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IMPULSIVE CONTROLLABILITY OF LINEAR DYNAMICAL SYSTEMS WITH APPLICATIONS TO MANEUVERS OF SPACECRAFT*

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Necessary and sufficient conditions for impulsive controllability of linear dynamical systems are obtained, which provide a novel approach to problems that are basically defined by continuous dynamical systems, but on which only discrete-time actions are exercised. As an application, impulsive maneuvering of a spacecraft is discussed.

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1 INTRODUCTION

In this paper, we study impulsive control problems. Such problems arise naturally from a wide variety of applications, such as spacecraft maneuver [2], ecosystems management and inspection processes in operations research. To control an object means to influence its behavior so as to achieve a desired result. There is a large body of literature on continuous control. For basic results of what constitutes the common core of control theory see [5]. Impulsive control is to solve problems that are basically defined by continuous dynamical systems, but on which only discrete-time actions are exercised. An essential benefit of the impulsive control approach may be derived from the fact that such controls are simpler to implement and involve cheaper control mechanisms. For example, in rocket control, impulsive corrections of trajectories may involve mechanisms that are less complex than mechanisms that monitor and correct on-line the flight of the rocket. Thus, if a mechanism for rocket trajectory control based on corrective impulses could be designed, such mechanisms would be less costly than continuous-time flight control mechanisms. Such considerations are also relevant in many economic and management situations where the structure of the control relates to implementation.

The objective of this paper is to develop a systematic mathematical theory of impulsive controllability of linear dynamical systems. We shall describe the problem in Section 2, and establish some preliminary results in Section 3. In Section 4 we shall state and prove our main theorems which give necessary and sufficient conditions for impulsive

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controllability of the system. Finally, in Section 5, we study as an application, the problem of impulsive maneuver of a spacecraft in near-circular orbit and obtain some criteria on controllability of the spacecraft based on various operative thrusters.

2 PROBLEM FORMULATION

Let a physical system be described by the differential equation

$$x' = A(t)x, \tag{2.1}$$

where $x \in \mathbb{R}^n$ and $A \in C[\mathbb{R}, \mathbb{R}^{n\times n}]$. It is desired that the solution $x(t)$ of (2.1) satisfy the following boundary conditions

$$x(t_0) = x_0, \quad x(t_f) = x_f. \tag{2.2}$$

It is well known, however, that a solution to problem (2.1) – (2.2) does not exist in general. Traditionally, the above problem is solved by adding a control term to the right-hand side of equation (2.1), which we refer to as continuous control. The idea of impulsive control is to achieve prescribed states in a prescribed time from arbitrary initial conditions by applying impulses to some components of the physical system.

Let x be partitioned into two parts so that $x^T = (z^T, y^T)$, where $z \in \mathbb{R}^{n-m}$ and $y \in \mathbb{R}^m$. The vector y represents the impulsive portion of the system, i.e. impulses can only be applied in the last m dimensions of the system. To avoid trivial solutions, we always assume $1 \leq m < n$. We say that system (2.1) is impulsively controllable if for any given $(t_0, x_0), (t_f, x_f) \in \mathbb{R}\times \mathbb{R}^n$ with $t_0 < t_f$, there exist $t_i \in (t_0, t_f)$, $i = 1, 2, \ldots, r$, with $t_1 < t_2 < \ldots < t_r$ and $\Delta y_i \in \mathbb{R}^m$, i = 1, 2,..., r such that the following impulsive differential system

$$\begin{cases} x' = A(t)x, & t \neq t_i, \\ x(t_i^+) = x(t_i^-) + \Delta x(t_i), \ i = 1, 2, \cdots, r, \\ x(t_0) = x_0, \end{cases} \tag{2.3}$$

where $\Delta x(t_i)^T = (0, \Delta y_i^T)$, has a solution $x(t)$ existing on $[t_0, t_f]$ such that $x(t_f) = x_f$. See [4] for a detailed discussion of impulsive differential systems.

Let $\Phi(t)$ be a fundamental matrix solution of the system (2.1). Then the system (2.3) has a unique solution $x(t)$ existing on $[t_0, t_f]$ given by

$$x(t) = \begin{cases} \Phi(t)\Phi^{-1}(t_0)x_0, & t \in [t_0, t_1], \\ \Phi(t)\Phi^{-1}(t_i)x(t_i^+), \ t \in (t_i, t_{i+1}], \ i = 1, 2, \cdots, r-1, \\ \Phi(t)\Phi^{-1}(t_r)x(t_r^+), \ t \in (t_r, t_f]. \end{cases} \tag{2.4}$$

It then follows from (2.3) that the system (2.1) is impulsively controllable if and only if

$$\sum_{i=1}^{r} \Phi_m^{-1}(t_i)\Delta y_i = b, \tag{2.5}$$

where $\Phi_m^{-1}(t)$ denotes the rightmost m columns of $\Phi^{-1}(t)$ and $b = \Phi^{-1}(t_f)x_f - \Phi^{-1}(t_0)x_0$. Setting $W = [\Phi_m^{-1}(t_1) | \Phi_m^{-1}(t_2) | \cdots | \Phi_m^{-1}(t_r)]$ and $Y^T = (\Delta y_1^T, \Delta y_1^T, \cdots, \Delta y_r^T)$, then (2.5) can be rewritten into the compact form

$$WY = b. \tag{2.6}$$

Since x_0, x_f, and therefore b, are arbitrary, it follows from elementary linear algebra that the system (2.1) is impulsively controllable if and only if

$$\text{Rank}(W) = n. \tag{2.7}$$

Equation (2.7) establishes a necessary and sufficient condition for impulsive controllability of system (2.1), but it requires the computation of a fundamental matrix and its inverse, which is often forbidding. In the following sections, we shall consider the case when the right-hand side of (2.1) is independent of t and obtain some necessary and sufficient conditions on the matrix A.

3 PRELIMINARY RESULTS

In this section, we consider the linear autonomous system

$$x' = Ax \tag{3.1}$$

where A is an $n \times n$ real matrix. The fundamental matrix solution $\Phi(t)$ of (3.1) is given by

$$\Phi(t) = e^{At}. \tag{3.2}$$

Let J be the upper triangular (complex) Jordan canonical form of A. Then $J = P^{-1}AP$, where P is a nonsingular matrix whose columns are an appropriate set of generalized eigenvectors of A. Clearly

$$\Phi_m^{-1}(t) = Pe^{-Jt}P_m^{-1},$$

and

$$W = [Pe^{-Jt_1}P_m^{-1} | Pe^{-Jt_2}P_m^{-1} | \cdots | Pe^{-Jt_r}P_m^{-1}] = PS,$$

where P_m^{-1} denotes the rightmost m columns of P^{-1} and

$$S = [e^{-Jt_1} P_m^{-1} | e^{-Jt_2} P_m^{-1} | \cdots | e^{-Jt_r} P_m^{-1}]. \tag{3.3}$$

Since P is nonsingular, it follows from (2.7) that the system (3.1) is impulsively controllable if and only if

$$\text{Rank}(S) = n. \tag{3.4}$$

Let the characteristic and minimal polynomials of A be given by

$$p(t) = (t - \lambda_1)^{\alpha_1}(t - \lambda_2)^{\alpha_2} \cdots (t - \lambda_s)^{\alpha_s}$$

and

$$q(t) = (t - \lambda_1)^{\beta_1}(t - \lambda_2)^{\beta_2} \cdots (t - \lambda_s)^{\beta_s}$$

respectively, where $\lambda_1, \lambda_2, \ldots, \lambda_s$ are the distinct eigenvalues of A. Then the Jordan canonical form, J, of A is block diagonal whose diagonal entries are of the form

$$J_{ij} = \begin{pmatrix} \lambda_i & 1 & 0 \cdots 0 & 0 \\ 0 & \lambda_i & 1 \cdots 0 & 0 \\ \cdots & & \cdots & \cdots \\ 0 & 0 & 0 \cdots \lambda_i & 1 \\ 0 & 0 & 0 \cdots 0 & \lambda_i \end{pmatrix} \tag{3.5}$$

with the following properties:

(i) corresponding to each λ_i, the number of J_{ij} equals the geometric multiplicity of λ_i, denoted by k_i;

(ii) denote by l_{ij} the order of J_{ij}, then $l_{ij} \le \beta_i$, $j = 1,2,\ldots,k_i$, and equality holds for at least one j. Furthermore, $\sum_{j=1}^{k_i} l_{ij} = \alpha_i$.

