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EIGENVALUES OF BOUNDARY VALUE PROBLEMS FOR HIGHER ORDER DIFFERENTIAL EQUATIONS

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We shall consider the boundary value problem

$$y^{(n)} + \lambda Q(t, y, y', \cdots, y^{(n-2)}) = \lambda P(t, y, y', \cdots, y^{(n-1)}), n \geq 2, t \in (0, 1),$$
$$y^{(i)}(0) = 0, 0 \leq i \leq n-3,$$
$$\alpha y^{(n-2)}(0) - \beta y^{(n-1)}(0) = 0,$$
$$\gamma y^{(n-2)}(1) + \delta y^{(n-1)}(1) = 0,$$

where $\lambda > 0$, α, β, γ and δ are constants satisfying $\alpha\gamma + \alpha\delta + \beta\gamma > 0$, $\beta, \delta \geq 0$, $\beta + \alpha > 0$ and $\delta + \gamma > 0$ to characterize the values of λ so that it has a positive solution. For the special case $\lambda = 1$, sufficient conditions are also established for the existence of positive solutions.

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1. INTRODUCTION

In this paper we shall consider the nth order differential equation

$$y^{(n)} + \lambda Q(t, y, y', \cdots, y^{(n-2)}) = \lambda P(t, y, y', \cdots, y^{(n-1)}), t \in (0, 1), \tag{1.1}$$

together with the boundary conditions

$$y^{(i)}(0) = 0, 0 \leq i \leq n-3, \tag{1.2}$$

$$\alpha y^{(n-2)}(0) - \beta y^{(n-1)}(0) = 0, \tag{1.3}$$

$$\gamma y^{(n-2)}(1) + \delta y^{(n-1)}(1) = 0, \tag{1.4}$$

where $n \geq 2$, $\lambda > 0$, α, β, γ and δ are constants so that

$$\rho = \alpha\gamma + \alpha\delta + \beta\gamma > 0, \tag{1.5}$$

and

$$\beta \geq 0, \quad \delta \geq 0, \quad \beta + \alpha > 0, \quad \delta + \gamma > 0. \tag{1.6}$$

We remark that condition (1.6) allows α and γ to be negative.

Further, we assume that there exist continuous functions f: $[0, \infty) \to (0, \infty)$ and p, p_1, q, q_1: $(0, 1) \to \Re$ such that

(A1) f is nondecreasing;
(A2) for $u \in [0, \infty)$,

$$q(t) \leq \frac{Q(t, u, u_1, \cdot\ \cdot\ \cdot, u_{n-2})}{f(u)} \leq q_1(t), \quad p(t) \leq \frac{P(t, u, u_1, \cdot\ \cdot\ \cdot, u_{n-1})}{f(u)} \leq p_1(t);$$

(A3) $q(t) - p_1(t)$ is nonnegative and is not identically zero on any subinterval of $(0, 1)$;
(A4) $\int_0^1 (\beta + \alpha t)[\delta + \gamma(1 - t)][q_1(t) - p(t)]dt < \infty$.

We shall establish upper and lower bounds for λ so that the boundary value problem (1.1)–(1.4) has a positive solution. By a *positive solution* y of (1.1)–(1.4), we mean $y \in C^{(n)}(0, 1) \cap C^{(n-1)}[0, 1]$, y satisfies (1.1) on $(0, 1)$, y fulfills (1.2)–(1.4), and y is nonnegative on $[0, 1]$, positive on some subinterval of $[0, 1]$. If, for a particular λ the boundary value problem (1.1)–(1.4) has a positive solution y, then we shall call λ an *eigenvalue* and y a corresponding *eigenfunction* of (1.1)–(1.4). Throughout, we shall let

$$E = \{\lambda > 0 \mid (1.1)\text{–}(1.4) \textit{ has a positive solution}\}.$$

We note that E is the set of eigenvalues of (1.1)–(1.4).

Next, for the special case $\lambda = 1$, we shall investigate the existence of positive solutions for the boundary value problem (1.1)–(1.4), assuming further that

(A5) f is either superlinear or sublinear.

To be precise, we introduce the notations

$$f_0 = \lim_{u \to 0} \frac{f(u)}{u}, \qquad f_\infty = \lim_{u \to \infty} \frac{f(u)}{u}.$$

The function f is said to be *superlinear* if $f_0 = 0, f_\infty = \infty$, and f is *sublinear* provided $f_0 = \infty, f_\infty = 0$.

The motivation for the present work stems from many recent investigations. In fact, when $n = 2$ the boundary value problem (1.1)–(1.4) arises in various physical phenomena such as gas diffusion through porous media [3], thermal self ignition of a chemically active mixture of gases in a vessel [7,18,21], catalysis theory [10], chemically reacting systems

[25], as well as adiabatic tubular reactor processes [9]. For the present work we refer particularly to [6,8,15–17]. In all these papers, eigenvalue characterizations for particular cases of (1.1)–(1.4) are discussed. For example in [16], Fink, Gatica and Hernandez deal with the boundary value problem

$$\begin{gathered} y'' + \lambda q(t) f(y) = 0, t \in (0, 1), \\ y(0) = y(1) = 0. \end{gathered} \tag{1.7}$$

Their results are extended in [17] to systems of second order boundary value problems. In [6] and [15], the authors tackle a different boundary value problem

$$\begin{gathered} y'' + \frac{N-1}{t} y' + \lambda q(t) f(y) = 0, t \in (0, 1), \\ y'(0) = y(1) = 0. \end{gathered} \tag{1.8}$$

Recently, Chyan and Henderson [8] have studied a more general problem than (1.7), namely,

$$\begin{gathered} y^{(n)} + \lambda q(t) f(y) = 0, t \in (0, 1), \\ y^{(i)}(0) = y^{(n-2)}(1) = 0, 0 \leq i \leq n-2. \end{gathered} \tag{1.9}$$

Our results not only generalize and extend the known eigenvalue theorems for (1.7)–(1.9), but also complement the discrete analog studied by Wong and Agarwal [29], as well as include several other known criteria offered in [1].

In the special case that $\lambda = 1$, applications of (1.1)–(1.4) and its discrete version have been made to singular boundary value problems by Agarwal and Wong [2,27]. Other particular cases of (1.1)–(1.4) and their discrete analogs have also been the subject matter of several recent publications on singular boundary value problems, e.g., see [1,11,20,22,23 and the references cited therein]. Further, in the particular case that $n = 2$, (1.1)–(1.4) arises in applications involving nonlinear elliptic problems in annular regions, for this we refer to [4,5,19,26]. In all these applications, it is frequent that only solutions that are positive are useful. We are particularly motivated by the work of [12–14], and our result is a generalization and extension of theirs. We further remark that other than the differential equation (1.1) considered is more general, the conditions on the coefficients in the boundary conditions are also weakened, and we allow some coefficients to be negative. Our work also complements naturally the discrete problem considered in [28].

The plan of this paper is as follows: In Section 2 we shall state a fixed point theorem due to Krasnosel'skii [24], and present some properties of a Green's function which will be used later. In Section 3 we define an appropriate Banach space and cone so that the set E can be characterized. Finally, in Section 4 we consider the special case $\lambda = 1$ and apply the fixed point theorem from [24] to yield a positive solution for (1.1)–(1.4).

2. PRELIMINARIES

THEOREM 2.1. [24] *Let B be a Banach space, and let $C \subset B$ be a cone in B. Assume Ω_1, Ω_2 are open subsets of B with $0 \in \Omega_1, \overline{\Omega}_1 \subset \Omega_2$, and let*

$$S : C \cap (\overline{\Omega}_2 \backslash \Omega_1) \to C$$

be a completely continuous operator such that, either

(a) $\|Sy\| \leq \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \geq \|y\|$, $y \in C \cap \partial\Omega_2$, *or*
(b) $\|Sy\| \geq \|y\|$, $y \in C \cap \partial\Omega_1$, *and* $\|Sy\| \leq \|y\|$, $y \in C \cap \partial\Omega_2$.

Then, S has a fixed point in $C \cap (\overline{\Omega}_2 \backslash \Omega_1)$.

To obtain a solution for (1.1)–(1.4), we need a mapping whose kernel $g(t, s)$ is the Green's function of the boundary value problem

$$-y^{(n)} = 0,$$

$$y^{(i)}(0) = 0, 0 \leq i \leq n-3,$$

$$\alpha y^{(n-2)}(0) - \beta y^{(n-1)}(0) = 0,$$

$$\gamma y^{(n-2)}(1) + \delta y^{(n-1)}(1) = 0.$$

It can be verified that

$$G(t, s) = \frac{\partial^{n-2}}{\partial t^{n-2}} g(t, s)$$

is the Green's function of the boundary value problem

$$-w'' = 0,$$

$$\alpha w(0) - \beta w'(0) = 0,$$

$$\gamma w(1) + \delta w'(1) = 0.$$

Further, we have [2]

$$G(t, s) = \frac{1}{\rho} \begin{cases} (\beta + \alpha s)[\delta + \gamma(1-t)], & 0 \leq s \leq t \\ (\beta + \alpha t)[\delta + \gamma(1-s)], & t \leq s \leq 1 \end{cases}. \tag{2.1}$$

We observe that the conditions (1.5) and (1.6) imply that $G(t, s)$ is nonnegative on $[0, 1] \times [0, 1]$, and positive on $(0, 1) \times (0, 1)$.

