MPE — Volume 1, pp. 1-10
Reprints available directly from the publisher
Photocopying permitted by license only

STABILITY OF CONDITIONALLY INVARIANT SETS AND CONTROLLED UNCERTAIN DYNAMIC SYSTEMS ON TIME SCALES

V. LAKSHMIKANTHAM

Florida Institute of Technology, Applied Mathematics Program, Melbourne, FL 32901 USA

Z. DRICI*

University of Texas at San Antonio, Division of CS, Math, and Statistics, San Antonio, TX 78249-0064 USA

(Received 4 August 1994)

A basic feedback control problem is that of obtaining some desired stability property from a system which contains uncertainties due to unknown inputs into the system. Despite such imperfect knowledge in the selected mathematical model, we often seek to devise controllers that will steer the system in a certain required fashion. Various classes of controllers whose design is based on the method of Lyapunov are known for both discrete [4], [10], [15], and continuous [3–9], [11] models described by difference and differential equations, respectively. Recently, a theory for what is known as dynamic systems on time scales has been built which incorporates both continuous and discrete times, namely, time as an arbitrary closed sets of reals, and allows us to handle both systems simultaneously [1], [2], [12], [13]. This theory permits one to get some insight into and better understanding of the subtle differences between discrete and continuous systems. We shall, in this paper, utilize the framework of the theory of dynamic systems on time scales to investigate the stability properties of conditionally invariant sets which are then applied to discuss controlled systems with uncertain elements. For the notion of conditionally invariant set and its stability properties, see [14]. Our results offer a new approach to the problem in question.

AMS Nos.: 34D20, 93C10, 93C55, 93D05, 93D15, 93D30
KEYWORDS: Stability, uncertain systems, invariant sets

1. PRELIMINARIES

Let $\mathbb{T}$ be a time scale (any subset of $\mathbb{R}$ with order and topological structure defined in a canonical way) with $t_0 \geq 0$ as a minimal element and no maximal element. Since a time scale $\mathbb{T}$ may or may not be connected, we need the concept of jump operators.

DEFINITION 1.1: The mappings σ, ρ: $\mathbb{T} \to \mathbb{T}$ defined as

$$\sigma(t) = inf\{s \in \mathbb{T}: s > t\} \text{ and } \rho(t) = sup\{s \in \mathbb{T}: s < t\}$$

are called jump operators.

*To whom correspondence should be addressed.

DEFINITION 1.2: A nonmaximal element $t \in \mathbb{T}$ is said to be *right-scattered* (*rs*) if $\sigma(t) > t$, and *right-dense* (*rd*) if $\sigma(t) = t$. A nonminimal element $t \in \mathbb{T}$ is called *left-scattered* (*ls*) if $\rho(t) < t$ and *left-dense* (*ld*) if $\rho(t) = t$.

DEFINITION 1.3: The mapping $\mu^*: \mathbb{T} \to \mathbb{R}_0^+$ defined by $\mu^*(t) = \sigma(t) - t$ is called *graininess*. When $\mathbb{T} = \mathbb{Z}$, $\mu^*(t) \equiv 1$ and when $\mathbb{T} = \mathbb{R}$, $\mu^*(t) \equiv 0$.

DEFINITION 1.4: The mapping $g: \mathbb{T} \to X$, where X is a Banach space, is called *rd*-continuous if

(*i*) it is continuous at each right-dense $t \in \mathbb{T}$,
(*ii*) at each left-dense point the left-sided limit $g(t^-)$ exists.

$C_{rd}[\mathbb{T}, X]$ will denote the set of *rd*-continuous mappings from $\mathbb{T}$ to X. It is clear that a continuous mapping is *rd*-continuous. However, if $\mathbb{T}$ contains left-dense and right-scattered (*ldrs*) points, then *rd*-continuity does not imply continuity, but on a discrete time scale the two notions coincide.

