \sin\left(\frac{|\sqrt{15}|}{2}t\right),\end{aligned}$$

$$\begin{aligned}\tilde{q}_2 &= \frac{4}{9} \sin\left(\frac{2}{5}\pi t\right), \\ \tilde{q}_3 &= \frac{11}{125} + \frac{1}{25}t - \frac{11}{125} \exp\left(-\frac{10}{3}t\right).\end{aligned}\quad (56)$$

As the second and third terms in $\tilde{q}_1$ are negative for $X_1^{\text{ref}} = 1/2$ and they are positive for $X_1^{\text{ref}} = -1/2$, then, the form of solution $\tilde{q}$ exists and is unique.

Note 2. Equations (55)–(56) are not included in the control method; they just show the ideal solution structure. For closed-loop control purposes, $\tilde{q}$ are obtained dynamically from sensor's output.

Even more, for selecting parameters of Table 3 and friction models, the actual robot shown in Figure 5 was used, but no other application was performed using the real robot.

The following friction values are used:

$$f_1 = 0.24 \dot{q}_1 \left[kg \cdot \frac{m^2}{s} \right] + 0.16 \operatorname{sgn} \dot{q}_1 \left[kg \cdot \frac{m^2}{s^2} \right],$$

$$\begin{aligned}f_2 &= 0.21 \dot{q}_2 \left[kg \cdot \frac{m^2}{s} \right] + 0.18 \operatorname{sgn} \dot{q}_2 \left[kg \cdot \frac{m^2}{s^2} \right], \\ f_3 &= 0.17 \dot{q}_3 \left[kg \cdot \frac{m^2}{s} \right] + 0.15 \operatorname{sgn} \dot{q}_3 \left[kg \cdot \frac{m^2}{s^2} \right],\end{aligned}\quad (57)$$

where it is possible to see that if any articular velocity of the robot becomes zero, then the corresponding friction equation is reduced to zero; the sign function (sgn) appears in the second terms. This vector includes friction forces produced by ball bearings, gear boxes, and drive belts. Thus, the approximated values can be obtained directly from the manufacturer's mechanical components.

Substituting the parameters of Table 3 in (33)–(39) considering the friction model, and including sensors (52), numerical values for M , C , G , and F_f^r are given by

$$M(\tilde{q}) = \begin{bmatrix} M_{11}^* & 2.04 & 1.08 \\ 2.04 & M_{22}^* & M_{23}^* \\ 1.08 & M_{32}^* & 1.0875 \end{bmatrix}, \quad (58)$$

where

$$\begin{aligned}M_{11}^* &= 2.92 + [17.3 \sin^2(0.9\tilde{q}_2) \\ &\quad + 9.984 \sin(0.9\tilde{q}_2) \sin(0.9\tilde{q}_2 + \tilde{q}_3 + 0.3\dot{\tilde{q}}_3) \\ &\quad + 7.488 \sin^2(0.9\tilde{q}_2 + \tilde{q}_3 + 0.3\dot{\tilde{q}}_3)] \times 10^{-3}, \\ M_{22}^* &= 2.044 + [9.984 \cos(\tilde{q}_3 + 0.3\dot{\tilde{q}}_3)] \times 10^{-3}, \\ M_{23}^* &= M_{32}^* = 1.0875 + [9.984 \cos(\tilde{q}_3 + 0.3\dot{\tilde{q}}_3)] \times 10^{-3}, \\ C(\tilde{q}, \dot{\tilde{q}}) &= \begin{bmatrix} 1.8\ddot{\tilde{q}}_2 A^* & 0 & (2\dot{\tilde{q}}_1 + \ddot{\tilde{q}}_1 + 0.2\ddot{\tilde{q}}_1)B^* \\ -(\dot{\tilde{q}}_1 + 0.5\ddot{\tilde{q}}_1 + 0.1\ddot{\tilde{q}}_1)A^* & -(2\dot{\tilde{q}}_3 + 0.6\ddot{\tilde{q}}_3)D^* & -(2\dot{\tilde{q}}_3 + 0.6\ddot{\tilde{q}}_3)D^* \\ -(\dot{\tilde{q}}_1 + 0.5\ddot{\tilde{q}}_1 + 0.1\ddot{\tilde{q}}_1)B^* & \ddot{\tilde{q}}_2 D^* & 0 \end{bmatrix},\end{aligned}\quad (59)$$

with the following numerical values

$$\begin{aligned}A^* &= [17.3 \sin(0.9\tilde{q}_2) \cos(0.9\tilde{q}_2) \\ &\quad + 4.992 \cos(0.9\tilde{q}_2) \sin \\ &\quad \cdot (0.9\tilde{q}_2 + \tilde{q}_3 + 0.3\dot{\tilde{q}}_3)] \\ &\quad \times 10^{-3} + B^*,\end{aligned}\quad (60)$$

$$\begin{aligned}B^* &= [4.992 \sin(0.9\tilde{q}_2) \cos(0.9\tilde{q}_2 + \tilde{q}_3 + 0.3\dot{\tilde{q}}_3) \\ &\quad + 7.488 \sin(0.9\tilde{q}_2 + \tilde{q}_3 + 0.3\dot{\tilde{q}}_3) \cos \\ &\quad \cdot (0.9\tilde{q}_2 + \tilde{q}_3 + 0.3\dot{\tilde{q}}_3)] \times 10^{-3},\end{aligned}\quad (61)$$

$$D^* = [4.992 \sin(\tilde{q}_3 + 0.3\dot{\tilde{q}}_3)] \times 10^{-3}, \quad (62)$$

FIGURE 9: The third joint of the robot solving the trajectory tracking of an upward sloping.

FIGURE 10: Enlarged view (zoom) of Figure 7 to show deeply the behaviour of the SMC.

joint impacts (or even shocks) the convergence of the adjacent ones. Also, it can be appreciated that the SMC is a robust control technique, because it never comes out from convergence once it is reached (after the transient state).

Even more, Figure 10 shows an enlarged view of Figure 7 from 6 to 6.5 [s], where the so-called chattering effect produced by the SMC appears.

4. Conclusions

As a result of this work, it is possible to affirm that serial robots can be controlled by taking advantage of the output of n -th order sensory systems, provided the following conditions and assumptions are fulfilled.

Conditions:

- (1) The solution of the differential equations that describe the dynamics of individual sensors must exist and be unique
- (2) The identification value of the actual robot's position should be negligible, $\varepsilon \rightarrow 0$

- (3) The dynamics of the electromechanical plant with n -th order sensor inclusion should be invariant with respect to the dynamics of the plant's output. In this work, it has been demonstrated that for the case of serial robot manipulators with k -DoF, this condition is always fulfilled

Assumptions:

- (1) The value of the plant's output $\tilde{q}$ is always available to be used by the closed-loop control system
- (2) Coefficients of terms that describe the sensor dynamics are known. In this work, these terms are represented by diagonal matrices $\hat{\beta}_i$ with $i = 1, \dots, n$ in (2).
- (3) Plant's dynamics is known, *i.e.*, the differential equation that model plant's dynamics is characterized

Then, serial robot manipulators are controllable using either continuous or variable structure-based controllers, and the differential equation that models sensor dynamics becomes the method to obtain the feedback for the closed-loop control.

Data Availability

The data used to support the findings of this study are included within the article.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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