LEMMA 2.1. *For* $(t, s) \in [1/4, 3/4] \times [0, 1]$, *we have*

$$G(t, s) \geq K\, G(s, s), \tag{2.2}$$

where $0 < K < 1$ is given by

$$K = \min \left\{ \frac{4\beta + \alpha}{4(\beta + \alpha)}, \frac{4\delta + \gamma}{4(\delta + \gamma)}, \frac{4\beta + 3\alpha}{4\beta + \alpha}, \frac{4\delta + 3\gamma}{4\delta + \gamma} \right\}. \tag{2.3}$$

Proof. For $0 \leq s \leq t$, using (2.1) the inequality (2.2) reduces to

$$\delta + \gamma(1 - t) \geq K[\delta + \gamma(1 - s)]. \tag{2.4}$$

In order that (2.4) holds, it is sufficient that K satisfies

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} [\delta + \gamma(1 - t)] \geq K \max_{s \in [0, \frac{3}{4}]} [\delta + \gamma(1 - s)]. \tag{2.5}$$

If $\gamma \geq 0$, then (2.5) gives

$$\delta + \frac{1}{4}\gamma \geq K(\delta + \gamma), \quad \text{or} \quad K \leq \frac{4\delta + \gamma}{4(\delta + \gamma)}. \tag{2.6}$$

If $\gamma < 0$, then it follows from (2.5) that

$$\delta + \frac{3}{4}\gamma \geq K\,(\delta + \frac{1}{4}\gamma), \quad \text{or} \quad K \leq \frac{4\delta + 3\gamma}{4\delta + \gamma}. \tag{2.7}$$

Next, for $t \leq s \leq 1$ the inequality (2.2) becomes

$$\beta + \alpha t \geq K(\beta + \alpha s).$$

Again, it suffices to find K such that

$$\min_{t \in [\frac{1}{4}, \frac{3}{4}]} (\beta + \alpha t) \geq K \max_{s \in [\frac{1}{4}, 1]} (\beta + \alpha s). \tag{2.8}$$

If $\alpha \geq 0$, then from (2.8) we obtain

$$\beta + \frac{1}{4}\alpha \geq K\,(\beta + \alpha), \quad \text{or} \quad K \leq \frac{4\beta + \alpha}{4(\beta + \alpha)}. \tag{2.9}$$

If $\alpha < 0$, then (2.8) yields

$$\beta + \frac{3}{4}\alpha \geq K\left(\beta + \frac{1}{4}\alpha\right), \quad \text{or} \quad K \leq \frac{4\beta + 3\alpha}{4\beta + \alpha}. \tag{2.10}$$

Taking into account (2.6), (2.7), (2.9) and (2.10), we immediately get (2.3).

LEMMA 2.2. *For $(t, s) \in [0, 1] \times [0, 1]$, we have*

$$G(t, s) \leq L\, G(s, s), \tag{2.11}$$

where $L \geq 1$ is given by

$$L = \max\left\{1, \frac{\beta}{\beta + \alpha}, \frac{\delta}{\delta + \gamma}\right\}. \tag{2.12}$$

Proof. For $0 \leq s \leq t$, using (2.1) the inequality (2.11) can be written as

$$\delta + \gamma(1 - t) \leq L[\delta + \gamma(1 - s)]. \tag{2.13}$$

In the case that $\gamma \geq 0$, it is clear that we may take $L = 1$ in (2.13). If $\gamma < 0$, in order that (2.13) holds it is sufficient that L satisfies

$$\max_{t\in[0,1]} [\delta + \gamma(1 - t)] \leq L \min_{s\in[0,1]} [\delta + \gamma(1 - s)],$$

which provides

$$L \geq \frac{\delta}{\delta + \gamma}. \tag{2.14}$$

Next, for $t \leq s \leq 1$, using (2.1) the inequality (2.11) reduces to

$$\beta + \alpha t \leq L(\beta + \alpha s). \tag{2.15}$$

If $\alpha \geq 0$, then we may take $L = 1$ in (2.15).
If $\alpha < 0$, then as before it suffices to have

$$\max_{t\in[0,1]} (\beta + \alpha t) \leq L \min_{s\in[0,1]} (\beta + \alpha s),$$

which yields

$$L \geq \frac{\beta}{\beta + \alpha}. \tag{2.16}$$

The expression (2.12) is immediate from (2.14) and (2.16).

LEMMA 2.3. *For $0 \le i \le n-2$ and $(t, s) \in [0, 1] \times [0, 1]$, we have*

$$0 \le \frac{\partial^i}{\partial t^i} g(t, s) \le L\, G(s, s) \frac{t^{n-2-i}}{(n-2-i)!}. \tag{2.17}$$

In particular, for $(t, s) \in [0, 1] \times [0, 1]$,

$$0 \le g(t, s) \le \frac{L}{(n-2)!} G(s, s). \tag{2.18}$$

Proof. We note that for $0 \le i \le n-3$,

$$\frac{\partial^i}{\partial t^i} g(t, s) = \int_0^t \frac{\partial^{i+1}}{\partial t^{i+1}} g(\tau, s) d\tau. \tag{2.19}$$

Hence, we find

$$\frac{\partial^{n-3}}{\partial t^{n-3}} g(t, s) = \int_0^t G(\tau, s) d\tau \le \int_0^t L\, G(s, s) d\tau = L\, G(s, s)\, t, \tag{2.20}$$

where we have used Lemma 2.2. It follows from (2.19) and (2.20) that

$$\frac{\partial^{n-4}}{\partial t^{n-4}} g(t, s) = \int_0^t \frac{\partial^{n-3}}{\partial t^{n-3}} g(\tau, s) d\tau \le \int_0^t L\, G(s, s)\, \tau d\tau = L\, G(s, s) \frac{t^2}{2!}.$$

Continuing the process, we obtain inequality (2.17) from which (2.18) is immediate.

We shall need the following notations later: Let

$$v(t) = q_1(t) - p(t), \quad \text{and} \quad u(t) = q(t) - p_1(t).$$

For a nonnegative y on $[0, 1]$, we denote

$$\theta = \int_0^1 G(s, s) v(s) f(y(s)) ds, \quad \text{and} \quad \Gamma = \int_0^1 G(s, s) u(s) f(y(s)) ds.$$

In view of (A2) and (A3), it is clear that $\theta \ge \Gamma > 0$. Further, we define the constant

$$\xi = \frac{K\Gamma}{L\theta}.$$

It is noted that $0 < \xi < 1$.

3. MAIN RESULTS

Let the Banach space

$$B = \{y \in C^{(n)}(0, 1) \cap C^{(n-1)}[0, 1] \mid y^{(i)}(0) = 0, 0 \le i \le n - 3\}$$

with norm $\|y\| = \sup_{t\in[0,1]} |y^{(n-2)}(t)|$, and let

$$C = \{y \in B \mid y^{(n-2)}(t) \textit{ is nonnegative and is not identically zero on } [0, 1];\ \min_{t\in[\frac{1}{4},\frac{3}{4}]} y^{(n-2)}(t) \ge \xi\|y\|\}.$$

It is noted that C is a cone in B. Further, we let

$$C_M = \{y \in C \mid \|y\| \le M\}.$$

LEMMA 3.1. *Let $y \in B$. For $0 \le i \le n - 2$, we have*

$$|y^{(i)}(t)| \le \frac{t^{n-2-i}}{(n-2-i)!}\|y\|,\ t \in [0, 1]. \tag{3.1}$$

In particular,

$$|y(t)| \le \frac{1}{(n-2)!}\|y\|,\ t \in [0, 1]. \tag{3.2}$$

Proof. For $y \in B$, we have

$$y^{(n-3)}(t) = \int_0^t y^{(n-2)}(s)ds,\ t \in [0, 1],$$

which implies

$$|y^{(n-3)}(t)| \le t\|y\|,\ t \in [0, 1]. \tag{3.3}$$

Next, since

$$y^{(n-4)}(t) = \int_0^t y^{(n-3)}(s)ds,\ t \in [0, 1],$$

on using (3.3) we get

$$|y^{(n-4)}(t)| \leq \int_0^t s\|y\|ds = \frac{t^2}{2!}\|y\|, t \in [0, 1].$$

Continuing in the same manner we obtain (3.1) and also (3.2).