DEFINITION 1.5: A mapping $u: \mathbb{T} \to X$ is said to be differentiable at $t \in \mathbb{T}$, if there exists an $a \in X$ such that for any $\epsilon > 0$, there exists a neighborhood N of t satisfying

$$|u(\sigma(t)) - u(s) - (\sigma(t) - s)a| \le \epsilon |\sigma(t) - s|$$

for all $s \in N$.

The derivative of u is denoted u^Δ. Note that if $\mathbb{T} = \mathbb{R}$, $a = du(t)/dt$, and if $\mathbb{T} = \mathbb{Z}$, $a = u(t+1) - u(t)$. In addition, the derivative has the following basic properties

(*i*) If u is differentiable at t, then it is continuous at t;
(*ii*) If u is continuous at t and t is *rs*, then u is differentiable and

$$u^\Delta(t) = [u(\sigma(t)) - u(t)]/\mu^*(t).$$

DEFINITION 1.6: For each $t \in \mathbb{T}$, let N be a neighborhood of t. Then, we define the generalized derivative (or Dini derivative), $D^+ u^\Delta(t)$, to mean that, given $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such that

$$\frac{u(\sigma(t)) - u(s)}{\mu(t,s)} < D^+u^\Delta(t) + \epsilon \text{ for } s \in N_\epsilon, \qquad s > t,$$

where $\mu(t,s) \equiv \sigma(t) - s$.

In case t is *rs* and u is continuous at t, we have, as in the case of the derivative,

$$D^+u^\Delta(t) = \frac{u(\sigma(t)) - u(t)}{\mu^*(t)}.$$

DEFINITION 1.7: Let g be a mapping from $\mathbb{T}$ to X. If the mapping $f: \mathbb{T} \to X$ is differentiable on $\mathbb{T}$ and satisfies $f^{\Delta}(t) = g(t)$ for $t \in \mathbb{T}$, then it is called the antiderivative of g on $\mathbb{T}$.

The antiderivative has the following properties:

(i) If $g: \mathbb{T} \to X$ is *rd*-continuous, then g has the antiderivative $f: t \to \int_r^t g(s)ds$, $r,t \in \mathbb{T}$.

(ii) If the sequence $\{g_n\}_{n \in \mathbb{N}}$ of *rd*-continuous functions $g_n: \mathbb{T} \to X$ converges uniformly on $[r,s]$ to the *rd*-continuous function g, then

$$\left(\int_r^s g_n(t)dt \right)_{n \in N} \to \int_r^s g(t)dt \ in\ X.$$

DEFINITION 1.8: The mapping $f: \mathbb{T} \times X \to X$ is *rd-continuous* if,

(*i*) it is continuous at each (t,x) with right-dense t, and

(*ii*) the limits $f(t^-,x) = \lim_{(s,y) \to (t^-,x)} f(s,y)$ and $f_{y \to x}(t,x)$ exist at each (t,x) with left-dense t.

A basic tool which is employed in the proofs is the following induction principle.

THEOREM 1.1 *Suppose that for any $t \in \mathbb{T}$, there is a statement $A(t)$ such that the following conditions are verified:*

(I) $A(t_0)$ is true;

(II) If t is right-scattered and $A(t)$ is true, then $A(\sigma(t))$ is also true;

(III) For each right-dense t, there exists a neighborhood N such that whenever $A(t)$ is true, $A(s)$ is also true for all $s \in N$, $s \geq t$;

(IV) For left-dense t, $A(s)$ is true for all $s \in [t_0,t)$ implies $A(t)$ is true.

Then the statement $A(t)$ is true for all $t \in \mathbb{T}$.

Note that in the case of the generalized derivative, $D^+u^{\Delta}(t)$, of Definition 1.6, we approach t only from the right. Therefore, for a statement $A(t)$ involving $D^+u^{\Delta}(t)$ condition *IV* is not needed.