Let $k = \sum_{j=1}^{s} k_i$ and the matrix P be divided correspondingly into k submatrices

$$P = [P_{11}P_{12} \cdots P_{1k_1}P_{21}P_{22} \cdots P_{2k_2} \cdots P_{s1}P_{s2} \cdots P_{sk_s}], \tag{3.6}$$

where P_{ij} is an $n \times l_{ij}$ matrix. Let us label the columns of P_{ij} by $P(i, j, l)$, $l = 1, 2, \ldots, l_{ij}$. Then $PJ = AP$ implies

$$P_{ij}J_{ij} = AP_{ij} \tag{3.7}$$

or equivalently

$$\begin{cases} (A - \lambda_i I)P(i, j, 1) = 0 \\ (A - \lambda_i I)P(i, j, 2) = P(i, j, 1) \\ \cdots \\ (A - \lambda_i I)P(i, j, l_{ij}) = P(i, j, l_{ij} - 1). \end{cases} \tag{3.8}$$

We shall next consider the form of the matrix S defined by (3.3). First we see that $e^{-Jt} = diag[e^{-J_{ij}t}]$, where

$$e^{-J_{ij}t} = e^{-\lambda_i t} \begin{bmatrix} 1 & -t & \frac{t^2}{2} & \cdots & \frac{(-t)^{l_{ij}-1}}{(l_{ij}-1)!} \\ 0 & 1 & -t & \cdots & \frac{(-t)^{l_{ij}-2}}{(l_{ij}-2)!} \\ \cdots & & \cdots & & \cdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}. \tag{3.9}$$

Let $P(i, j, l)_k$ denote the kth component of $P(i, j, l)$ and let $C(i, j, l)_k$ denote its corresponding cofactor of P. Then we have

$$P_m^{-1} = \frac{1}{|P|} \begin{bmatrix} C(1, 1, 1)_{n-m+1} & \cdots & C(1, 1, 1)_n \\ C(1, 1, 2)_{n-m+1} & \cdots & C(1, 1, 2)_n \\ \cdots & \cdots & \cdots \\ C(s, k_s, l_{sk_s})_{n-m+1} & \cdots & C(s, k_s, l_{sk_s})_n \end{bmatrix} \tag{3.10}$$

where $|P|$ is the determinant of the matrix P.

With this notation it is evident that in the product $e^{-Jt} P_m^{-1}$, the block-diagonal nature of e^{-Jt} separates $e^{-Jt} P_m^{-1}$ into k blocks of rows. That is

$$e^{-Jt} P_m^{-1} = \frac{1}{|P|} \begin{bmatrix} \vdots \\ e^{-J_{ij}t} C_{ij} \\ \vdots \end{bmatrix} \tag{3.11}$$

where

$$C_{ij} = \begin{bmatrix} C(i, j, 1)_{n-m+1} & \cdots & C(i, j, 1)_n \\ & \cdots & \cdots \\ C(i, j, l_{ij})_{n-m+1} & \cdots & C(i, j, l_{ij})_n \end{bmatrix}.$$

The last row in each of these blocks will play a crucial role in the following discussion, hence we define the $k_i \times m$ matrix

$$C(i) = \begin{bmatrix} C(i,1,l_{i1})_{n-m+1} & \cdots & C(i,1,l_{i1})_n \\ C(i,2,l_{i2})_{n-m+1} & \cdots & C(i,2,l_{i2})_n \\ \cdots & \cdots & \cdots \\ C(i,k_i,l_{ik_i})_{n-m+1} & \cdots & C(i,k_i,l_{ik_i})_n \end{bmatrix} \tag{3.12}$$

for each $i = 1,\ldots,s$. We note that $|P|^{-1}\ C(i)$ is a submatrix of P_m^{-1} of order $k_i \times m$. Finally, let $L(i)$ denote the submatrix of P of order $n \times (n - k_i)$ formed by eliminating the columns $P(i, j, l_{ij})$, $j = 1,\ldots,k_i$. (These columns are the last columns in each of the submatrices of P_{ij}, $j = 1,\ldots,k_i$, and correspond to the rows of $C(i)$). Also, let $M(i)$ and $N(i)$ denote the first $n - m$ and last m rows of $L(i)$ respectively so that

$$L(i) = \begin{bmatrix} M(i) \\ N(i) \end{bmatrix} \tag{3.13}$$

where $M(i)$ is of order $(n - m) \times (n - k_i)$ and $N(i)$ is of order $m \times (n - k_i)$.

We are ready to prove the following lemmas.

Lemma 3.1 *Rank*$\{M(i)\} = n - m$ *if and only if Rank*$\{C(i)\} = k_i$, *where* $C(i)$ *and* $M(i)$ *are defined by (3.12) and (3.13) respectively.*

Proof: Suppose Rank$\{M(i)\} = n - m$. Since P is nonsingular, Rank$\{L(i)\} = n - k_i$. Thus there must be at least $m - k_i$ rows, $r_1,\ldots,r_{m-k_i}$, of $N(i)$ such that the square matrix

$$M = \begin{bmatrix} M(i) \\ r_1 \\ \vdots \\ r_{m-k_i} \end{bmatrix}$$

is nonsingular, i.e. $|M| \neq 0$. Since M is a $(n - k_i) \times (n - k_i)$ submatrix of P, it follows from Jacobi's theorem [A] that there is a $k_i \times k_i$ complimentary submatrix of $(P^{-1})^T$ formed by a certain k_i columns of the matrix $|P|^{-1}\ C(i)$ with a nonzero determinant. Conversely, if Rank$\{C(i)\} = k_i$, then we must have Rank$\{M(i)\} = n - m$, for otherwise, we would have $|M| = 0$ for any rows $r_1,\ldots,r_{m-k_i}$ of $N(i)$ which implies, by Jacobi's theorem [1], that every minor of order k_i in $C(i)$ is zero. This contradiction completes the proof of Lemma 3.1.

Lemma 3.2 *If Rank*$\{M(i)\} = n - m$, *then for any scalar row vector,* $a = (a_1,\ldots,a_n)$, $ae^{-Jt}P_m^{-1} = 0\ \forall t \in (t_0, t_f)$ *implies* $a = 0$; i.e. *the rows of the matrix* $e^{-Jt}P_m^{-1}$ *are linearly independent as functions of t.*

Proof: We partition the vector a corresponding to (3.6) into k parts, i.e. $a = (a_{11}, a_{12},\ldots,a_{1k_1}, a_{21}, a_{22},\ldots,a_{2k_2},\ldots, a_{s1}, a_{s2},\ldots,a_{sk_2})$. Then by (3.11) $ae^{-Jt}P_m^{-1} = 0$ implies

$$\sum_{i=1}^{s}\sum_{j=1}^{k_i} a_{ij}e^{-J_{ij}t}C_{ij} = 0. \tag{3.14}$$

Since $\lambda_1,\ldots,\lambda_s$ are distinct eigenvalues of A, it follows from (3.9) and (3.14) that

$$\sum_{j=1}^{k_i} a_{ij}e^{-J_{ij}t}C_{ij} = 0, \quad \forall\, i = 1,\cdot\cdot\cdot, s. \tag{3.15}$$

From (3.9) we see that the component functions of $e^{-J_{ij}t}C_{ij}$ are linear combinations of terms of the form $e^{-\lambda_i t}t^{\alpha}$; α is said to be the degree of the term, $0 \le \alpha \le l_{ij} - 1$. From property (ii) of the Jordan canonical form (3.5), we have $0 \le \alpha \le \beta_i - 1$. A row in $e^{-J_{ij}t}C_{ij}$ is said to be of degree d in t if the highest degree non-zero term in the row is of degree d in t.

