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SURFACE NORMALS AND BARYCENTRIC COORDINATES

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The normal to a triangular parametric surface is investigated where the parameters used are barycentric coordinates. Formulae for the normal are obtained for non-rational and rational surfaces.

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1. INTRODUCTION

Normally, the large free-form surfaces used in geometric modelling systems are built up from simpler patches. Usually these patches have four sides. They are defined parametrically with two parameters which effectively represent the coordinates of a typical point within a rectagular region of the plane.

In some cases, large surfaces are built up from three sided patches or involve a mixture of such patches and others. For triangular patches, the parameters used relate to a triangular region in the plane. Often *barycentric coordinates* with respect to the triangle are used for the parameters. These are not independent; their sum is unity.

Often there is a requirement to evaluate surface normals. This paper looks at this evaluation when barycentric coordinates are used. What is obtained is a symmetric formula involving the partial derivatives with respect to the three coordinates (treated as being independent).

The ideas are also taken a stage further to rational forms of surfaces. Here an extra fourth coordinate is used, with the cartesian point being obtained by dividing each of the first three by the fourth. This division complicates the evaluation of partial derivatives.

What is found is that when the surface normal is formed some of this complexity disappears. The normal is expressed as a 5×5 determinant which involves the unit vectors along the three main axes and the four rational coordinates and their partial derivatives with respect to the barycentric coordinates. No quotient is present except for a factor which divides the whole determinant. This can be omitted if there is interest only in the direction of the normal.

The case for non-rational surfaces is given in section 2 using direct vector methods. An alternative approach is presented in section 3 using matrices and determinants and this is extended to the rational case in section 4. An example using a representation of an octant of a sphere as a fourth degree Bézier patch is given in section 5.

2. SURFACE NORMALS

Consider a function $\mathbf{r}(\mathbf{t})$ which a mapping from the plane, $\Re^2$, into three dimensional space, $\Re^3$. In particular, attention is given to the case when $\mathbf{t}$ lies within a triangular region of the plane whose vertices are $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$.

The notation (p, q) is used for the cartesian position of a typical point $\mathbf{t}$ in the plane. Similarly, the coordinates of the vertices of the triangle are given as follows.

$$\mathbf{r}_1 = (p_1, q_1) \qquad \mathbf{r}_2 = (p_2, q_2) \qquad \mathbf{r}_3 = (p_3, q_3)$$

In addition, *barycentric coordinates* are used relative to the triangle. These are denoted by u, v, w. If $\mathbf{t}$ is the typical point in the triangle (or in the plane generally), then its barycentric coordinates are given by

$$\begin{aligned} u &= \Delta(\mathbf{t}, \mathbf{r}_2, \mathbf{r}_3)/\Delta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ v &= \Delta(\mathbf{r}_1, \mathbf{t}, \mathbf{r}_3)/\Delta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \\ w &= \Delta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{t})/\Delta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) \end{aligned} \tag{1}$$

where the Δ notation is used for the area of the triangle formed by its three arguments (taken in anticlockwise order around the triangle). For example, the area of the triangle formed by $\mathbf{r}_1$, $\mathbf{r}_2$ and $\mathbf{r}_3$ is given by

$$\begin{aligned} \Delta(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3) &= \frac{1}{2}\begin{vmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{vmatrix} \\ &= \frac{1}{2}(p_1q_2 + p_2q_3 + p_3q_1 - p_2q_1 - p_3q_2 - p_1q_3) \end{aligned} \tag{2}$$

It is clear that the sum of the barycentric coordinates is unity,

$$u + v + w = 1$$

and the following equations relate them to the cartesian coordinates of the vertices of the triangle

$$\begin{aligned} p &= up_1 + vp_2 + wp_3 \\ q &= uq_1 + vq_2 + wq_3 \end{aligned}$$

In order to investigate the normal to the surface, partial derivatives of $\mathbf{r}$ with respect to p and q are required. The chain rule is used to relate these to the partial derivatives with

respect to u, v, w. It is important to note, that when this latter set of partial derivatives is used, u, v, w are being treated as *independent*, even though it is known that their sum is unity. The chain rule gives

$$\mathbf{r}_p = \mathbf{r}_u u_p + \mathbf{r}_v v_p + \mathbf{r}_w w_p$$

$$\mathbf{r}_q = \mathbf{r}_u u_q + \mathbf{r}_v v_q + \mathbf{r}_w w_q$$

Here the subscript notation is used for partial derivatives. Thus $\mathbf{r}_p$ means $\partial\mathbf{r}/\partial p$, and so on. The next step is to express the barycentric coordinates u, v, w in terms of p and q, so that their partial derivatives can be found. From the above, it follows that

$$\begin{bmatrix} 1 & 1 & 1 \\ p_1 & p_2 & p_3 \\ q_1 & q_2 & q_3 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \begin{bmatrix} 1 \\ p \\ q \end{bmatrix}$$

The determinant of the 3×3 matrix is twice the area, $\Delta = \Delta\,(\mathbf{r}_1, \mathbf{r}_2, \mathbf{r}_3)$, of the triangle. Inverting the matrix shows that

$$\begin{bmatrix} u \\ v \\ w \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} p_2q_3 - p_3q_2 & q_2 - q_3 & p_3 - p_2 \\ p_3q_1 - p_1q_3 & q_3 - q_1 & p_1 - p_3 \\ p_1q_2 - p_2q_1 & q_1 - q_2 & p_2 - p_1 \end{bmatrix} \begin{bmatrix} 1 \\ p \\ q \end{bmatrix}$$