Following Definition 1.6, define, for $V \in C_{rd}[\mathbb{T} \times R^n, R_+]$, $D^+V^{\Delta}(t,x(t))$ to mean that, given $\epsilon > 0$, there exists a right neighborhood $N_\epsilon \subset N$ of t such that

$$\frac{1}{\mu(t,s)}[V(\sigma(t), x(\sigma(t))) - V(s, x(\sigma(t)) - \mu(t,s)f(t,x(t)))] < D^+V^{\Delta}(t, x(t)) + \epsilon$$

for each $s \in N_\epsilon$, $s > t$. As before, if t is rs and $V(t,x(t))$ is continuous at t, this reduces to

$$D^+V^{\Delta}(t,x(t)) = \frac{V(\sigma(t),x(\sigma(t)) - V(t,x(t))}{\mu^*(t)}.$$

THEOREM 1.2: *Let* $V \in C_{rd}[\mathbf{T} \times R^n, R_+]$, $V(t,x)$ *be locally Lipschitzian in* x *for each* $t \in \mathrm{T}$ *which is rd, and let*

$$D^+V^\Delta(t,\ x) \leq g(t, V(t,x))$$

where $g \in C_{rd}\,[\mathrm{T} \times R_+, R]$, $g(t,u)\mu^*(t)$ *is nondecreasing in* u *for each* $t \in \mathrm{T}$ *and* $r(t) = r(t, t_0, u_0)$ *is the maximal solution of* $u^\Delta = g(t, u)$, $u(t_0) = u_0 \geq 0$, *existing on* T. *Then,* $V(t_0, x_0) \leq u_0$ *implies that* $V(t, x(t)) \leq r(t, t_0, u_0)$, $t \in \mathrm{T}$, $t \geq t_0$.

Proof We apply the induction principle of Theorem 1.1 to the statement

$$A(t):\ V(t,\ x(t)) \leq r(t), \qquad t \in \mathrm{T}, \qquad t \geq t_0.$$

(*I*) $A(t_0)$ is true since $V(t_0, x(t_0)) \leq u_0$.

(*II*) Let t be *rs* and $A(t)$ be true. We need to show that $A(\sigma(t))$ is true. By definition, if we set $m(t) = V(t,x(t))$, we see that

$$m(\sigma(t)) - r(\sigma(t)) = [D^+m^\Delta(t) - r^\Delta(t)]\mu^*(t) + m(t) - r(t),$$

which, because of $g(t,u)\mu^*(t)$ being nondecreasing in u and $A(t)$ being true, reduces to

$$m(\sigma(t)) - r(\sigma(t)) \leq [g(t,m(t)) - g(t,r(t))]\mu^*(t) + m(t) - r(t) \leq 0.$$

In view of the fact that

$$\frac{m(\sigma(t)) - m(t)}{\mu^*(t)} = \frac{V(\sigma(t), x(\sigma(t)) - V(t,x(t))}{\mu^*(t)},$$

we see that $A(\sigma(t))$ is true.

(*III*) Let t be *rd* and N be a right neighborhood of t. Assume $A(t)$ is true. We need to show that $A(s)$ is true for $s > t$, $s \in N$. This follows from the comparison theorem for differential equations relative to Lyapunov functions, see [14].

Since in evaluating $D^+V^\Delta(t,x)$ we are interested only in the case where t is approached from the right, consideration of item *IV* of Theorem 1.2 for left-dense points is not needed. Hence $A(t)$ is true for $t \in \mathrm{T}$.

2. STABILITY OF CONDITIONALLY INVARIANT SETS

Consider the differential system

$$x^\Delta = f(t,x,\lambda), \qquad x(t_0)=x_0 \qquad t_0 \in \mathrm{T} \tag{2.1}$$

where $f \in C_{rd}[\mathrm{T} \times \mathrm{R}^m \times \mathrm{R}^d, \mathrm{R}^n]$ and $\lambda \in \mathrm{R}^d$ is a parameter. Also consider the differential equation

$$u^{\Delta} = g(t, u, |\lambda|), \qquad u(t_0) = u_0 \geq 0, \qquad t_0 \in \mathrm{T}(2.2) \tag{2.2}$$

where $g \in C_{rd}[\mathrm{T} \times \mathrm{R}_+, \mathrm{R}]$.