LEMMA 3.2. *Let $y \in C$. For $0 \leq i \leq n-2$, we have*

$$y^{(i)}(t) \geq 0, t \in [0, 1], \tag{3.4}$$

and

$$y^{(i)}(t) \geq \left(t - \frac{1}{4}\right)^{n-2-i} \frac{\xi}{(n-2-i)!}\|y\|, t \in \left[\frac{1}{4}, \frac{3}{4}\right]. \tag{3.5}$$

In particular,

$$y(t) \geq \frac{\xi}{4^{n-2}(n-2)!}\|y\|, t \in \left[\frac{1}{2}, \frac{3}{4}\right]. \tag{3.6}$$

Proof. Inequality (3.4) is obvious from the fact that

$$y^{(i)}(t) = \int_0^t y^{(i+1)}(s)ds, t \in [0, 1], 0 \leq i \leq n-3.$$

To prove (3.5), we note that

$$y^{(n-3)}(t) = \int_0^t y^{(n-2)}(s)ds \geq \int_{\frac{1}{4}}^t \xi\|y\|ds = \xi\|y\|\left(t - \tfrac{1}{4}\right), t \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right]. \tag{3.7}$$

Next, on using (3.7), we find

$$\begin{aligned} y^{(n-4)}(t) &= \int_0^t y^{(n-3)}(s)ds \geq \int_{\frac{1}{4}}^t \xi\|y\|\left(s - \tfrac{1}{4}\right) ds \\ &= \left(t - \tfrac{1}{4}\right)^2 \frac{\xi}{2!}\|y\|, t \in \left[\tfrac{1}{4}, \tfrac{3}{4}\right]. \end{aligned}$$

Continuing the process we obtain (3.5). Inequality (3.6) is immediate from (3.5) by taking $i = 0$ and substituting $t = 1/2$ in the right side of (3.5).

Remark 3.1. If $y \in C$ is a solution of (1.1)–(1.4), then (3.4) and (3.6) imply that y is a positive solution of (1.1)–(1.4).

To obtain a positive solution of (1.1)–(1.4), we shall seek a fixed point of the operator λS in the cone C, where S: $C \to B$ is defined by

$$Sy(t) = \int_0^1 g(t, s)[Q(s, y, y', \cdots, y^{(n-2)}) - P(s, y, y', \cdots, y^{(n-1)})]ds, \; t \in [0, 1]. \tag{3.8}$$

In view of condition (A2) we get

$$Uy(t) \leq Sy(t) \leq Vy(t), \; t \in [0, 1], \tag{3.9}$$

where

$$Uy(t) = \int_0^1 g(t, s)u(s)f(y(s))ds, \tag{3.10}$$

and

$$Vy(t) = \int_0^1 g(t, s)v(s)f(y(s))ds. \tag{3.11}$$

Since for $t \in [0, 1]$,

$$(Sy)^{(n-2)}(t) = \int_0^1 G(t, s)[Q(s, y, y', \cdots, y^{(n-2)}) - P(s, y, y', \cdots, y^{(n-1)})]ds,$$

$$(\mathrm{U}y)^{(n-2)}(t) = \int_0^1 G(t, s)u(s)f(y(s))ds,$$

and

$$(Vy)^{(n-2)}(t) = \int_0^1 G(t, s)v(s)f(y(s))ds,$$

using (A2) again gives

$$(Uy)^{(n-2)}(t) \leq (Sy)^{(n-2)}(t) \leq (Vy)^{(n-2)}(t), \; t \in [0, 1]. \tag{3.12}$$

We shall now show that the operator S is compact on the cone C. Let us consider the case when $u(t)$ is unbounded in a deleted right neighborhood of 0 and also in a deleted left neighborhood of 1. Clearly, $v(t)$ is also unbounded near 0 and 1. For $m \in \{1, 2, 3, \ldots\}$, define u_m, v_m: $[0, 1] \to \Re$ by

$$u_m(t) = \begin{cases} u\left(\dfrac{1}{m+1}\right), & 0 \leq t \leq \dfrac{1}{m+1} \\ u(t), & \dfrac{1}{m+1} \leq t \leq \dfrac{m}{m+1} \\ u\left(\dfrac{m}{m+1}\right), & \dfrac{m}{m+1} \leq t \leq 1, \end{cases} \tag{3.13}$$

$$v_m(t) = \begin{cases} v\left(\dfrac{1}{m+1}\right), & 0 \le t \le \dfrac{1}{m+1} \\ v(t), & \dfrac{1}{m+1} \le t \le \dfrac{m}{m+1} \\ v\left(\dfrac{m}{m+1}\right), & \dfrac{m}{m+1} \le t \le 1, \end{cases} \tag{3.14}$$

and the operators U_m, V_m: $C \to B$ by

$$U_m y(t) = \int_0^1 g(t, s) u_m(s) f(y(s)) ds,$$

$$V_m y(t) = \int_0^1 g(t, s) v_m(s) f(y(s)) ds.$$

It is standard that for each m, both U_m and V_m are compact operators on C. Let $M > 0$ and $y \in C_M$. Then, in view of (2.18) and (3.2), we find

$$\begin{aligned} &|V_m y(t) - Vy(t)| \\ &= \int_0^1 g(t, s)|v_m(s) - v(s)| f(y(s)) ds \\ &= \int_0^{\frac{1}{m+1}} g(t, s)|v_m(s) - v(s)| f(y(s)) ds \\ &\qquad + \int_{\frac{m}{m+1}}^1 g(t, s)|v_m(s) - v(s)| f(y(s)) ds \\ &\le \frac{L}{(n-2)!} f\left(\frac{M}{(n-2)!}\right) \left[\int_0^{\frac{1}{m+1}} G(s, s)|v_m(s) - v(s)| ds \right. \\ &\qquad \left. + \int_{\frac{m}{m+1}}^1 G(s, s)|v_m(s) - v(s)| f(y(s)) ds \right] \\ &= \frac{\mathrm{L}}{(n-2)!} f\left(\frac{\mathrm{M}}{(n-2)!}\right) \left[\int_0^{\frac{1}{m+1}} \mathrm{G}(s, s) \left| v\left(\frac{1}{m+1}\right) - v(s) \right| ds \right. \\ &\qquad \left. + \int_{\frac{m}{m+1}}^1 \mathrm{G}(s, s) \left| v\left(\frac{m}{m+1}\right) - v(s) \right| f(y(s)) ds \right]. \end{aligned}$$

The integrability of $G(t, t)v(t)$ (condition (A4)) implies that V_m converges uniformly to V on C_M. Hence, V is compact on C. Similarly, we can verify that U_m converges uniformly to U on C_M and therefore U is compact on C. It follows from inequality (3.9) that the operator S is compact on C.

THEOREM 3.1. *There exists a $c > 0$ such that the interval $(0, c] \subseteq E$.*

Proof. Let $M > 0$ be given. Define

$$c = M\left\{\frac{L}{\rho} f\left(\frac{M}{(n-2)!}\right)\int_0^1 (\beta + \alpha s)[\delta + \gamma(1-s)]v(s)ds\right\}^{-1}. \tag{3.15}$$

Let $y \in C_M$ and $0 < \lambda \leq c$. We shall prove that $\lambda Sy \in C_M$. For this, first we shall show that $\lambda Sy \in C$. From (3.12) and (A3), we find

$$(\lambda Sy)^{(n-2)}(t) \leq \lambda \int_0^1 G(t, s)u(s) f(y(s))ds \geq 0, t \in [0, 1]. \tag{3.16}$$

Further, it follows from (3.12) and Lemma 2.2 that

$$\begin{aligned}(Sy)^{(n-2)}(t) &\leq \int_0^1 G(t, s)v(s) f(y(s))ds \\ &\leq L\int_0^1 G(s, s)v(s) f(y(s))ds, t \in [0, 1].\end{aligned}$$

Therefore,

$$\|Sy\| \leq L\int_0^1 G(s, s)v(s) f(y(s))ds = L\theta. \tag{3.17}$$

Now, on using (3.12), Lemma 2.1 and (3.17), we find for $t \in \left[\frac{1}{4}, \frac{3}{4}\right]$,

$$\begin{aligned}(\lambda Sy)^{(n-2)}(t) &\geq \lambda \int_0^1 G(t, s)u(s) f(y(s))ds \\ &\geq \lambda K \int_0^1 G(s, s)u(s) f(y(s))ds \\ &= \lambda K\Gamma \geq \lambda\xi\|Sy\| = \xi\|\lambda Sy\|.\end{aligned}$$

Hence,

$$\min_{t\in[\frac{1}{4},\frac{3}{4}]} (\lambda Sy)^{(n-2)}(t) \geq \xi\|\lambda Sy\|. \tag{3.18}$$

It follows from (3.16) and (3.18) that $\lambda Sy \in C$.

Next, on using (3.12), Lemma 2.2, (3.2), (2.1) and (3.15) successively, we get

$$\begin{aligned}(\lambda Sy)^{(n-2)}(t) &\leq \lambda \int_0^1 G(t, s)v(s)f(y(s))ds \\ &\leq L\lambda \int_0^1 G(s, s)v(s)f(y(s))ds \\ &\leq L\lambda \int_0^1 G(s, s)v(s)f\left(\frac{M}{(n-2)!}\right) ds\end{aligned}$$

$$= \frac{L\lambda}{\rho} \int_0^1 (\beta + \alpha s)[\delta + \gamma(1 - s)]v(s) f\left(\frac{M}{(n-2)!}\right) ds$$
$$\leq M, t \in [0, 1],$$

which implies

$$\|\lambda S y\| \leq M.$$

Hence, $(\lambda S)(C_M) \subseteq C_M$. Also, the standard arguments yield that λS is completely continuous. By Schauder fixed point theorem, λS has a fixed point in C_M. Clearly, this fixed point is a positive solution of (1.1)–(1.4) and therefore λ is an eigenvalue of (1.1)–(1.4). Since $0 < \lambda \leq c$ is arbitrary, it follows immediately that $(0, c] \subseteq E$.