Now, since $\text{Rank}\{M(i)\} = n - m$ it follows from Lemma 3.1 that $\text{Rank}\{C(i)\} = k_i$, i.e. the rows of $C(i)$ are independent and consequently non-zero. An analysis of (3.9), (3.11) and (3.12) then shows that row l of $e^{-J_{ij}t}C_{ij}$, $1 \le l \le l_{ij}$ is of degree $l_{ij} - l$ in t and the coefficients of the terms of degree $l_{ij} - l$ in this row are given by the j-th row of $C(i)$. Note that for each set (i, j) all the rows of $e^{-J_{ij}t}C_{ij}$ are of different degrees so that for a given value i, if two rows of $e^{-J_{ij}t}C_{ij}$, $j = 1,\ldots,k_i$, have the same degree, the coefficients of their highest degree terms are given by *different* rows in $C(i)$. It then follows from the linear independence of the rows of $C(i)$ that all rows of $e^{-J_{ij}t}C_{ij}$, $1 \le j \le k_i$ of the same degree d in t are linearly independent as functions of t; this is true for each value $d = 0, 1,\ldots\beta_i - 1$. Consequently, since rows of different degree are clearly independent of each other it follows that all the rows of $e^{-J_{ij}t}C_{ij}$, $1 \le j \le k_i$ are linearly independent as functions of t, i.e. for fixed i

$$\sum_{j=1}^{k_i} a_{ij}e^{-J_{ij}t}C_{ij} = 0 \ \text{ implies } \ a_{ij} = 0\ \forall\, j = 1,\cdot\cdot\cdot,k_i. \tag{3.16}$$

Hence by (3.15) and (3.16), $ae^{-Jt}P_m^{-1} = 0$ implies $a = 0$.

Our final Lemma in this section is a result from linear algebra.

Lemma 3.3 *Let A and B be subspaces of an inner product space V, then*

$$(A \cap B)^{\perp} = A^{\perp} + B^{\perp}. \tag{3.17}$$

Proof: Let $v \in A^{\perp} + B^{\perp}$, then $v = x + y$, where $x \in A^{\perp}$, $y \in B^{\perp}$. Let $w \in A \cap B$, then $\langle v, w\rangle = \langle x, w\rangle + \langle y, w\rangle$, but $w \in A$ and $w \in B$ implies $\langle x, w\rangle = 0 = \langle y, w\rangle$. Therefore $\langle v, w\rangle = 0\ \forall v \in A^{\perp} + B^{\perp}$ and $\forall w \in A \cap B$. Hence

$$A^{\perp} + B^{\perp} \subseteq (A \cap B)^{\perp}. \tag{3.18}$$

Now V may be expressed as the direct sum of a subspace and its orthogonal complement, in particular we may write

$$V = (A^{\perp} + B^{\perp})\oplus(A^{\perp} + B^{\perp})^{\perp}. \tag{3.19}$$

Let $v \in (A \cap B)^{\perp}$; by (3.19) we write v uniquely as $v = u + z$ where $u \in A^{\perp} + B^{\perp}$ and $z \in (A^{\perp} + B^{\perp})^{\perp}$. Since $v \in (A \cap B)^{\perp}$ we have

$$0 = \langle v, w \rangle = \langle u, w \rangle + \langle z, w \rangle, \forall\, w \in A \cap B. \tag{3.20}$$

By (3.18), since $u \in A^{\perp} + B^{\perp}$ we have $\langle u, w \rangle = 0$ which substituted into (3.20) equation gives

$$\langle z, w \rangle = 0 \quad \forall\, w \in A \cap B. \tag{3.21}$$

However, $z \in (A^{\perp} + B^{\perp})^{\perp}$ hence $\langle z, x + y \rangle = 0\ \forall x \in A^{\perp}, \forall y \in B^{\perp}$. In particular choosing $x = 0$ and $y = 0$ respectively gives

$$\langle z, y \rangle = 0 \quad \forall\, y \in B^{\perp} \text{ and } \langle z, x \rangle = 0 \quad \forall\, x \in A^{\perp}.$$

Hence $z \in B$ and $z \in A$, *i.e.* $z \in A \cap B$. But by (3.21) z is orthogonal to every member of $A \cap B$ hence $z = 0$. Therefore $v = u \in A^{\perp} + B^{\perp}$; thus

$$(A \cap B)^{\perp} \subseteq A^{\perp} + B^{\perp}, \tag{3.22}$$

and by (3.18) and (3.22) the Lemma is proved.

4 NECESSARY AND SUFFICIENT CONDITIONS

In this section we present two theorems that give necessary and sufficient conditions for impulsive controllability of a system. The first theorem is a condition on the generalized eigenvectors of the matrix A, the second is a condition on the generalized eigenvectors of a certain submatrix of A. We also give several corollaries that apply to special cases.

Theorem 4.1 *Let P be a nonsingular matrix such that $J = P^{-1}AP$ is the upper triangular (complex) Jordan canonical form of A. A necessary and sufficient condition for the system (3.1) to be controllable by impulses in the last m dimensions, $1 \leq m < n$, is*

$$\text{Rank}\{M(i)\} = n - m, \quad \forall\, i = 1, \cdots, s, \tag{4.1}$$

where $M(i)$ is defined by (3.13) and s is the number of distinct eigenvalues of A. Furthermore, the number of impulses, r, required to achieve any desired final state for a controllable system is in the range $\frac{n}{m} \leq r \leq n - m + 1$.

An immediate corollary of Theorem 4.1 provides a necessary condition on the matrix A.

Corollary 4.1 *If the geometric multiplicity of an eigenvalue of A exceeds the number of impulsive dimensions m then the system is not controllable.*

Proof: $M(i)$ is of order $(n - m) \times (n - k_i)$ where k_i is the geometric multiplicity of λ_i, hence if $k_i > m$ then $\text{Rank}\{M(i)\} \leq n - k_i < n - m$.

Proof of Theorem 4.1 Consider the rows $(i, j, l_{ij}), j = 1, \ldots k_i$ of $e^{-Jt}P_m^{-1}$, i.e. the last row in each of the (i, j) blocks of $e^{-Jt}P_m^{-1}$ for some $i = 1, \ldots, s$. From (3.9) we see that the last row of $e^{-J_{ij}t}$ is $[0, 0, \ldots 0, e^{-\lambda_i t}]$ hence by (3.11) and (3.12) the k_i rows of $e^{-Jt}P_m^{-1}$ we wish to consider are

$$\frac{1}{|P|} e^{-\lambda_i t} C(i), \tag{4.2}$$

and the corresponding k_i rows of S, defined by (3.3), are

$$\frac{1}{|P|} C(i) \, [e^{-\lambda_i t_1} I \mid e^{-\lambda_i t_2} I \mid \cdot \ \cdot \ \cdot e^{-\lambda_i t_r} I]. \tag{4.3}$$

Therefore, if $\text{Rank}\{C(i)\} < k_i$, then the k_i rows of S given by (4.3) are linearly dependent implying $\text{Rank}\{S\} < n$ hence by (3.4) the system is uncontrollable. The necessity of the theorem condition (4.1) then follows from Lemma 3.1.

To establish the sufficiency of (4.1) we shall show, with the aid of Lemma 3.2, that it is possible to choose times $t_i \in (t_0, t_f)$, $i = 1, \ldots, r$ with $t_1 < t_2 < \ldots < t_r$ and $\frac{n}{m} \leq r \leq n - m + 1$, such that $\text{Rank}\{S\} = n$.