Let us begin by defining the notions of self-invariant and conditionally invariant sets, as well as relevant stability concepts. For that purpose, let $0 < r < \rho$ and

$$A=\{x \in \mathrm{R}^n: |x| \leq r\} \text{ and } B = \{x \in \mathrm{R}^n: |x| \leq p\}. \tag{2.3}$$

Also let $x(t) = x(t, \lambda, t_0, x_0)$ be any solution of (2.1).

Definition 2.1

(1) $A \subset \mathrm{R}^n$ is said to be *self-invariant* for the differential system (2.1) if

$$x_0 \in A \Rightarrow x(t) \in A \qquad t \geq t_0, \ t \in \mathrm{T};$$

(2) B is said to be conditionally-invariant with respect to A for the differential system (2.1) if

$$x_0 \in A \Rightarrow x(t) \in B, \qquad t \geq t_0, \ t \in \mathrm{T}.$$

Definition 2.2

The conditionally invariant set B relative to A is said to be uniformly asymptotically stable (U.A.S.) if

(*i*) it is uniformly stable (U.S.), i.e., if given any $\epsilon > 0$ and $t_0 \in \mathrm{T}$, there is a $\delta = \delta(\epsilon) > 0$ such that $x_0 \in S(A, \delta)$ implies that $x(t) \in S(B, \epsilon)$, $t \geq t_0$ where $S(A, \delta) = \{x \in \mathrm{R}^n: \|x\| \leq r + |\delta\}$, and $S(B, \epsilon) = \{x \in \mathrm{R}^n: \|x\| \leq \rho + \epsilon\}$;

(*ii*) it is uniformly quasi-asymptotically stable, i.e., given $\eta > 0$, $t_0 \in \mathrm{R}_+$, there exist a $\delta_0 > 0$ and $T = T(\eta)$ such that

$$\| x_0 \| < S(A, \delta_0) \text{ implies that } \| x(t) \| < S(B, \eta) \ t \geq t_0 + T.$$

We need corresponding definitions relative to the comparison equation (2.2). Let $\kappa = \{a \in C_{rd}[\mathrm{T}, \mathrm{R}_+]: a(u)$ is strictly increasing in u, $a(0) = 0$ and $a(u) \to \infty$ as $u \to \infty\}$

Definition 2.4 Let $\Omega=\{u \in R_+: u \leq r_0\}$, for some $r_0 > 0$.

(1) Ω is said to be self-invariant if

$$u_0 \leq r_0 \text{ implies that } u(t) \leq r_0, \qquad t \geq t_0, \qquad t \in \mathrm{T}.$$

(2) Ω is said to be U.A.S. if

(*i*) it is U.S., i.e., given any $\epsilon > 0$ and $t_0 \in \mathbb{T}$, there exists a $\delta = \delta(\epsilon) > 0$ such that $u_0 < r_0 + \delta$ implies $u(t) < r_0 + \epsilon, \quad t \geq t_0$, and

(*ii*) given $\eta > 0$ and $t_0 \in \mathbb{R}_+$, there exists a $\delta_0 > 0$ such $T = T(\eta) > 0$ such that $u_0 > r + \delta_0$ implies that $u(t) < r_0 + \eta, \quad t \geq t_0 + T$.

We now prove the following theorem on U.A.S.