The next theorem makes use of the monotonicity and compactness of the operator S on the cone C. We refer to [16, Theorem 3.2] for its proof.

THEOREM 3.2. [16] *Suppose that $\lambda_0 \in E$. Then, for each $0 < \lambda < \lambda_0$, $\lambda \in E$.*

The following corollary is immediate from Theorem 3.2.

COROLLARY 3.1. *E is an interval.*

We shall establish conditions under which E is a bounded or unbounded interval. For this, we need the following results.

THEOREM 3.3. *Let λ be an eigenvalue of* (1.1)–(1.4) *and $y \in C$ be a corresponding eigenfunction.*

(a) Suppose that $\delta = \beta = 0$ and $\gamma = \alpha = 1$. If

$$y^{(n-1)}(0) = \nu \tag{3.19}$$

for some $\nu > 0$, then λ satisfies

$$a(v)\nu \left[f\left(\frac{\nu}{(n-1)!}\right)\right]^{-1} \leq \lambda \leq a(u)\nu[f(0)]^{-1}, \tag{3.20}$$

where

$$a(z) = \left\{ \int_0^1 (1 - s)z(s)ds \right\}^{-1}. \tag{3.21}$$

(b) Suppose that $\delta > 0$, $\beta = 0$ and (3.19) holds for some $\nu > 0$.

(i) If $\gamma \geq 0$, then λ satisfies

$$b(v)\nu(\gamma + \delta) \left[f\left(\frac{\nu}{(n-1)!}\right)\right]^{-1} \leq \lambda \leq b(u)\nu(\gamma + \delta)[f(0)]^{-1}, \tag{3.22}$$

where

$$b(z) = \left\{ \int_0^1 [\gamma(1-s) + \delta] z(s) ds \right\}^{-1}. \tag{3.23}$$

(ii) If $\gamma < 0$, then λ satisfies

$$c\left(u, v, f, 0, \frac{\nu}{(n-1)!}\right) \nu(\gamma + \delta) \leq \lambda \leq c\left(v, u, f, \frac{\nu}{(n-1)!}, 0\right) \nu(\gamma + \delta), \tag{3.24}$$

where

$$c\,(z, w, f, \theta_1, \theta_2) = \left\{ \gamma f(\theta_1) \int_0^1 (1-s) z(s) ds + \delta f(\theta_2) \int_0^1 w(s) ds \right\}^{-1}. \tag{3.25}$$

(c) Suppose that $\delta = 0$ and $\beta > 0$. If

$$y^{(n-2)}(0) = \mu, \qquad y^{(n-1)}(0) = \nu \tag{3.26}$$

for some μ, $\nu > 0$ such that $\alpha\mu = \beta\nu$, then λ satisfies

$$a(\nu)\,(\mu + \nu) \left[f\left(\frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!} \right) \right]^{-1} \leq \lambda \leq a(u)\,(\mu + \nu)[f(0)]^{-1}, \tag{3.27}$$

where $a(\cdot)$ is defined in (3.21).

(d) Suppose that $\delta > 0$, $\beta > 0$ and (3.26) holds for some μ, $\nu > 0$ such that $\alpha\mu = \beta\nu$.
(i) If $\gamma \geq 0$, then λ satisfies

$$\begin{aligned} b(\nu)\,[\gamma(\mu + \nu) + \delta\nu] &\left[f\left(\frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right) \right]^{-1} \\ &\leq \lambda \leq b(u)\,[\gamma(\mu + \nu) + \delta\nu][f(0)]^{-1}, \end{aligned} \tag{3.28}$$

where $b(\cdot)$ is defined in (3.23).

(ii) If $\gamma < 0$, then λ satisfies

$$\begin{aligned} c\left(u, v, f, 0, \frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right) &[\gamma(\mu + \nu) + \delta\nu] \\ \leq \lambda \leq c\left(v, u, f, \frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}, 0\right) &[\gamma(\mu + \nu) + \delta\nu], \end{aligned} \tag{3.29}$$

where $c\,(\cdot, \cdot, \cdot, \cdot, \cdot)$ is defined in (3.25).

Proof. (a) In this case, the boundary conditions (1.2)–(1.4) reduce to

$$\begin{aligned} y^{(i)}(0) &= 0, 0 \le i \le n-2, \\ y^{(n-2)}(1) &= 0. \end{aligned} \tag{3.30}$$

For $m \in \{1, 2, 3, \ldots\}$, we define $f_m = f * \eta_m$, where η_m is a standard mollifier [8, 16] such that f_m is Lipschitz and converges uniformly to f.

For a fixed m, let λ_m be an eigenvalue and y_m, with $y_m^{(n-1)}(0) = \nu$, be a corresponding eigenfunction of the following boundary value problem

$$\begin{aligned} y_m^{(n)} + \lambda_m Q_m(t, y_m, y_m', \cdots, y_m^{(n-2)}) &= \lambda_m P_m(t, y_m, y_m', \cdots, y_m^{(n-1)}), \\ t \in [0, 1], \end{aligned} \tag{3.31}$$

$$\begin{aligned} y_m^{(i)}(0) &= 0, 0 \le i \le n-2, \\ y_m^{(n-2)}(1) &= 0, \end{aligned} \tag{3.32}$$

where Q_m and P_m converge uniformly to Q and P respectively, and

$$u_m(t) \le \frac{Q_m(t, z, z_1, \cdots, z_{n-2}) - P_m(t, z, z_1, \cdots, z_{n-1})}{f_m(z)} \le v_m(t) \tag{3.33}$$

(see (3.13) and (3.14) for the definitions of $u_m(t)$ and $v_m(t)$).

Clearly, y_m is the unique solution of the initial value problem (3.31),

$$\begin{aligned} y_m^{(i)}(0) &= 0, 0 \le i \le n-2, \\ y_m^{(n-1)}(0) &= \nu. \end{aligned} \tag{3.34}$$

Since

$$\begin{aligned} y_m^{(n)}(t) &= \lambda_m[P_m(t, y_m, y_m', \cdots, y_m^{(n-1)}) - Q_m(t, y_m, y_m', \cdots, y_m^{(n-2)})] \\ &\le -\lambda_m u_m(t) f_m(y_m(t)) \\ &\le 0, \end{aligned}$$

we have $y_m^{(n-1)}$ is nonincreasing and hence

$$y_m^{(n-1)}(t) \le y_m^{(n-1)}(0) = \nu, t \in [0, 1]. \tag{3.35}$$

Using the initial conditions (3.34) and (3.35), we find for $t \in [0, 1]$,

$$y_m^{(n-2)}(t) = \int_0^t y_m^{(n-1)}(s)ds \leq \int_0^t \nu ds = \nu t.$$

This in turn leads to

$$y_m^{(n-3)}(t) = \int_0^t y_m^{(n-2)}(s)ds \leq \int_0^t \nu s ds = \nu \frac{t^2}{2!}, \; t \in [0, 1].$$

Continuing the process we obtain for $t \in [0, 1]$,

$$y_m(t) \leq \nu \frac{t^{n-1}}{(n-1)!} \leq \frac{\nu}{(n-1)!}. \tag{3.36}$$

Now, from (3.31), (3.33) and (3.36) we get for $t \in [0, 1]$,

$$\lambda_m u_m(t) f_m(0) \leq -y_m^{(n)}(t) \leq \lambda_m v_m(t) f_m\left(\frac{\nu}{(n-1)!}\right). \tag{3.37}$$

An integration of (3.37) from 0 to t provides

$$\phi_1(t) \leq y_m^{(n-1)}(t) \leq \phi_2(t), \; t \in [0, 1], \tag{3.38}$$

where

$$\phi_1(t) = \nu - \lambda_m f_m\left(\frac{\nu}{(n-1)!}\right) \int_0^t v_m(s)ds,$$

and

$$\phi_2(t) = \nu - \lambda_m f_m(0) \int_0^t u_m(s)ds.$$

Again, we integrate (3.38) from 0 to t, and subsequently change the order of integration, to obtain

$$\phi_3(t) \leq y_m^{(n-2)}(t) \leq \phi_4(t), \; t \in [0, 1], \tag{3.39}$$

where

$$\phi_3(t) = \nu t - \lambda_m f_m\left(\frac{\nu}{(n-1)!}\right) \int_0^t (t-s) v_m(s)ds,$$

and

$$\phi_4(t) = \nu t - \lambda_m f_m(0) \int_0^t (t-s) u_m(s)ds.$$

From inequality (3.39), in order to have $y_m^{(n-2)}(1) = 0$ (see (3.32)), it is necessary that

$$\phi_3(1) \leq 0, \quad \text{and} \quad \phi_4(1) \geq 0,$$

or equivalently,

$$\lambda_m \geq a(v_m)\nu \left[f_m \left(\frac{\nu}{(n-1)!} \right) \right]^{-1}, \tag{3.40}$$

and

$$\lambda_m \leq a(u_m)\nu \, [f_m(0)]^{-1}. \tag{3.41}$$

A combination of (3.40) and (3.41) yields

$$a(v_m)\nu \left[f_m \left(\frac{\nu}{(n-1)!} \right) \right]^{-1} \leq \lambda_m \leq a(u_m)\nu \, [f_m(0)]^{-1}. \tag{3.42}$$