The number of columns in S is rm hence $\text{Rank}\{S\} = n$ implies $r \geq n/m$; this provides the lower bound for r. For fixed time t_i, we view $e^{-Jt_i}P_m^{-1}$ as a linear transformation T_i: $\mathbb{C}^n \rightarrow \mathbb{C}^m$ defined by $T_i z = [e^{-Jt_i}P_m^{-1}]^T z$, $\forall z \in \mathbb{C}^n$. Similarly we view S as a linear transformation S_r: $\mathbb{C}^n \rightarrow \mathbb{C}^{rm}$ defined by the Cartesian product $S_r z = T_1 z \times \ldots \times T_r z$, $\forall z \in \mathbb{C}^n$. From these definitions it is clear that $KerS_r = \cap_{i=1}^r KerT_i$. The requirement that S have rank n is equivalent to $\dim\{\text{Ker}S_r\} = 0$.

We claim that for any non-zero $b \in \mathbb{C}^n$, $\exists t_i \in (t_0, t_f)$ such that $b \notin \text{Ker}T_i$. If no such t_i exists then $b \in \text{Ker}T_i \ \forall t_i \in (t_0, t_f)$ hence $[e^{-Jt}P_m^{-1}]^T b = 0 \ \forall \, t \in (t_0, t_f)$ and by Lemma 3.2 we must have $b = 0$, a contradiction.

We now construct S. Choose $t_1 \in (t_0, t_f)$. Since e^{-Jt_1} and P^{-1} are nonsingular, $e^{-Jt_1}P_m^{-1}$ has rank m hence $\dim\{\text{Image}T_1\} = m$ and $\dim\{KerT_1\} = n - m = d_1$. Choose nonzero $b \in \text{Ker}T_1$. By our above claim, $\exists t_2 \in (t_0, t_f)$, $t_2 \neq t_1$, such that $b \notin \text{Ker}T_2$. With this choice of t_2, $KerS_2 = \cap_{i=1}^2 KerT_i$ is a *proper* subspace of $\text{Ker}T_1$, hence $d_2 = \dim\{\text{Ker}S_2\} < \dim\{\text{Ker}T_1\} = d_1$. If $d_2 > 0$ we chose a new nonzero $b \in \text{Ker}S_2$ and $t_3 \in (t_0, t_f)$, $t_3 \neq t_1, t_2$, such that $b \notin \text{Ker}T_3$ giving $KerS_3 = \cap_{i=1}^3 KerT_i = KerS_2 \cap KerT_3 \subset KerS_2$ and $d_3 = \dim\{\text{Ker}S_3\} < \dim\{\text{Ker}S_2\} = d_2$. Carrying on in this fashion we obtain $d_r < d_{r-1} < \ldots < d_2 < d_1 = n - m$ which shows that for some $r \leq n - m + 1$, $d_r = 0$. Thus for these times $\{t_1, \ldots, t_r\}$ (which we may rearrange so that $t_1 < t_2, \ldots < t_r$) where $\frac{n}{m} \leq r \leq n - m + 1$, $d_r = 0$ implying $\text{Rank}\{S\} = n$ and by (3.4) the system is controllable.

Remark: Theorem 4.1 does not say that any set of times $\{t_i\}$, $i = 1,\ldots, n - m + 1$, will control a system satisfying (4.1), only that a set $\{t_1,\ldots t_r\}$, $\frac{n}{m} \leq r \leq n - m + 1$, exists which renders the system controllable. In particular, when the component functions of $e^{-Jt}P_m^{-1}$ are periodic, it may be necessary to avoid choosing times that correspond to the period of one of the functions. For example if

$$A = \begin{pmatrix} 0 & 1 & 1 \\ -1 & 1 & 2 \\ 0 & 0 & 1 \end{pmatrix},$$

$m = 1$, then the system $x = Ax$ is controllable with

$$e^{-Jt}P^{-1} = \frac{1}{4}\begin{pmatrix} 2e^{-t} \\ e^{-it}(-3 + i) \\ e^{it}(-3 - i) \end{pmatrix}$$

from which it is seen that for $t_i = n\pi$,

$$\begin{pmatrix} 0 \\ 3 + i \\ -3 + i \end{pmatrix} \in \mathrm{Ker}T_i,$$

hence for $\{0, \pi, 2\pi,\ldots, r\pi\}$, $\dim\{\mathrm{Ker}S_r\} > 0$ and $\mathrm{Rank}\{S\} < n\ \forall r$. However, for $\{0, \pi/3, \pi/2\}$ $\dim\{\mathrm{Ker}S_r\} = 0$ as can be checked.

Theorem 4.1 provides a necessary and sufficient condition for controllability based on the generalized eigenvectors of A; as such, it is a considerable improvement to the original condition given by equation (2.7). Corollary 4.1 provides a necessary condition that requires even less information. One need only compute the eignevalues, λ_i, of A and then determine $\mathrm{Rank}\{A - \lambda_i I\}$. If $n - \mathrm{Rank}\{A - \lambda I\} > m$ then the system is uncontrollable.

We now present a necessary and sufficient condition that requires less information, or at least information easier to obtain, than Theorem 4.1. Let us write the matrix A for the n-dimensional system with impulses in the last m dimensions as

$$A = \left(\begin{array}{c|c} A_1 & A_2 \\ \hline A_3 & A_4 \end{array}\right), \tag{4.4}$$

where A_1, A_2, A_3, and A_4 are $(n - m) \times (n - m)$, $(n - m) \times m$, $m \times (n - m)$ and $m \times m$ matrices. As might be anticipated, the submatrices A_1 and A_2 play a crucial role in the controllability of the system which is revealed by the following theorem.

Theorem 4.2 *Let impulses be applied in the last m dimensions, $1 \leq m < n$, of system (3.1), and A be subdivided according to (4.4). Let $\{\lambda_1,\ldots, \lambda_s\}$ be the set of distinct eigenvalues of A and let $\{\mu_1,\ldots, \mu_\sigma\}$ be the set of distinct eigenvalues of A_1. Define Ψ as $\Psi = \{\lambda_1,\ldots, \lambda_s\} \cap \{\mu_1,\ldots, \mu_\sigma\}$* and for $\nu \in \Psi$ let $E^T(\nu)$ *denote the eigenspace of A_1^T corresponding to* *eigenvalue ν. Finally let $col(A_2)^{\perp}$ denote the orthogonal complement of the column space of A_2. The system is controllable if and only if*

$$\Psi = \varnothing, \tag{4.5}$$

$$\text{or } E^T(\nu) \cap col(A_2)^{\perp} = \{0\} \quad \forall\, \nu \in \Psi. \tag{4.6}$$

Equivalently, if we let $E_h(\nu)$ denote the space spanned by the set of "highest order" generalized eigenvectors of A_1 of ν, then the system is controllable if and only if

$$\Psi = \varnothing,$$

$$\text{or } \mathrm{col}(Q/E_h(\nu)) + col(A_2) = \mathbb{C}^{n-m} \quad \forall\, \nu \in \Psi, \tag{4.7}$$

where Q is a non-singular matrix whose columns are the generalized eigenvectors of A_1 such that $Q^{-1} A_1 Q$ is the upper triangular Jordan canonical form of A_1 and *$Q/E_h(\nu)$ denotes all the columns of Q except those in $E_h(\nu)$.*

Remark: Equation (4.7) simply provides an alternative method for testing a given system for controllability. Whereas (4.6) is in terms of eigenspaces of A_1^T and the orthogonal complement of the column space of A_2, (4.7) is in terms of the generalized eigenvectors of A_1 and the column space of A_2. The method that will require the least amount of work for a given matrix A, will depend partly on the size of the set Ψ and on the number of columns in A_2. The matrix Q has the same format as that described in Section 3 for P, although in this context Q relates to the matrix A_1, rather than the entire matrix A. If we let $\xi(i, j, l)$ denote the columns of Q using the notation of Section 3, then

$$E_h(\nu) = \mathrm{span}\{\xi(i_0, j, l_{i_0 j}) \mid 1 \le j \le k_{i_0}, \lambda_{i_0} = \nu\}. \tag{4.8}$$