THEOREM 2.1 *Assume that*

(A_0) *There exists* $V \in C_{rd}[\mathbb{T} \times \mathbb{R}^n, R_+]$, $V(t,x)$ *locally Lipschitzian in* x *for each right-dense* $t \in \mathbb{T}$, *such that, for* $(t, x) \in \mathbb{T} \times \mathbb{R}^n$,

$$b(|x|) \leq V(t, x) \leq a(|x|)$$

where $a, b \in \mathcal{K}$, *and*

$$D^+V^\Delta(t, x) \leq g(t, V(t, x), |\lambda|)$$

where $g \in C_{rd}[\mathbb{T} \times \mathbb{R}_+ \times \mathbb{R}_+, \mathbb{R}]$;

(A_1) *For each* $\lambda \in \mathbb{R}^d$, *there exists* $r = r(\lambda) > 0$, $\rho = \rho(\lambda) > 0$, *such that* $a(r) = b(\rho)$, *and* $r(\lambda)$, $\rho(\lambda) \to 0$ *as* $|\lambda| \to 0$;

(A_2) *The set* $\Omega = \{u \in \mathbb{R}_+ : 0 \leq u \leq a(r)\}$ *is invariant and U.A.S.*

Then, the set $B = \{x \in \mathbb{R}^n: |x| \leq \rho\}$ *is conditionally invariant with respect to* $A = \{x \in \mathbb{R}^n: |x| \leq r\}$ *and it is U.A.S.*

Proof Suppose that the set Ω is invariant with respect to (2.2) and U.A.S. We shall first show that the set $B = \{x \in \mathbb{R}^n: |x| \leq \rho\}$ is conditionally invariant with respect to $A = \{x \in \mathbb{R}^n: |x| \leq r\}$ relative to (2.1).

Since Ω is invariant relative to (2.2), we have

$$u_0 \leq a(r) \text{ implies that } u(t) \leq a(r), \qquad t \geq t_0 \tag{2.4}$$

where $u(t)$ is any solution of (2.2). We need to prove that

$$|x_0| \leq r \text{ implies that } |x(t)| \leq \rho, t \geq t_0, \ t \in \mathbb{T}.$$

If not, there would exist a solution $x(t)$ of (2.1) with $|x_0| \leq r$ and $t_2 > t_0$, $t_2 \in \mathbb{T}$, such that $|x_0| \leq r$ but $|x(t_2)| > \rho$ and $|x(t)| \leq \rho$.

By Theorem 1.2 we have,

$$V(t, x(t)) \leq w(t), \ t_0 \leq t_0 \leq t_2$$

where $w(t) = w(t, |\lambda|, t_0, u_0)$ is the maximal solution of (2.2). Take $u_0 = V(t_0, x_0)$ so that when $|x_0| \leq r$, we have

$$u_0 = V(t_0, x_0) \leq a(|x_0|) \leq a(r).$$

As a result, it follows, in view of (2.4) and (A_1), that

$$b(\rho) < b(|x(t_2)|) \leq V(t_2, x(t_2)) \leq w(t_2) \leq a(r) = b(\rho),$$

which is a contradiction. Hence, B is conditionally invariant with respect to A.

Next, we need to prove U.S. of the conditionally invariant set B with respect to A. Let $\epsilon > 0$ and $t_0 \in \mathrm{T}$ be given. Choose $\delta_1 = \delta_1(\epsilon) > 0$ such that $a(r + \delta_1) < b(\rho + \epsilon)$. Since the invariant set Ω is U.S., we have, for some $\delta \leq \delta_1$,

$$u_0 < a(r + \delta) \Rightarrow u(t) < a(r + \delta_1) \qquad \text{for } t \geq t_0,\ t \in \mathrm{T}.$$

Claim $|x_0| \leq r + \delta \Rightarrow |x(t)| < \rho + \epsilon$, $t \geq t_0$. If this is not true, then there exists a solution of (2.1) and $t_1 > t_0$ such that

$$|x(t_1)| \geq \rho + \epsilon.$$

Choose $u_0 = V(t_0, x_0)$. Then, as before, by Theorem 2.4 we have, since $u_0 = V(t_0, x_0) \leq a(|x_0|) < a(r + \delta)$,

$$b(\rho + \epsilon) \leq b(|x(t_1)| \leq V(t_1, x(t_1)) \leq w(t_1) < a(r + \delta_1) < b(\rho + \epsilon).$$

This contradiction proves U.S. of Ω.