It follows from (3.38) that $\{y_m^{(n-1)}\}_{m=1}^{\infty}$ is a uniformly bounded sequence on [0, 1]. Using the initial conditions (3.34) and repeated integrations, we find that $\{y_m^{(i)}\}_{m=1}^{\infty}$, $0 \leq i \leq n-1$ is a uniformly bounded sequence. Thus, there exists a subsequence, which can be relabelled as $\{y_m\}_{m=1}^{\infty}$, that converges uniformly (in fact, in $C^{(n-1)}$-norm) to some y on [0, 1]. We note that each $y_m(t)$ can be expressed as

$$\begin{aligned} y_m(t) = \lambda_m \int_0^1 g(t, s)[Q_m(s, y_m, y_m', \cdots, y_m^{(n-2)}) \\ - P_m(s, y_m, y_m', \cdots, y_m^{(n-1)})]ds, \; t \in [0, 1]. \end{aligned} \tag{3.43}$$

Since $\{\lambda_m\}$ is a bounded sequence (from (3.42)), there is a subsequence, which can be relabelled as $\{\lambda_m\}$, that converges to some λ. Then, letting $m \to \infty$ in (3.43) yields

$$\begin{aligned} y(t) = \lambda \int_0^1 g(t, s)[Q(s, y, y', \cdots, y^{(n-2)}) \\ - P(s, y, y', \cdots, y^{(n-1)})]ds, \; t \in [0, 1]. \end{aligned}$$

This means that y is an eigenfunction of (1.1)–(1.4) corresponding to the eigenvalue λ. Further, $y^{(n-1)}(0) = \nu$, and inequality (3.20) follows from (3.42) immediately.

(b) Here, the boundary conditions (1.2)–(1.4) reduce to

$$y^{(i)}(0) = 0, \; 0 \leq i \leq n-2,$$

$$\gamma y^{(n-2)}(1) + \delta y^{(n-1)}(1) = 0.$$

Using a similar technique as in case (a), for a fixed m we let λ_m be an eigenvalue and y_m, with $y_m^{(n-1)}(0) = \nu$, be a corresponding eigenfunction of the boundary value problem (3.31),

$$\begin{aligned} &y_m^{(i)}(0) = 0, 0 \le i \le n-2, \\ &\gamma y_m^{(n-2)}(1) + \delta y_m^{(n-1)}(1) = 0. \end{aligned} \tag{3.44}$$

It is obvious that the eigenfunction y_m is the unique solution of the initial value problem (3.31), (3.34). As before we get the inequalities (3.38) and (3.39).

If $\gamma \ge 0$, then

$$\gamma\phi_3(t) + \delta\phi_1(t) \le \gamma y_m^{(n-2)}(t) + \delta y_m^{(n-1)}(t) \le \gamma\phi_4(t) + \delta\phi_2(t). \tag{3.45}$$

If $\gamma < 0$, then (3.38) and (3.39) lead to

$$\gamma\phi_4(t) + \delta\phi_1(t) \le \gamma y_m^{(n-2)}(t) + \delta y_m^{(n-1)}(t) \le \gamma\phi_3(t) + \delta\phi_2(t). \tag{3.46}$$

Since y_m satisfies $\gamma y_m^{(n-2)}(1) + \delta y_m^{(n-1)}(1) = 0$ (from (3.44)), in inequality (3.45) it is necessary that

$$\gamma\phi_3(1) + \delta\phi_1(1) \le 0, \quad \text{and} \quad \gamma\phi_4(1) + \delta\phi_2(1) \ge 0,$$

which provide

$$b(v_m)\nu(\gamma+\delta)\left[f_m\left(\frac{\nu}{(n-1)!}\right)\right]^{-1} \le \lambda_m \le b(u_m)\nu(\gamma+\delta)[f_m(0)]^{-1}, \gamma \ge 0. \tag{3.47}$$

Likewise, in inequality (3.46) we must have

$$\gamma\phi_4(1) + \delta\phi_1(1) \le 0, \text{ and } \gamma\phi_3(1) + \delta\phi_2(1) \ge 0,$$

which give

$$\begin{aligned} c\left(u_m, v_m, f_m, 0, \frac{\nu}{(n-1)!}\right)\nu(\gamma+\delta) \le \lambda_m \le c\left(v_m, u_m, f_m, \frac{\nu}{(n-1)!}, 0\right) \\ \times \nu(\gamma+\delta), \gamma < 0. \end{aligned} \tag{3.48}$$

Using a similar argument as in case (a), we get y_m converges to y (satisfying (3.19)), λ_m converges to λ, and y is a eigenfunction of (1.1)–(1.4) corresponding to the eigenvalue λ. Further, inequalities (3.22) and (3.24) follow from (3.47) and (3.48), respectively.

(c) In this case, the boundary conditions (1.2)–(1.4) become

$$\begin{aligned} &y^{(i)}(0) = 0, 0 \le i \le n-3, \\ &y^{(n-2)}(1) = 0, \\ &\alpha y^{(n-2)}(0) - \beta y^{(n-1)}(0) = 0. \end{aligned}$$

For a fixed m, let λ_m be an eigenvalue and y_m, with $y_m^{(n-2)}(0) = \mu, y_m^{(n-1)}(0) = \nu, \alpha\mu = \beta\nu$, be a corresponding eigenfunction of the boundary value problem (3.31),

$$\begin{aligned} &y_m^{(i)}(0) = 0, 0 \le i \le n-3, \\ &y_m^{(n-2)}(1) = 0, \\ &\alpha y_m^{(n-2)}(0) - \beta y_m^{(n-1)}(0) = 0. \end{aligned} \tag{3.49}$$

Clearly, the eigenfunction y_m is the unique solution of the differential equation (3.31), together with the initial conditions

$$\begin{aligned} &y_m^{(i)}(0) = 0, 0 \le i \le n-3, \\ &y_m^{(n-2)}(0) = \mu, \\ &y_m^{(n-1)}(0) = \nu. \end{aligned} \tag{3.50}$$

As in case (a), we see that $y_m^{(n-1)}$ is nonincreasing and hence (3.35) holds. In view of the initial conditions (3.50) and (3.35), we find

$$y_m^{(n-2)}(t) = \mu + \int_0^t y_m^{(n-1)}(s)ds \le \mu + \int_0^t \nu ds = \mu + \nu t, t \in [0, 1].$$

It follows that

$$y_m^{(n-3)}(t) = \int_0^t y_m^{(n-2)}(s)ds \le \int_0^t (\mu + \nu s)ds = \mu t + \nu\frac{t^2}{2!}, t \in [0, 1].$$

Continuing the process we obtain for $t \in [0, 1]$,

$$y_m(t) \le \mu\frac{t^{n-2}}{(n-2)!} + \nu\frac{t^{n-1}}{(n-1)!} \le \frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}. \tag{3.51}$$

Now, it follows from (3.31), (3.33) and (3.51) that for $t \in [0, 1]$,

$$\lambda_m u_m(t) f_m(0) \le -y_m^{(n)}(t) \le \lambda_m v_m(t) f_m\left(\frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right). \tag{3.52}$$

An integration of (3.52) from 0 to t gives

$$\phi_5(t) \le y_m^{(n-1)}(t) \le \phi_6(t),\ t \in [0, 1], \tag{3.53}$$

where

$$\phi_5(t) = \nu - \lambda_m f_m \left(\frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right) \int_0^t v_m(s)ds,$$

and

$$\phi_6(t) = \nu - \lambda_m f_m(0) \int_0^t u_m(s)ds.$$

Once again, we integrate (3.53) from 0 to t, to get

$$\phi_7(t) \le y_m^{(n-2)}(t) \le \phi_8(t),\ t \in [0, 1], \tag{3.54}$$

where

$$\phi_7(t) = \mu + \nu t - \lambda_m f_m \left(\frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right) \int_0^t (t-s)v_m(s)ds,$$

and

$$\phi_8(t) = \mu + \nu t - \lambda_m f_m(0) \int_0^t (t-s)u_m(s)ds.$$

Since y_m satisfies the boundary condition $y_m^{(n-2)}(1) = 0$ (see (3.49)), in inequality (3.54) we must have

$$\phi_7(1) \le 0, \quad \text{and} \quad \phi_8(1) \ge 0,$$

or equivalently,

$$\lambda_m \ge a(v_m)(\mu + \nu)\left[f_m\left(\frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right)\right]^{-1}, \tag{3.55}$$

and

$$\lambda_m \le a(u_m)(\mu + \nu)[f_m(0)]^{-1}. \tag{3.56}$$

Again, using a similar argument as in case (a), we find that y_m converges to y (satisfying (3.26)), λ_m converges to λ, and y is a eigenfunction of (1.1)–(1.4) corresponding to the eigenvalue λ. Further, inequality (3.27) follows immediately from (3.55) and (3.56). (d)