Proof of Theorem 4.2 Taking the orthogonal complement of (4.6) and applying Lemma 3.3 to the left side gives

$$E^T(\nu)^{\perp} + \mathrm{col}(A_2) = \mathbb{C}^{n-m}\, \forall\, \nu \in \Psi. \tag{4.9}$$

The equivalence of (4.6) and (4.7) will follow from (4.9) when we have showed that

$$E^T(\nu)^{\perp} = \mathrm{col}(Q/E_h(\nu)). \tag{4.10}$$

Let f_ν be the number of independent eigenvectors of A_1^T of ν, then

$$\dim\{E^T(\nu)\} = f_\nu \tag{4.11}$$

$$\dim\{E^T(\nu)^{\perp}\} = (n - m) - f_\nu \tag{4.12}$$

Now, from (4.8)

$$\dim\{E_h(\nu)\} = k_{i_0} \tag{4.13}$$

where k_{i_0} is the geometric multiplicity of eigenvalue $\lambda_{i_0} = \nu$; however, since $\text{Rank}\{A_1 - \nu I\} = \text{Rank}\{A_1^T - \nu I\}$ we must have $k_{i_0} = f_\nu$. Since Q is nonsingular, $Q/E_h(\nu)$ has $n - m - f_\nu$ independent columns. To confirm (4.10) we will show that all of these columns are orthogonal to the space $E^T(\nu)$.

Let v be an eigenvector of A_1^T of ν, i.e. $v \in E^T(\nu)$, and let $\xi(i, j, l)$ be the generalized eigenvectors of A_1, then

$$(A_1^T - \nu I)v = 0, \tag{4.14}$$

$$(A_1^T - \lambda_i I)\xi(i, j, l) = \xi(i, j, l - 1) \text{ where } \xi(i, j, 0) = 0. \tag{4.15}$$

Transposing (4.14) and multiplying by $\xi(i, j, l)$ gives

$$v^T(A_1 - \nu I)\xi(i, j, l) = 0. \tag{4.16}$$

Substituting for $A_1\xi(i, j, l)$ from (4.15) gives

$$v^T((\lambda_i - \nu)\xi(i, j, l) + \xi(i, j, l - 1)) = 0. \tag{4.17}$$

If $\lambda_i \neq \nu$ then induction on l applied to (4.17) gives

$$v^T\xi(i, j, l) = \quad 0, \forall\, i, j, l \text{ such that } \lambda_i \neq \nu. \tag{4.18}$$

If $\lambda_i = \nu$ (which must occur since the eigenvalues of A_1 and A_1^T are identical) then (4.17) collapses to

$$v^T\xi(i, j, l - 1) = 0 \qquad \lambda_i = \nu, \quad \forall\, j, l. \tag{4.19}$$

Equations (4.18) and (4.19) hold for all $v \in E^T(\nu)$, and show that $E^T(\nu)$ is orthogonal to all the columns of Q except the "highest order" generalized eigenvectors of A_1 of ν, i.e. all columns of Q except $\{\xi(i_0, j, 1_{i_0}j) \mid 1 \leq j \leq k_{i_0}, \lambda_{i_0} = \nu\}$. This confirms (4.10) and hence establishes the equivalence of (4.6) and (4.7).

We are now ready to prove the theorem itself. Our approach will be to show that uncontrollability implies that (4.5), (4.6) must fail and that failure of (4.5), (4.6) implies uncontrollability.

Again we use the notation of the previous section and write the generalized eigenvalues of A, $p(i, j, l)$, $1 \leq i \leq s$, $1 \leq j \leq k_i$, $1 \leq l \leq l_{ij}$; as

$$p(i, j, l) = \begin{pmatrix} u(i, j, l) \\ w(i, j, l) \end{pmatrix} \text{ where } u(i, j, l) \in \mathbb{C}^{n-m} \text{ and } w(k, j, l) \in \mathbb{C}^{\mathrm{m}}.$$

The generalized eigenvectors satisfy

$$(A - \lambda_i I)\begin{pmatrix} u(i,j,l) \\ w(i,j,l) \end{pmatrix} = \begin{pmatrix} u(i,j,l-1) \\ w(i,j,l-1) \end{pmatrix}, \tag{4.20}$$

where we define $\begin{pmatrix} u(i,j,0) \\ w(i,j,0) \end{pmatrix} = 0$. The first $n - m$ equations of (4.20) may be written as

$$A_2 w(i,j,l) = (\lambda_i I - A_1)u(i,j,l) + u(i,j,l-1). \tag{4.21}$$

Suppose the system is uncontrollable. Then by Theorem 4.1, $\text{Rank}\{M(i_0)\} < n - m$ for some i_0, $1 < i_0 \leq s$. $M(i_0)$ has $n - m$ rows, since $\text{Rank}\{M(i_0)\} < n - m$, $\exists$ nonzero $b \in \mathbb{C}^{n-m}$ such that $b^T M(i_0) = 0$, i.e. the rows of $M(i_0)$ are dependent. Define

$$a = \begin{pmatrix} (A_1^T - \lambda_{i_0} I)b \\ A_2^T b \end{pmatrix} \in \mathbb{C}^n, \tag{4.22}$$

and consider the product $a^T P$ where P is the matrix of generalized eigenvectors as in Theorem 4.1,

$$\begin{aligned} a^T P &= (b^T A_1 - \lambda_{i_0} b^T, b^T A_2)\left[\cdots \begin{vmatrix} u(i,j,l) \\ w(i,j,l) \end{vmatrix} \cdots\right] \\ &= [\cdots | b^T A_1 u(i,j,l) - \lambda_{i_0} b^T u(i,j,l) + b^T A_2 w(i,j,l) | \cdots]. \end{aligned} \tag{4.23}$$

Substituting for $A_2 \omega(i,j,l)$ from (4.21) gives

$$a^T P = [\cdots | (\lambda_i - \lambda_{i_0}) b^T u(i,j,l) + b^T u(i,j,l-1) | \cdots]. \tag{4.24}$$

Now $M(i_0)$ contains all the vectors $u(i,j,l)$ as columns except the set $\{u(i_0, j, l_{i_0} j) \mid 1 \leq j \leq k_{i_0}\}$, i.e. all vectors u except the last vector in each block corresponding to eigenvalue λ_{i_0}. Therefore, since $b^T M(i_0) = 0$, we have, in particular, $b^T u(i,j,l-1) = 0\ \forall i,j,l$ and $b^T u(i,j,l) = 0\ \forall i,j,l;\ i \neq i_0$; and, since $\lambda_i - \lambda_{i_0} = 0$ for $i = i_0$, the right side of (4.24) vanishes. Thus uncontrollability implies

$$a^T P = 0, \tag{4.25}$$

and, since P is nonsingular, (4.25) implies

$$a = 0. \tag{4.26}$$

Substituting (4.22) into (4.26) we require

$$b^T A_2 = 0 \tag{4.27}$$

$$\text{and } (A_1^T - \lambda_{i_0} I)b = 0 \tag{4.28}$$

Equation (4.27) says that $b \in \text{col}(A_2)^\perp$, while equation (4.28) says that λ_{i_0} is an eigenvalue of A_1^T, say $\lambda_{i_0} = \nu$, and $b \in E^T(\nu)$. Therefore uncontrollability implies $\Psi \neq \emptyset$ and $E^T(\nu) \cap col\{A_2\}^\perp \neq \{0\}$, i.e. (4.5) and (4.6) fail.