Finally, to prove U.A.S., fix $\epsilon = \rho_0$ and designate $\delta_0^0 = \delta(\rho_0)$. Since Ω is U.A.S., given $0 < \eta < \rho_0$ and $t_0 \in \mathrm{T}$, there exists a $\delta_0 > 0$ and $T = T(\eta) > 0$ such that

$$u_0 < a(r + \delta_0^*) \text{ implies that } u(t) < a(r + \eta), \qquad t \geq t_0 + T.$$

Choose $\delta_0 = \min(\delta_0^0, \delta_0^*)$ so that we have, by U.S. of B, $|x_0|$ $r + \delta_0$ implies that $|x(t)| < \rho + \rho_0$, $t \geq t_0$. Consequently, with $u_0 \equiv V(t_0, x_0) \leq a(r + \delta)$, we arrive at, for $t \geq t_0 + T$,

$$b(|x(t)|) \leq V(t, x(t)) \leq r(t) < a(r + \eta) < b(\rho + \eta)$$

which implies that $|x(t)| < \rho + \eta$, $t \geq t_0 + T$, provided $|x_0| < r + \delta_0$. The proof is therefore complete.

As an application of Theorem 2.1, we shall study the control of uncertain systems. Consider the dynamical system

$$x^{\Delta} = f(t, x, \mathrm{w}) + B(t, x)F(t, x, u, \mathrm{w}) \tag{2.5}$$

$$x(t_0) = x_0, \qquad t \in \mathrm{T},$$

where $u \in \mathbb{R}^m$ is the control and $w \in \mathrm{O}$ is an uncertain parameter. Let us list the following conditions for convenience.

(*i*) $f \in C_{rd}[\mathrm{T} \times R^n \times \mathrm{O}, R^n]$, $\quad \mathrm{B} \in C_{rd}[\mathrm{T} \times R^n, R^{n \times m}]$ is a matrix and $F \in C_{rd}[\mathrm{T} \times R^n \times R^m \times \mathrm{O}, \mathbb{R}^n]$.

(*ii*) $V \in C^1_{\mathrm{rd}}[\mathrm{T} \times R^n, \mathrm{R}_+]$, and for $(t,x) \in \mathrm{T} \times R^n$

$$V^{\Delta}_{\mathrm{f}}(t,x) \le -c(|x|)$$

$$b(|x|) \le V(t, x) \le a(|x|),$$

where a, b, $c \in k$.

(*iii*) For each $\mathrm{w} \in \mathrm{O}$

$$u^T F(t, x, u, \omega) \ge -\beta_1(t, x, \omega)\, |u| + \beta_2(t, x, \omega)|u|^2$$

where β_1, $\beta_2 \in C_{\mathrm{rd}}[\mathrm{T} \times R^n \times \mathrm{O}, \mathrm{R}_+]$ satisfying

$$\beta_1(t, x, \omega) \le \beta_2(t, x, \omega)\rho(t,x),$$

$$\beta_1(t,x,w) \le k(t, x),$$

p, $k \in C_{\mathrm{rd}}[\mathrm{T} \times R^n, R_+]$.

(*iv*) $p = \{ p_\lambda \in C_{rd}[\mathrm{T} \times R^n, R^n] : \lambda > 0\}$ is the stabilizing family of controllers satisfying

$$|\alpha(t, x)|p_\lambda(t, x) = -|p_\lambda(t, x)|\ \alpha(t, x)$$

(i.e., p_λ is of opposite direction to α), and

$$|\eta| > 0 \Rightarrow |p_\lambda| \ge p[1 - \frac{\lambda}{|\eta|}]$$

where $\alpha = B^{\mathrm{T}} V_x^T$ and $\eta(t, x) = k(t,x) \cdot \alpha(t, x)$.