For a fixed m, let λ_m be an eigenvalue and y_m, with $y_m^{(n-2)}(0) = \mu$, $y_m^{(n-1)}(0) = \nu$, $\alpha\mu = \beta\nu$, be a corresponding eigenfunction of the boundary value problem (3.31),

$$y_m^{(i)}(0) = 0, 0 \le i \le n-3,$$

$$\alpha y_m^{(n-2)}(0) - \beta y_m^{(n-1)}(0) = 0,$$

$$\gamma y_m^{(n-2)}(1) + \delta y_m^{(n-1)}(1) = 0.$$

It is obvious that the eigenfunction y_m is the unique solution of the initial value problem (3.31), (3.50). As in case (c), we get the inequalities (3.53) and (3.54) which lead to

$$\gamma\phi_7(t) + \delta\phi_5(t) \le \gamma y_m^{(n-2)}(t) + \delta y_m^{(n-1)}(t) \le \gamma\phi_8(t) + \delta\phi_6(t), \gamma \ge 0, \tag{3.57}$$

and

$$\gamma\phi_8(t) + \delta\phi_5(t) \le \gamma y_m^{(n-2)}(t) + \delta y_m^{(n-1)}(t) \le \gamma\phi_7(t) + \delta\phi_6(t), \gamma < 0. \tag{3.58}$$

Since y_m satisfies the boundary condition $\gamma y_m^{(n-2)}(1) + \delta y_m^{(n-1)}(1) = 0$, in inequality (3.57) it is necessary that

$$\gamma\phi_7(1) + \delta\phi_5(1) \le 0, \quad \text{and} \quad \gamma\phi_8(1) + \delta\phi_6(1) \ge 0,$$

which reduce to

$$\begin{aligned} b(v_m)\,[\gamma(\mu+\nu) + \delta\nu]\left[f_m\left(\frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right)\right]^{-1} \\ \le \lambda_m \le b(u_m)\,[\gamma(\mu+\nu) + \delta\nu][f_m(0)]^{-1}, \gamma \ge 0. \end{aligned} \tag{3.59}$$

Likewise, in inequality (3.58) we must have

$$\gamma\phi_8(1) + \delta\phi_5(1) \le 0, \quad \text{and} \quad \gamma\phi_7(1) + \delta\phi_6(1) \ge 0,$$

which provide

$$\begin{aligned} c\left(u_m, v_m, f_m, 0, \frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}\right)[\gamma(\mu+\nu) + \delta\nu] \\ \le \lambda_m \le c\left(v_m, u_m, f_m, \frac{\mu}{(n-2)!} + \frac{\nu}{(n-1)!}, 0\right)[\gamma(\mu+\nu) + \delta\nu], \\ \gamma < 0. \end{aligned} \tag{3.60}$$

Once again, using a similar argument as in case (a), we find that y_m converges to y (satisfying (3.26)), λ_m converges to λ, and y is a eigenfunction of (1.1)–(1.4) corresponding to the eigenvalue λ. Further, inequalities (3.28) and (3.29) follow immediately from (3.59) and (3.60) respectively.

THEOREM 3.4. *Let λ be an eigenvalue of (1.1)–(1.4) and $y \in C$ be a corresponding eigenfunction. Further, let $\eta = \|y\|$. Then,*

$$\lambda \geq \frac{\eta\rho}{L}\left\{f\left(\frac{\eta}{(n-2)!}\right)\int_0^1 (\beta + \alpha s)[\delta + \gamma(1-s)]v(s)ds\right\}^{-1}. \tag{3.61}$$

Also, there exists a $c > 0$ such that

$$\lambda \leq \frac{\eta\rho}{f(c\eta)}\left\{\int_{1/4}^{1/2}\left(\delta + \frac{\gamma}{2}\right)(\beta + \alpha s)u(s)ds\right\}^{-1}. \tag{3.62}$$

Proof. We observe that $y^{(n)}$ is nonpositive and hence $y^{(n-2)}$ is concave on [0, 1]. This, together with the fact that $y^{(n-2)}$ is nonnegative, implies the existence of a unique $t_0 \in (0, 1)$ such that

$$\eta = \|y\| = y^{(n-2)}(t_0).$$

To prove that (3.61) holds, we use (3.12), Lemma 2.2, (3.2) and (2.1) sucessively, to get

$$\begin{aligned}
\eta &= y^{(n-2)}(t_0) = (\lambda Sy)^{(n-2)}(t_0)\\
&\leq \lambda\int_0^1 G(t_0, s)v(s)f(y(s))ds\\
&\leq \lambda L\int_0^1 G(s, s)v(s)f(y(s))ds\\
&\leq \lambda L\int_0^1 G(s, s)v(s)f\left(\frac{\eta}{(n-2)!}\right)ds\\
&= \frac{\lambda L}{\rho}f\left(\frac{\eta}{(n-2)!}\right)\int_0^1 (\beta + \alpha s)[\delta + \gamma(1-s)]v(s)ds.
\end{aligned}$$

The inequality (3.61) now follows immediately.

Next, to prove (3.62) we shall consider four cases.

Case 1 $\delta = \beta = 0$, $\gamma = \alpha = 1$

Here, $y^{(n-2)}(0) = y^{(n-2)}(1) = 0$. By the concavity of $y^{(n-2)}$, we find

$$\begin{aligned}
y^{(n-2)}(t) &\geq \begin{cases} \dfrac{\eta}{t_0}t, & t \in [0, t_0]\\[2ex] \dfrac{\eta}{1-t_0}(1-t), & t \in [t_0, 1]\end{cases}\\
&\geq \eta t(1-t),\ t \in [0, 1].
\end{aligned} \tag{3.63}$$

Thus, on using (1.2) and (3.63) we get for $t \in [0, 1]$,

$$y^{(n-3)}(t) = \int_0^t y^{(n-2)}(s)ds \geq \int_0^t \eta s(1-s)ds = \eta\left(\frac{t^2}{2} - \frac{t^3}{3}\right).$$

Continuing the integration process we obtain

$$y(t) \geq \eta\, \psi(t),\ t \in [0, 1], \tag{3.64}$$

where

$$\psi(t) = \frac{t^{n-1}}{(n-1)!} - 2\frac{t^n}{n!}.$$

We note that

$$\psi'(t) = \frac{t^{n-2}}{(n-2)!}\left(1 - \frac{2t}{n-1}\right)$$

is nonnegative for $t \in I \equiv \left[0, \frac{n-1}{2}\right]$. Hence, in particular $\psi(t)$ is nondecreasing for $t \in J \equiv \left[\frac{1}{4}, \frac{1}{2}\right] \subseteq I$. It follows from (3.64) that

$$y(t) \geq c\eta,\ t \in J, \tag{3.65}$$

where

$$c = \psi\left(\frac{1}{4}\right) = \frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!} > 0. \tag{3.66}$$

Now, in view of (3.12), (3.65) and (2.1), we find

$$\begin{aligned}
\eta &\geq y^{(n-2)}\left(\frac{1}{2}\right) = (\lambda Sy)^{(n-2)}\left(\frac{1}{2}\right) \\
&\geq \lambda \int_0^1 G\left(\frac{1}{2}, s\right) u(s) f(y(s))ds \\
&\geq \lambda \int_{1/4}^{1/2} G\left(\frac{1}{2}, s\right) u(s) f(y(s))ds \\
&\geq \lambda \int_{1/4}^{1/2} G\left(\frac{1}{2}, s\right) u(s) f(c\eta)ds \\
&= \frac{\lambda}{\rho} f(c\eta) \int_{1/4}^{1/2} \left(\delta + \frac{\gamma}{2}\right)(\beta + \alpha s)u(s)ds
\end{aligned}$$

from which (3.62) follows immediately.

Case 2 $\delta > 0$, $\beta = 0$

In this case, $y^{(n-2)}(0) = 0$, $y^{(n-2)}(1) \neq 0$. Hence, for $t \in [0, 1]$,

$$y^{(n-2)}(t) \geq y^{(n-2)}(1)\, t \geq y^{(n-2)}(1)\, t(1-t). \tag{3.67}$$

Using a similar technique as in Case 1, it follows from (3.67) and successive integrations that

$$y(t) \geq y^{(n-2)}(1)\, \psi(t),\ t \in [0, 1]. \tag{3.68}$$

This leads to (3.65), where

$$c = \frac{y^{(n-2)}(1)}{\eta}\left[\frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!}\right] > 0. \tag{3.69}$$

The rest of the proof is similar to that of Case 1.