Conversely, suppose $\Psi \neq \emptyset$ and $E^T(\nu) \cap \text{col}(A_2)^\perp \neq \{0\}$ for some $\nu \in \Psi$, i.e. $\exists$ nonzero $b \in E^T(\nu) \cap \text{col}(A_2)^\perp$. Define a as in (4.22) with $\lambda_{i_0} = \nu$. Since ν is an eigenvalue of A_1^T and $b \in E^T(\nu)$, we have $(A_1^T - \lambda_{i_0} I)b = 0$, also since $b \in \text{col}(A_2)^\perp$ we have $A_2^T b = 0$, hence $a = 0$ which substituted into (4.24) gives

$$0 = [\cdot\ \cdot\ \cdot|(\lambda_i - \lambda_{i_0})b^T u(i,j,l) + b^T u(i,j,l-1)|\cdot\ \cdot\ \cdot]. \tag{4.29}$$

For the columns in (4.29) where $i = i_0$ we have $\lambda_i - \lambda_{i_0} = 0$ and therefore

$$b^T u(i_0, j, l-1) = 0 \ \ \forall\, 1 \leq j \leq k_{i_0},\ 1 \leq l \leq l_{ij}. \tag{4.30}$$

For columns where $i \neq i_0$ then $\lambda_i - \lambda_{i_0} \neq 0$ since the eigenvalues are distinct. Then, since $u(i, j, 0) = 0$ by definition, we have, for $l = 1$

$$b^T u(i,j,l) = 0 \quad \forall\, i \neq i_0,\ 1 \leq j \leq k_i. \tag{4.31}$$

Using (4.31) as the base case, we have by induction on l that

$$b^T u(i,j,l) = 0, \quad \forall\, i \neq i_0,\ 1 \leq j \leq k_i,\ 1 \leq l \leq l_{ij}. \tag{4.32}$$

Equations (4.30) and (4.32) together show that b is orthogonal to all vectors $u(i, j, l)$ except the set $\{u(i_0, j, l_{i_0 j}) \mid 1 \leq j \leq k_{i_0}\}$, i.e. $b^T M(i_0) = 0$. Therefore, since $b \neq 0$, the $n - m$ rows of $M(i_0)$ are dependent, hence $\text{Rank}\{M(i_0)\} < n - m$ and by Theorem 4.1 the system is uncontrollable.

An important corollory that provides a sufficient condition for controllability follows from Theorem 4.2.

Corollary 4.2 *If Rank$\{A_2\} = n - m$ then the system is controllable.*

Proof: $\text{Rank}\{A_2\} = n - m$ implies $\dim\{\text{col}(A_2)\} = n - m$ and since $\mathbb{C}^{n-m} = \text{col}(A_2) \oplus \text{col}(A_2)^\perp$, we must have $\dim\{\text{col}(A_2)^\perp\} = 0$ and $\text{col}(A_2)^\perp = \{0\}$ hence (4.6) will hold whenever (4.5) fails and by Theorem 4.2 the system is therefore controllable.

Note that since A_2 is $(n - m) \times m$, the condition $\text{Rank}\{A_2\} = n - m$ implies $n - m \leq m$, i.e. $m \geq \frac{n}{2}$ so that Corollary 4.2 only applies to systems where impulses are allowed in at least half of the dimensions. An important application of Corollory 4.2 can be stated as follows.

A second order linear system of the form

$$x'' = Ax + Bx'$$

is always controllable when impulses are allowed in all of the velocity variables. To see this, simply reduce the system to a first order system of the form

$$\begin{pmatrix} x \\ v \end{pmatrix}' = \begin{pmatrix} 0 & I \\ A & B \end{pmatrix} \begin{pmatrix} x \\ v \end{pmatrix} \text{ where } v = x'.$$

Since I is nonsingular, Corollory 4.2 immediately shows that the system is controllable when impulses are allowed in all of the velocity dimensions.

An expected result for the special case $A_2 = 0$ can also be stated as a corollary of Theorem 4.2.

Corollory 4.3 *If $A_2 = 0$ then the system is uncontrollable.*

Proof:

$$det(A - \lambda I) = det \left(\begin{array}{c|c} A_1 - \lambda I & 0 \\ \hline A_3 & A_4 - \lambda I \end{array} \right) = det(A_1 - \lambda I) \cdot det(A_4 - \lambda I)$$

thus all eigenvalues of A_1 are also eigenvalues of A, i.e. (4.5) fails. Furthermore $A_2 = 0$ $\Rightarrow \text{col}(A_2) = \{0\} \Rightarrow \text{col}(A_2)^\perp = \mathbb{C}^{n-m}$ hence $E^T(\nu) \cap \text{col}\,(A_2)^\perp = E^T(\nu) \neq \{0\}$, i.e. (4.6) fails. Therefore by Theorem 4.2 the system is uncontrollable.

We now turn our consideration to the special cases $m = n - 1$, and $m = n - 2$.

Corollory 4.4 *Let impulses be applied in the last $n - 1$ dimensions. Then system (3.1) is controllable if and only if at least one of the entries $a_{12},\ldots a_{1n}$ of A is nonzero.*

Proof: $A_2 = [a_{12},\ldots,a_{1n}]$ If $A_2 = 0$ then the system is uncontrollable by Corollory 4.3, if one of $a_{12},\ldots, a_{1n}$ is non-zero then $\text{Rank}\{A_2\} = 1 = n - m$ and the system is controllable by Corollory 4.2.

Corollory 4.5 *Let $m = n - 2$. The system (3.1) is controllable if and only if $col(A_2) \not\subseteq span\{v\}$ where v is an eigenvector of A_1.*

Proof: A_2 is a $2 \times (n - 2)$ matrix. If $\text{Rank}\{A_2\} = 2$ then $\dim\{\text{col}(A_2)\} = 2$ hence $\text{col}(A_2) \not\subseteq \text{span}\{v\}$ and by Corollory 4.2 the system is controllable. If $\text{Rank}\{A_2\} = 0$ then clearly $\text{col}(A_2) = \{0\} \subseteq \text{span}\{v\}$ and by Corollory 4.3 the system is uncontrollable. It remains to consider $\text{Rank}\{A_2\} = 1$. If $\text{Rank}\{A_2\} = 1$ and the system is uncontrollable then $\text{col}(Q/E_h(\lambda_1)) + \text{col}(A_2) \neq \mathbb{C}^2$ for some $\lambda_1 \in \Psi$. Since $\dim\{\text{col}(A_2)\} = 1$ and $\dim\{\mathbb{C}^2\} = 2$ this means that

$$\text{col}(Q/E_h(\lambda_1)) \subseteq \text{col}(A_2) \tag{4.33}$$

and

$$\dim\{\text{col}(Q/E_h(\lambda_1))\} \leq \dim\{\text{col}(A_2)\} = 1. \tag{4.34}$$

Now Q and $E_h(\lambda_1)$ are of three possible forms

	2 distinct eigenvalues	1 eigenvalue-1 eigenvector	1 eigenvalue-2 eigenvectors
Q	$[v(1, 1, 1) \;\; v(2, 1, 1)]$	$[v(1, 1, 1) \;\; v(1, 1, 2)]$	$[v(1, 1, 1) \;\; v(1, 2, 1)]$
$E_h(\lambda_1)$	$\text{span}\{v(1, 1, 1)\}$	$\text{span}\{v(1, 1, 2)\}$	$\text{span}\{v(1, 1, 1) \;\; v(1, 2, 1)\}$

(4.35)

where we use the notation of Section 3. Hence $\text{col}(Q/E_h(\lambda_1)) = \text{span}\{v(i, 1, 1)\}$ $i = 1$ or $i = 2$, or $\text{col}(Q/E_h(\lambda_1)) = \{0\}$. If $\text{col}(Q/E_h(\lambda_1)) = \text{span}\{v(i, 1, 1)\}$ then (4.34) and (4.33) imply $\text{col}(A_2) = \text{span}\{v(i, 1, 1)\}$. If $\text{col}(Q/E_h(\lambda_1)) = \{0\}$ then all non-zero vectors in $\mathbb{C}^2$ are eigenvectors of A_1 and again (4.34) gives $\text{col}(A_2) = \text{span}\{v\}$ where v is some eigenvector of A_1.