We can now prove the following result as an application of Theorem 2.1. Let A and B be the sets defined in (2.3).

THEOREM 2.2: *Suppose that the conditions (i)–(iv) hold. Then the set B is conditionally invariant with respect to the set and it is U.A.S.*

Proof. From (*ii*), it follows that $V_f^{\Delta}(t, x) \leq -\gamma(V(t,x))$ where $\gamma(u) = c(a^{-1}(u))$. Using conditions (*iii*) and (*iv*), we obtain

$$\begin{aligned}\alpha(t, x)F(t, x, p_\lambda, \omega) &= -\frac{|\alpha|}{|p_\lambda|} p_\lambda F(t, x, p_\lambda, \omega)\\ &\leq |\alpha| [\beta_1(t, x) - \beta_2(t, x) |p_\lambda|]\\ &\leq |\alpha| [\beta_1(t, x) - \beta_2(t, x)\rho(t, x)(1 - \frac{\lambda}{|\eta|})\\ &\leq \frac{|\alpha| \beta_1(t, x)}{|\eta|}\lambda \leq \lambda.\end{aligned}$$

Hence, it follows that

$$V_f^{\Delta}(t, x) \leq -\gamma(V(t, x)) + \lambda$$

and consequently, $g(t, w, \lambda) = -\gamma(w) + \lambda$. Clearly, $a(r) = a(r(\lambda)) = \gamma^{-1}(\lambda)$, and because of (*ii*), we see that (A_0) and (A_1) of Theorem 2.1 are satisfied. It is therefore enough to show that (A_2) is true relative to (2.2.)

To prove that $0 \leq u_0 \leq a(r) \Rightarrow u(t) \leq a(r)$ for $t \geq t_0$, $t \in \mathbb{T}$, suppose that it is not true. Then, there exists $t^* > t_0$ such that $u(t^*) > a(r)$ and $t_1 \in (t_0, t^*)$ such that $u(t_1) \leq a(r)$ and $u(t_1) \geq a(r)$ for $t_1 \leq t < t^*$. Hence,

$$a(r) < u(t^*) = u(t_1) + \int_{t_1}^{t^*} g(s, u(s), \lambda)ds \leq a(r)$$

since $u(t_1) \leq a(r)$ and $g(t, u, |\lambda|) \leq 0$ if $u \geq a(r)$. This contradiction proves that the set $\Omega = \{u \in \mathbb{R}_+ : 0 \leq u \leq a(r)\}$ is self-invariant.

To prove U.S. of Ω, let $\epsilon > 0$, $t_0 \in \mathbb{T}$ be given. Choose $\delta = \delta(\epsilon)$ such that $a(r + \delta) < a(+\epsilon)$. Then, we claim

$$u_0 < a(r + \delta) \Rightarrow u(t) < a(r + \epsilon), \qquad t \in \mathbb{T}.$$

If not, there exists, $t_1 \geq t_0$ such that $u_0 < a(r + \delta)$ and $u(t_1) \geq a(r + \epsilon)$. This, in turn, yields

$$a(r + \epsilon) \leq u(t_1) < a(r + \delta) < a(r + \epsilon).$$

This contradiction proves U.S. of Ω.

To show that Ω is U.A.S., set $\epsilon = \rho_0$ and $\delta_0 = \delta(\rho_0)$, so that by U.S. of Ω

$$u_0 < a(r + \delta_0) \Rightarrow u(t) < a(r + \rho_0), \qquad t \in \mathbb{T}.$$

Assume $u_0 < a(r + \delta_0)$. Let $0 < \eta < \rho_0$ and $\delta = \delta(\eta)$, $0 < \delta = \delta(\eta) < \delta_0$ of U.S. Let $T > \frac{a(r+\delta_0)}{\gamma(a(r+\delta(\eta)) - \lambda}$. Then, we claim that, for some $t^* \in [t_0, t_0 + T]$, we have $u(t^*) < a(r + \delta)$.