Case 3 $\delta = 0$, $\beta > 0$

In this case, $y^{(n-2)}(0) \neq 0$, $y^{(n-2)}(1) = 0$. Thus, for $t \in [0, 1]$,

$$y^{(n-2)}(t) \geq y^{(n-2)}(0)\,(1-t) \geq y^{(n-2)}(0)\, t(1-t). \tag{3.70}$$

Again, as in Case 1 it follows from (3.70) and successive integrations that

$$y(t) \geq y^{(n-2)}(0)\, \psi(t),\ t \in [0, 1]. \tag{3.71}$$

Inequality (3.71) implies (3.65), where

$$c = \frac{y^{(n-2)}(0)}{\eta}\left[\frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!}\right] > 0. \tag{3.72}$$

The rest of the proof is similar to that of Case 1.

Case 4 $\delta > 0$, $\beta > 0$

Here, $y^{(n-2)}(0) \neq 0$, $y^{(n-2)}(1) \neq 0$. Let

$$m = \min\{y^{(n-2)}(0), y^{(n-2)}(1)\}.$$

Then,

$$y^{(n-2)}(t) \geq m \geq mt(1-t),\ t \in [0, 1]. \tag{3.73}$$

Once again, it follows from (3.73) and successive integrations that

$$y(t) \geq m\,\psi(t),\ t \in [0, 1], \tag{3.74}$$

which in turn leads to inequality (3.65), where

$$c = \frac{m}{\eta}\left[\frac{1}{4^{n-1}(n-1)!} - \frac{2}{4^n n!}\right] > 0. \tag{3.75}$$

The rest of the proof is similar to that of Case 1.

This completes the proof of the theorem.

THEOREM 3.5. *Let*

$$F_B = \left\{ f \,\middle|\, \frac{u}{f(u)} \text{ is bounded for } u \in [0, \infty) \right\},$$

$$F_0 = \left\{ f \,\middle|\, \lim_{u\to\infty} \frac{u}{f(u)} = 0 \right\},$$

$$F_\infty = \left\{ f \,\middle|\, \lim_{u\to\infty} \frac{u}{f(u)} = \infty \right\}.$$

(a) If $f \in F_B$, then $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$.
(b) If $f \in F_0$, then $E = (0, c]$ for some $c \in (0, \infty)$.
(c) If $f \in F_\infty$, then $E = (0, \infty)$.

Proof. (a) This is immediate from (3.62).

(b) Since $F_0 \subseteq F_B$, it follows from case (a) that $E = (0, c)$ or $(0, c]$ for some $c \in (0, \infty)$. In particular,

$$c = \sup E. \tag{3.76}$$

Let $\{\lambda_m\}_{m=1}^{\infty}$ be a monotonically increasing sequence in E which converges to c, and let $\{y_m\}_{m=1}^{\infty}$ in C be a corresponding sequence of eigenfunctions. Further, let $\eta_m = \|y_m\|$. Then, (3.61) implies that no subsequence of $\{\eta_m\}_{m=1}^{\infty}$ can diverge to infinity. Thus, there exists $M > 0$ such that $\eta_m \leq M$ for all m. In view of (3.2), we find that y_m is uniformly bounded. Hence, there is a subsequence of $\{y_m\}$, relabelled as the original sequence, which converges uniformly to some $y \in C$.

Noting that $\lambda_m S y_m = y_m$, we have

$$cSy_m = \frac{c}{\lambda_m} y_m. \tag{3.77}$$

Since $\{cSy_m\}_{m=1}^{\infty}$ is relatively compact, y_m converges to y and λ_m converges to c, it follows from (3.77) that

$$cSy = y,$$

i.e., $c \in E$. This completes the proof for case (b).

(c) This follows from Corollary 3.1 and (3.61).

Example 3.1. Consider the boundary value problem

$$y^{(3)} + \lambda \left\{ \phi(t, y, y') + \frac{2t}{[t(7 - t - t^3) + 5]^r} \right\} (y + 5)^r$$
$$= \lambda\phi(t, y, y')\,(y + 5)^r,\ t \in (0, 1),$$
$$y(0) = 0,$$
$$-2y'(0) - 7y''(0) = 0,$$
$$14y'(1) + y''(1) = 0,$$

where $\lambda > 0$, $0 \le r < 1$, and $\phi(t, y, y')$ is any function of t, y and y'.

Taking $f(y) = (y + 5)^r$, we find

$$\frac{Q(t, y, y')}{f(y)} = \phi(t, y, y') + \frac{2t}{[t(7 - t - t^3) + 5]^r}$$

and

$$\frac{P(t, y, y', y'')}{f(y)} = \phi(t, y, y').$$

Hence, we may take

$$q(t) = \phi(t, y, y') + \frac{t}{[t(7 - t - t^3) + 5]^r},$$

$$q_1(t) = \phi(t, y, y') + \frac{2t}{[t(7 - t - t^3) + 5]^r}$$

and

$$p(t) = p_1(t) = \phi(t, y, y').$$

Since $f \in F_\infty$, by Theorem 3.5(c) the set $E = (0, \infty)$. As an example when $\lambda = 12$, the boundary value problem has a positive solution given by $y(t) = t(7 - t - t^3)$.

Example 3.2. Consider the boundary value problem

$$y'' + \lambda \left\{ \phi(t, y) + \frac{1}{[t(1-t)+5]^r} \right\} (y+3)^r = \lambda \phi(t, y)\,(y+3)^r,\ t \in (0, 1),$$

$$y(0) - 2y'(0) = 0,$$

$$y(1) + 2y'(1) = 0,$$

where $\lambda > 0$, $0 \leq r < 3/2$, and $\phi(t, y)$ is any function of t and y.

Taking $f(y) = (y+3)^r$, we find

$$\frac{Q(t, y)}{f(y)} = \phi(t, y) + \frac{1}{[t(1-t)+5]^r},$$

and

$$\frac{P(t, y, y')}{f(y)} = \phi(t, y).$$

Hence, we may choose

$$q(t) = \phi(t, y) + \frac{1}{2[t(1-t)+5]^r}, \quad q_1(t) = \phi(t, y) + \frac{1}{[t(1-t)+5]^r},$$

and

$$p(t) = p_1(t) = \phi(t, y).$$

Case 1 $0 \leq r < 1$

Since $f \in F_\infty$, by Theorem 3.5(c) the set $E = (0, \infty)$. For example when $\lambda = 2$, the boundary value problem has a positive solution given by $y(t) = t(1 - t) + 2$.

Case 2 $r = 1$

Since $f \in F_B$, by Theorem 3.5(a) the set E is an open or half-closed interval. Further, we note from Case 1 and Theorem 3.2 that E contains the interval $(0, 2]$.

Case 3 $1 < r < \frac{3}{2}$

Since $f \in F_0$, by Theorem 3.5(b) the set E is a half-closed interval. Again, it is noted that $(0, 2] \subseteq E$.

4. SPECIAL CASE: $\lambda = 1$

THEOREM 4.1. *Suppose that* (A1)–(A5) *hold. Then,* (1.1)–(1.4) *has a positive solution.*

Proof. To obtain a positive solution of (1.1)–(1.4), we shall seek a fixed point of the operator S (defined in (3.8)) in the cone C. We have seen that S is compact on the cone C. Further, we observe from the proof of Theorem 3.1 that S maps C into itself. Also, the standard arguments yield that S is completely continuous.

Case 1 Suppose that f is superlinear. Since $f_0 = 0$, we may choose $a_1 > 0$ such that $f(u) \leq \epsilon\, u$ for $0 < u \leq a_1$, where $\epsilon > 0$ satisfies

$$\frac{L\epsilon}{(n-2)!}\int_0^1 G(s, s)v(s)ds \leq 1. \tag{4.1}$$

Let $y \in C$ be such that $\|y\| = a_1(n-2)!$. Then, from (3.2) we have $|y(t)| \leq a_1$, $t \in [0, 1]$. Hence, applying (3.12), Lemma 2.2, (3.2) and (4.1) successively gives for $t \in [0, 1]$,

$$\begin{aligned}(Sy)^{(n-2)}(t) &\leq \int_0^1 G(t, s)v(s)f(y(s))ds \\ &\leq L\int_0^1 G(s, s)v(s)f(y(s))ds \\ &\leq L\epsilon\int_0^1 G(s, s)v(s)y(s)ds \\ &\leq L\epsilon\int_0^1 G(s, s)v(s)\frac{1}{(n-2)!}\|y\|ds \leq \|y\|.\end{aligned}$$

Consequently,

$$\|Sy\| \leq \|y\|. \tag{4.2}$$

If we set

$$\Omega_1 = \{y \in B \mid \|y\| < a_1(n-2)!\},$$

then (4.2) holds for $y \in C \cap \partial\Omega_1$.