Conversely, suppose $\text{col}(A_2) = \text{span}\{v\}$ where $A_1 v = \lambda_1 v$, i.e. $v = v(1, 1, 1)$. If A_1 has only one eigenvalue but two independent eigenvectors, then $\text{Rank}\{A_1 - \lambda_1 I\} = 0$ and since $\text{Rank}\{A_2\} = 1$ we have

$$\text{Rank}\{A_1 - \lambda_1 I | A_2\} = 1$$

which implies

$$\text{Rank}\{A - \lambda_1 I\} < n.$$

Thus λ_1 is an eigenvalue of A, i.e. $\Psi \neq \emptyset$, (4.5) fails. Furthermore, (4.35) shows that $\text{col}(Q/E_h(\lambda_1)) = \{0\}$ and hence

$$\text{col}(Q/E_h(\lambda_1)) + \text{col}(A_2) = \text{span}\{v\} \neq \mathbb{C}^2$$

so that (4.7) fails and by Theorem 4.2 the system is uncontrollable. If A_1 has one eigenvalue and only one eigenvector then $\exists$ nonzero $v(1, 1, 2)$ satisfying

$$(A_1 - \lambda_1 I)v(1, 1, 2) = v(1, 1, 1) = v \neq 0. \tag{4.36}$$

If A_1 has two eigenvalues $\lambda_2 \neq \lambda_1$ then $v = v(1, 1, 1)$ satisfies

$$(A_1 - \lambda_2 I)v(1, 1, 1) = (\lambda_1 - \lambda_2)v(1, 1, 1) \neq 0. \tag{4.37}$$

Equations (4.36) and (4.37) imply respectively that

$$1 = \text{Rank}\{A_1 - \lambda_i I\} = \text{Rank}\{A_1 - \lambda_i I \mid v(1, 1, 1)\}, i = 1, 2, \tag{4.38}$$

hence

$$\text{Rank}\{A - \lambda_i I\} < n, i = 1, 2, \tag{4.39}$$

so that λ_1, (λ_2) is an eigenvalue of A as well as A_1, i.e. (4.5) fails. Furthermore, for the one eigenvalue-one eigenvector case (4.35) shows that

$$\operatorname{col}(Q/E_h(\lambda_1)) = \operatorname{span}\{v(1, 1, 1)\},$$

while for the two eigenvalue case

$$\operatorname{col}(Q/E_h(\lambda_2)) = \operatorname{span}\{v(1, 1, 1)\}.$$

Thus for these two cases

$$\operatorname{col}(Q/E_h(\lambda_i)) + \operatorname{col}(A_2) = \operatorname{span}\{v(1, 1, 1)\} \neq \mathbb{C}^2,$$

$i = 1, 2$ respectively so that (4.7) fails and the system is uncontrollable.

We remark that the special cases $n = 2$ and $n = 3$ are completely described by the preceding corollories. In particular, the only non-trivial case for $n = 2$ is $m = 1$ which is covered by Corollory 4.4, whereas the two cases of interest ($m = 1$, $m = 2$) for $n = 3$ are covered by Corollory 4.5 and Corollory 4.4 respectively.

To complete this section, we give a result on impulsive controllability of nonhomogeneous systems

$$x' = A(t)x + B(t), \tag{4.40}$$

where B: $\mathbb{R} \to \mathbb{R}^n$ is continuous.

Theorem 4.3 *The system (4.40) is controllable if and only if the system (2.1) is controllable.*

Proof: The solution to (4.40) with initial value $x(t_0) = x_0$ is given by

$$x(t) = \Phi(t)\left[\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds\right], \tag{4.41}$$

where $\Phi(t)$ is a fundamental matrix of $A(t)$. Equation (2.4) is then replaced by

$$x(t) = \begin{cases} \Phi(t)[\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t} \Phi^{-1}(s)B(s)ds], & t \in [t_0, t_1] \\ \Phi(t)[\Phi^{-1}(t_i)x(t_i^+) + \int_{t_i}^{t} \Phi^{-1}(s)B(s)ds], & t \in (t_i, t_{i+1}], i = 1,\cdots, r-1,. \\ \Phi(t)[\Phi^{-1}(t_r)x(t_r^+) + \int_{t_r}^{t} \Phi^{-1}(s)B(s)ds], & t \in (t_r, t_f] \end{cases} \tag{4.42}$$

Applying the impulses at times $t_1,\ldots, t_r$ as given by (2.3) yields

$$x(t_r^+) = \Delta x_r + x(t_r^-) = \Phi(t_r)\left[\sum_{i=1}^{r} \Phi^{-1}(t_i)\Delta x_i + \Phi^{-1}(t_1)x(t_1^-) + \int_{t_1}^{t_r} \Phi^{-1}(s)B(s)ds\right]. \tag{4.43}$$

Extending (4.43) to the terminals (t_0, t_f) by

$$x_f = \Phi(t_f)\left[\Phi^{-1}(t_r)x(t_r^+) + \int_{t_r}^{t_f} \Phi^{-1}(s)B(s)ds\right]$$

$$x(t_1^-) = \Phi(t_1)\left[\Phi^{-1}(t_0)x_0 + \int_{t_0}^{t_1} \Phi^{-1}(s)B(s)ds\right], \tag{4.44}$$

gives

$$\sum_{i=1}^{r} \Phi^{-1}(t_i)\Delta x_i = \Phi^{-1}(t_f)x_f - \Phi^{-1}(t_0)x_0 - \int_{t_0}^{t_f} \Phi^{-1}(s)B(s)ds, \tag{4.45}$$

which is identical to equation (2.5) with the addition of the integral on the right hand side. As in Section 2 this collapses to

$$Wy = b - \int_{t_0}^{t_f} \Phi(s)B(s)ds. \tag{4.46}$$

The quantities on the right side of (4.46) are known hence, since b is arbitrary, the condition for a solution y, $\forall b \in \mathbb{C}^n$ is

$$\mathrm{Rank}\{W\} = n, \tag{4.47}$$

which is identical to (2.7), the condition for controllability of the homogeneous system (2.1).

5 APPLICATIONS

As an application to the preceding theory we shall consider impulsive maneouver of a space-craft in near-circular orbit. The linearized equations of motion for such an orbit are usually referred to as the Clohessy-Wiltshire equations [3] and are given by

$$\begin{cases} \delta r'' = 3\omega^2 \delta r + 2aw\delta\theta' \\ a\delta\theta'' = -2\omega\delta r' \\ \delta z'' = -\omega^2 \delta z \end{cases} \tag{5.1}$$

where δr, $a\delta\theta$, and δz are the first-order variations in the radical, circumferential and out-of-plane components of position variation, a is the radius of the reference orbit, and ω is the mean motion. The validity of the linearization requires that the velocity variations as well as the position components $\delta r/a$ and δz be small. Since the gravitational attraction is independent of θ, $\delta\theta$ is unrestricted. Consequently, the domain of validity for equations (5.1) is a torus about the circular reference orbit.