If this is false then we have $u(t) \geq a(r + \delta)$ for $t \in [t_0, t_0 + T]$, *which yields*

$$0 \leq a(r + \delta(\eta)) \leq u(t_0) + T) = u(t_0) + \int_{t_0}^{t_0+T} (- \gamma(u(s)) + \lambda) ds$$

$$\leq a\,(r + \delta_0) + [- \gamma\,(a(r + \delta(\eta) + \lambda]T.$$

In view of the choice of T, this leads to a contradiction. Hence, it follows that, if $u_0 < a(r + \delta_0)$, then $u(t) < \eta$ for $t \geq t_0 + T$, which implies that Ω is U.A.S. and the proof of the theorem is complete.

References

1. A. Aulbach and S. Hilger, A unified approach to continuous and discrete dynamics, *Qualitative Theory of Differential Equations*, **53** Szeged (Hungary), (1988).
2. ———, Linear Dynamic Processes with Inhomogeneous Time-Scales, *Non-linear Dynamics and Quantum Dynamical Systems*, Akademie-Verlag, Berlin, (1990).
3. B. R. Barmish and G. Leitmann, On ultimate boundedness control of uncertain systems in the absence of matching conditions, *IEEE Transactions on Automatic Control*, **AC-27** (1982), 153.
4. M. Corless, and G. Leitmann, Continuous state feedback guaranteeing uniform ultimate boundedness for uncertain dynamical systems, *IEEE Transactions on Automatic Control*, **AC-26** (1981), 1139.
5. ———, Adaptive control of systems in the absence containing uncertain functions and unknown functions with uncertain bounds, *Journal of Optimization Theory and Applications*, **41** (1983), 155.
6. ———, Deterministic control of uncertain systems, *Proceedings of the Conference on Modeling and Adaptive Control*, Sopron, Hungary, *Lecture Notes in Control and Information Sciences*, **105**, Springer Verlag, Berlin (1988), 108.
7. ———, Controller design for uncertain systems via Lyapunov functions, *Proceedings of the American Control Conference*, Atlanta, Georgia (1988), 2019–2025.
8. ———, Deterministic control of uncertain systems via a constructive use of Lyapunov stability theory, *Lecture Notes in Control Theory and Information Sciences*, **143**, Springer Verlag, Berlin, (1989).
9. ———, Deterministic control of uncertain Systems: A Lyapunov theory approach, Chapter 11 of *Deterministic Control of Uncertain Systems*, Edited by A.S.I. Zinober, Peter Peregrinus, London, (1990).
10. M. Corless and J. Manela, Control of uncertain discrete-time systems, *Proceedings of the American Control Conference* Seattle, Washington, (1986).
11. S. Gutman and G. Leitmann, Stabilizing feedback control for dynamical systems with bounded uncertainty, *Proceedings of the IEEE Conference on Decision Control*, Clearwater, Florida (1976).
12. B. Kaymakcalan and S. Leela, A survey of dynamic systems on time scales, *Nonlinear Times and Digest*, **1** (1994), 37–60.
13. V. Lakshmikantham and B. Kaymakcalan, Monotone flows and fixed points for dynamic systems on time scales, to appear in *Computer and Mathematics Applications*.
14. V. Lakshmikantham and S. Leela, *Differential and Integral Inequalities*, Vol. I., Academic Press, New York (1969).
15. M. E. Magana and S.H. Zak, Robust output feedback stabilization of discrete-time uncertain dynamical systems, *IEEE Transactions on Automatic Control* **AC-33** (1988), 1082.