Next, since $f_\infty = \infty$, we may choose $\bar{a}_2 > 0$ such that $f(u) \geq Mu$ for $u \geq \bar{a}_2$, where $M > 0$ satisfies

$$\frac{\xi M}{4^{n-2}(n-2)!}\int_{1/2}^{3/4} G\left(\tfrac{1}{2}, s\right) u(s)ds \geq 1. \tag{4.3}$$

Let

$$a_2 = \max\left\{2a_1(n-2)!, \frac{4^{n-2}(n-2)!}{\xi}\bar{a}_2\right\},$$

and let $y \in C$ be such that $\|y\| = a_2$. Then, from (3.6) we have

$$y(t) \geq \frac{\xi}{4^{n-2}(n-2)!}\|y\| \geq \frac{\xi}{4^{n-2}(n-2)!} \cdot \frac{4^{n-2}(n-2)!}{\xi}\bar{a}_2 = \bar{a}_2, t \in \left[\frac{1}{2}, \frac{3}{4}\right].$$

Hence, $f(y(t)) \geq My(t)$ for $t \in [\frac{1}{2}, \frac{3}{4}]$. In view of (3.12), (3.6) and (4.3), we find

$$\begin{aligned}(Sy)^{(n-2)}\left(\frac{1}{2}\right) &\geq \int_0^1 G\left(\frac{1}{2}, s\right) u(s)f(y(s))ds \\ &\geq \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) u(s)f(y(s))ds \\ &\geq M\int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) u(s)y(s)ds \\ &\geq M\int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) u(s)\frac{\xi}{4^{n-2}(n-2)!}\|y\|\, ds \geq \|y\|.\end{aligned}$$

Therefore,

$$\|Sy\| \geq \|y\|. \tag{4.4}$$

If we set

$$\Omega_2 = \{y \in B \mid \|y\| < a_2\},$$

then (4.4) holds for $y \in C \cap \partial\Omega_2$.

In view of (4.2) and (4.4), it follows from Theorem 2.1 that S has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$, such that

$$a_1(n-2)! \leq \|y\| \leq a_2.$$

This y is a positive solution of (1.1)–(1.4).

Case 2 Suppose that f is sublinear. Since $f_0 = \infty$, there exists $a_3 > 0$ such that $f(u) \geq \bar{M}u$ for $0 < u \leq a_3$, where $\overline{M} > 0$ satisfies

$$\frac{\xi\bar{M}}{4^{n-2}(n-2)!}\int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) u(s)ds \geq 1. \tag{4.5}$$

Let $y \in C$ be such that $\|y\| = a_3(n-2)!$. Then, from (3.2) we have $|y(t)| \leq a_3$, $t \in [0, 1]$. Hence, on using (3.12), (3.6) and (4.5) successively, we get

$$
\begin{aligned}
(Sy)^{(n-2)}\left(\frac{1}{2}\right) &\geq \int_0^1 G\left(\frac{1}{2}, s\right) u(s) f(y(s)) ds \\
&\geq \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) u(s) f(y(s)) ds \\
&\geq \overline{M} \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) u(s) y(s) ds \\
&\geq \overline{M} \int_{1/2}^{3/4} G\left(\frac{1}{2}, s\right) u(s) \frac{\xi}{4^{n-2}(n-2)!} \|y\| \, ds \geq \|y\|.
\end{aligned}
$$

from which the inequality (4.4) follows immediately. If we set

$$\Omega_1 = \{y \in B \mid \|y\| < a_3(n-2)!\},$$

then (4.4) holds for $y \in C \cap \partial\Omega_1$.

Next, in view of $f_\infty = 0$, we may choose $\bar{a}_4 > 0$ such that $f(u) \leq \bar{\epsilon}\, u$ for $u \geq \bar{a}_4$, where $\bar{\epsilon} > 0$ satisfies

$$\frac{L\bar{\epsilon}}{(n-2)!} \int_0^1 G(s, s) v(s) ds \leq 1. \tag{4.6}$$

There are two cases to consider, namely, f is bounded and f is unbounded.

Case (i) Suppose that f is bounded, i.e., $f(u) \leq R$, $u \in [0, \infty)$ for some $R > 0$. Let

$$a_4 = \max\left\{2a_3(n-2)!, \frac{LR}{(n-2)!} \int_0^1 G(s, s) v(s) ds\right\},$$

and let $y \in C$ be such that $\|y\| = a_4(n-2)!$. For $t \in [0, 1]$, from (3.12) and Lemma 2.2, we find

$$
\begin{aligned}
(Sy)^{(n-2)}(t) &\leq \int_0^1 G(t, s) v(s) f(y(s)) ds \\
&\leq R \int_0^1 G(t, s) v(s) ds \\
&\leq LR \int_0^1 G(s, s) v(s) ds \\
&\leq a_4(n-2)! = \|y\|.
\end{aligned}
$$

Hence, (4.2) holds.

Case (ii) Suppose that f is unbounded, i.e., there exists

$$a_4 > \max\{2a_3(n-2)!, \bar{a}_4\},$$

such that $f(u) \leq f(a_4)$ for $0 < u \leq a_4$. Let $y \in C$ be such that $\|y\| = a_4\,(n-2)!$. Then, from (3.2) we have $|y(t)| \leq a_4$, $t \in [0, 1]$. Hence, applying (3.12), Lemma 2.2 and (4.6) successively gives for $t \in [0, 1]$,

$$\begin{aligned}(Sy)^{(n-2)}(t) &\leq \int_0^1 G(t, s)v(s)f(y(s))ds \\ &\leq L\int_0^1 G(s, s)v(s)f(y(s))ds \\ &\leq L\int_0^1 G(s, s)v(s)f(a_4)ds \\ &\leq L\int_0^1 G(s, s)v(s)\bar{\epsilon}\, a_4\, ds \\ &\leq a_4(n-2)! = \|y\|\end{aligned}$$

from which (4.2) follows immediately.

In both Cases (i) and (ii), if we set

$$\Omega_2 = \{y \in B \mid \|y\| < a_4(n-2)!\},$$

then (4.2) holds for $y \in C \cap \partial\Omega_2$.

Now that we have obtained (4.4) and (4.2), it follows from Theorem 2.1 that S has a fixed point $y \in C \cap (\bar{\Omega}_2 \backslash \Omega_1)$, such that

$$a_3(n-2)! \leq \|y\| \leq a_4(n-2)!.$$

This y is a positive solution of (1.1)–(1.4).

The proof of the theorem is complete.

The following two examples illustrate Theorem 4.1.

Example 4.1. Consider the boundary value problem

$$\begin{gathered} y^{(4)} + \left\{\phi(t, y, y', y'') + \frac{24}{[t^2(9-t-t^2)+1]^r}\right\}(y+1)^r \\ = \phi(t, y, y', y'')\,(y+1)^r,\ t \in (0, 1), \\ y(0) = y'(0) = y''(1) = 0, \\ -y''(0) - 3y^{(3)}(0) = 0, \end{gathered}$$

where $0 \leq r < \frac{3}{4}$ and $\phi(t, y, y', y'')$ is any function of t, y, y' and y''.

Taking $f(y) = (y+1)^r$ (which is sublinear), we find

$$\frac{Q(t, y, y', y'')}{f(y)} = \phi(t, y, y', y'') + \frac{24}{[t^2(9 - t - t^2) + 1]^r},$$

and

$$\frac{P(t, y, y', y'', y^{(3)})}{f(y)} = \phi(t, y, y', y'').$$

Hence, we may choose

$$q(t) = \phi(t, y, y', y'') + \frac{2}{[t^2(9 - t - t^2) + 1]^r},$$

$$q_1(t) = \phi(t, y, y', y'') + \frac{24}{[t^2(9 - t - t^2) + 1]^r},$$

and

$$p(t) = p_1(t) = \phi(t, y, y', y'').$$

All the conditions of Theorem 4.1 are fulfilled and therefore the boundary value problem has a positive solution. One such solution is given by $y(t) = t^2(9 - t - t^2)$.

Example 4.2. Consider the boundary value problem

$$y'' + \left\{\phi(t, y) + \frac{6t}{(10 - t - t^3)^r}\right\}(y + 7)^r = \phi(t, y)(y + 7)^r, t \in (0, 1),$$

$$-y(0) - 3y'(0) = 0,$$

$$4y(1) + y'(1) = 0,$$

where $0 \leq r < \frac{4}{3}$, $r \neq 1$, and $\phi(t, y)$ is any function of t and y.

Taking $f(y) = (y + 7)^r$ (which is superlinear if $r > 1$, and sublinear if $r < 1$), we find

$$\frac{Q(t, y)}{f(y)} = \phi(t, y) + \frac{6t}{(10 - t - t^3)^r},$$

and

$$\frac{P(t, y, y')}{f(y)} = \phi(t, y).$$

Hence, we may take

$$q(t) = \phi(t, y) + \frac{t}{(10 - t - t^3)^r}, \; q_1(t) = \phi(t, y) + \frac{8t}{(10 - t - t^3)^r},$$

and

$$p(t) = p_1(t) = \phi(t, y).$$

Again, all the conditions of Theorem 4.1 are satisfied and so the boundary value problem has a positive solution. Indeed, $y(t) = 3 - t - t^3$ is one such solution.

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