Written as a first order system, $y' = Ay$, (5.1) becomes

$$\begin{pmatrix} x \\ \\ \\ \\ v \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 3\omega^2 & 0 & 0 & 0 & 2\omega & 0 \\ 0 & 0 & 0 & -2\omega & 0 & 0 \\ 0 & 0 & -\omega^2 & 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} x \\ \\ \\ \\ v \end{pmatrix}, \tag{5.2}$$

where $x = (\delta r \ a\delta\theta \ \delta z)^{\mathrm{T}}$ and $v = x'$. In this situation, an impulse, Δv in the velocity dimensions represents an instantaneous change in velocity with no change in position and as such is an approximation to the effect of a short-duration thrust applied to the spacecraft. This approximation is commonly used when dealing with spacecraft maneouver problems for several reasons. First, fuel burns are usually short when compared to the length of time of the maneouver so that the idealization does not embody a significant deviation from reality. Second, the trajectory of the spacecraft between impulses is given by the free-fall orbit of a body in the gravitational field which is in general much easier to determine than the orbit of a body under the influence of a thrusting force. For example, the free-fall orbit of a body in an inverse square field is one of the well-known Keplerian conics whereas the orbit of a thrusting body in the same field is much more complicated. Third, spacecraft maneouver problems are usually concerned with the optimization of some quantity. Since the optimization process is generally insensitive to details, even in situations where the idealized model is poor, the analysis of the model can provide a good first estimate and the results can be used as initial data for iterative schemes that utilize more accurate models.

If we allow impulses in all three velocity dimensions then Corollory 4.2 applied to our system, (5.2), immediately shows that the system is controllable. What if the radial thruster failed? Could the spacecraft still achieve any desired final state, $(x_f^T, v_f^T)^T$, (within the validity of the linearization) using only the tangential thruster and out-of-plane thruster? For impulses in the last two dimensions only, we have

$$A_1 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 3\omega^2 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \\ 2\omega & 0 \end{pmatrix}. \tag{5.3}$$

Computation shows that the eignevalues of A and A_1 are respectively $\{0, i\omega, -i\omega\}$ and $\{0, \sqrt{3}\omega, -\sqrt{3}\omega\}$ so that $\Phi = \{0\}$ is non-empty. The eigenspace of A_1^T of eigenvalue 0 is given by

$$E^T(0) = \text{span}\left\{\begin{pmatrix}0\\1\\0\\0\end{pmatrix}, \begin{pmatrix}0\\0\\1\\0\end{pmatrix}\right\}.$$

¿From (5.3) it is also clear that $\text{col}(A_2)^{\perp}$ is a two-dimensional space given by

$$\text{col}(A_2)^{\perp} = \text{span}\left\{\begin{pmatrix}1\\0\\0\\0\end{pmatrix}, \begin{pmatrix}0\\-2\omega\\0\\1\end{pmatrix}\right\}.$$

It is now easy to see that $E^T(0) \cap \text{col}(A_2)^{\perp} = \{0\}$, hence equation (4.6) is satisfied and by Theorem 4.2 the system is controllable.

What if the tangential or out-of-plane thrusters failed? By re-arranging the variables i.e. switching some rows and columns in the matrix A, we could obtain a new A_1 and A_2 and proceed as in the previous case using the test of Theorem 4.2 However, to show the utility of Theorem 4.1 we shall keep the same matrix and calculate its generalized eigenvectors. Label the eigenvalues of A as $\lambda_1 = 0$, $\lambda_2 = i\omega$, $\lambda_3 = -i\omega$. Each of these eigenvalues has algebraic multiplicity two, however λ_1 has only one independent eigenvector. Some computation gives the generalized eigenvectors of A as $P = [P_{11} \mid P_{21} \mid P_{22} \mid P_{31} \mid P_{32}]$

where

$$P_{11} = \begin{pmatrix}0 & -2/3\omega\\1 & 1\\0 & 0\\0 & 0\\0 & 1\\0 & 0\end{pmatrix} P_{21} = \begin{pmatrix}0\\0\\1\\0\\0\\i\omega\end{pmatrix} P_{22} = \begin{pmatrix}1\\2i\\0\\i\omega\\-2\omega\\0\end{pmatrix}$$

$$P_{31} = \begin{pmatrix}0\\0\\1\\0\\0\\-i\omega\end{pmatrix} P_{32} = \begin{pmatrix}1\\-2i\\0\\-i\omega\\-2\omega\\0\end{pmatrix} \tag{5.4}$$

For impulses is only the tangential and out-of-plane directions we have

$$M(1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2i & 0 & -2i \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & i\omega & 0 & -i\omega \end{bmatrix}, \tag{5.5}$$

$$M(2) = \begin{bmatrix} 0 & -2/3\omega & 0 & 1 \\ 1 & 1 & 0 & -2i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -i\omega \end{bmatrix}, \tag{5.6}$$

$$M(3) = \begin{bmatrix} 1 & -2/3\omega & 0 & 1 \\ 1 & 1 & 0 & 2i \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & i\omega \end{bmatrix}. \tag{5.7}$$

Analysis of (5.5)–(5.7) shows that Rank $\{M(i)\} = 4$, $\forall i = 1, 2, 3$ hence by Theorem 4.1 the system is controllable; this agrees with our conclusion obtained earlier from Theorem 4.2. The advantage of using Theorem 4.1 is that the relations (5.4) contain *all* the information we need to determine whether the system is controllable by impulses in *any* of the dimensions. The theorems of §4 are stated in terms of the last m dimensions being impulsive. If we wanted to have only the fourth and sixth dimensions impulsive in a 6-d system then we would have to switch rows 4 and 5 and columns 4 and 5 of the matrix A before applying the theorems. Fortunately, the operation of switching two rows of a matrix and the same two columns does not affect the eigenvalues of the matrix and has only the affect of switching the same two components in the generalized eigenvectors. Returning then to our example, if the tangential thruster failed so that impulses could only be applied in the fourth and sixth dimensions, then from (5.4) we would get

$$M(1) = \begin{bmatrix} 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 2i & 0 & -2i \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & -2\omega & 0 & -2\omega \end{bmatrix}, \tag{5.8}$$

$$M(2) = \begin{bmatrix} 0 & -2/3\omega & 0 & 1 \\ 1 & 1 & 0 & -2i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2\omega \end{bmatrix}, \tag{5.9}$$

$$M(3) = \begin{bmatrix} 0 & -2/3\omega & 0 & 1 \\ 1 & 1 & 0 & 2i \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & -2\omega \end{bmatrix}. \tag{5.10}$$

However, from (5.8) we see that Rank $\{M(1)\} = 3$ since

$$-4i\begin{pmatrix}0\\1\\0\\0\end{pmatrix}+\begin{pmatrix}1\\2i\\0\\-2\omega\end{pmatrix}-\begin{pmatrix}1\\-2i\\0\\-2\omega\end{pmatrix}=0,$$

so that by Theorem 4.1 the system is uncontrollable. Similarly if the out-of-plane thruster failed and only the radial and tangential thrusters were operative, we would have from (5.4)

$$M(1)=\begin{bmatrix}0&0&1&0&1\\1&0&2i&0&-2i\\0&1&0&1&0\\0&i\omega&0&-i\omega&0\end{bmatrix}, \tag{5.11}$$

$$M(2)=\begin{bmatrix}0&-2/3\omega&0&1\\1&1&0&-2i\\0&0&1&0\\0&0&-i\omega&0\end{bmatrix}, \tag{5.12}$$

$$M(3)=\begin{bmatrix}0&-2/3\omega&0&1\\1&1&0&2i\\0&0&1&0\\0&0&i\omega&0\end{bmatrix}. \tag{5.13}$$

However, we see from (5.12) and (5.13) that Rank $\{M(2)\}$ = Rank $\{M(3)\} = 3$ so that by Theorem 4.1 the system is uncontrollable. If two thrusters failed so that impulses could only be applied in one dimension then since the geometric multiplicity of eigenvalues $i\omega$ and $-i\omega$ is 2, by Corollory 4.1 the system is uncontrollable. To summarize then, any desired final state $(x_f^T, v_f^T)^T$ (within the validity of the linearization) can be achieved by the spacecraft provided the tangential and out-of-plane thrusters are operative.

With regard to whether it is easier to use the method of Theorem 4.1 or 4.2 when testing a given system, from this example we saw that for a given set of impulsive dimensions the method of Theorem 4.2 was easier to use; however, when we wanted to determine *which* dimensions needed to be impulsive for the system to be controllable, the method of Theorem 4.1 was more appropriate.

References

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