Differentiating partially with respect to p and q yields

$$\begin{bmatrix} u_p \\ v_p \\ w_p \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} q_2 - q_3 \\ q_3 - q_1 \\ q_1 - q_2 \end{bmatrix}$$

$$\begin{bmatrix} u_q \\ v_q \\ w_q \end{bmatrix} = \frac{1}{2\Delta} \begin{bmatrix} p_3 - p_2 \\ p_1 - p_3 \\ p_2 - p_1 \end{bmatrix}$$

and so

$$\mathbf{r}_p = [(q_2 - q_3)\mathbf{r}_u + (q_3 - q_1)\mathbf{r}_v + (q_1 - q_2)\mathbf{r}_w] / 2\Delta \tag{3}$$

$$\mathbf{r}_q = [(p_3 - p_2)\mathbf{r}_u + (p_1 - p_3)\mathbf{r}_v + (p_2 - p_1)\mathbf{r}_w] / 2\Delta$$

The normal to the surface is taken to be the vector product of $\mathbf{r}_p$ and $\mathbf{r}_q$. This is denoted by $\mathbf{n}$. Thus

$$4\Delta^2(\mathbf{r}_p \times \mathbf{r}_q) = [(q_2 - q_3)\,(p_1 - p_3) - (q_3 - q_1)\,(p_3 - p_2)]\mathbf{r}_u \times \mathbf{r}_v + \ldots$$

Here the dots denote a sum over two additional summands. The term in the square brackets is seen to be 2Δ (cf. equation (2)). Hence the following result holds which gives the normal in terms of the partial derivatives with respect to the barycentric coordinates.

$$\mathbf{n} = \mathbf{r}_p \times \mathbf{r}_q = [(\mathbf{r}_u \times \mathbf{r}_v) + (\mathbf{r}_v \times \mathbf{r}_w) + (\mathbf{r}_w \times \mathbf{r}_u)] / 2\Delta \tag{4}$$

3. MATRICES AND DETERMINANTS

In this section, an alternative derivation of equation (4) is obtained. This is carried out using determinants. The techniques used are generalised for rational surfaces in the next section.

The approach springs from the fact that a vector product can be expressed as a determinant. Given two vectors

$$\mathbf{a} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$$

$$\mathbf{b} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$$

their vector product is given by

$$\mathbf{a} \times \mathbf{b} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} \tag{5}$$

The point $\mathbf{r}(\mathbf{t})$ lies in three dimensional space. Suppose that it has cartesian coordinates x, y, z so that

$$\mathbf{r}(\mathbf{t}) = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

Consider the matrix A given by

$$A = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{0} \\ x_u & y_u & z_u & 1 \\ x_v & y_v & z_v & 1 \\ x_w & y_w & z_w & 1 \end{bmatrix}$$

Taking the determinant of A and expanding by the last column, shows that

$$\det(A) = (\mathbf{r}_v \times \mathbf{r}_w) + (\mathbf{r}_w \times \mathbf{r}_u) + (\mathbf{r}_u \times \mathbf{r}_v)$$

Matrix B is defined as

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & q_2 - q_3 & q_3 - q_1 & q_1 - q_2 \\ 0 & p_3 - p_2 & p_1 - p_3 & p_2 - p_1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and it is clear that, as in equation (2),

$$\det(B) = 2\Delta$$

Now form the product BA. Equation (3) is used to show that

$$BA = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{0} \\ x_p\Delta & y_p\Delta & z_p\Delta & 0 \\ x_q\Delta & y_q\Delta & z_q\Delta & 0 \\ x_w & y_w & z_w & 1 \end{bmatrix}$$

The determinant of this matrix is clearly that of the upper left 3×3 minor. This has a factor of Δ in two rows, so that, using equation (5), the determinant is

$$\det(BA) = 4\Delta^2(\mathbf{r}_p \times \mathbf{r}_q)$$

Since $\det(BA) = \det(B)\det(A)$, it follows that

$$\mathbf{r}_p \times \mathbf{r}_q = [(\mathbf{r}_u \times \mathbf{r}_v) + (\mathbf{r}_v \times \mathbf{r}_w) + (\mathbf{r}_w \times \mathbf{r}_u)] / 2\Delta$$

which is equation (4) again.

4. RATIONAL SURFACES

Consider now the case of a rational surface. Here there is a map $\mathbf{R}$ from the plane, $\Re^2$, to four dimensional space, $\Re^4$. Again the image of a triangle in the plane is generally what is required.

Each point $\mathbf{R} = [X, Y, Z, W]$ in $\Re^4$ corresponds to the point

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k}$$

in $\Re^3$, where

$$x = X/W \qquad y = Y/W \qquad z = Z/W$$

The partial derivatives of $\mathbf{r}$ with respect to p and q are complicated because of the divisions. They are as follows.

$$\mathbf{r}_p = \frac{(X_pW - XW_p)\mathbf{i} + (Y_pW - YW_p)\mathbf{j} + (Z_pW - ZW_p)\mathbf{k}}{W^2}$$

$$\mathbf{r}_q = \frac{(X_qW - XW_q)\mathbf{i} + (Y_qW - YW_q)\mathbf{j} + (Z_qW - ZW_q)\mathbf{k}}{W^2}$$

The vector product of these gives the normal $\mathbf{n}$. After simplification, the component in the $\mathbf{i}$ direction is as follows.

$$[(Y_pZ_q - Y_qZ_p)W + (Z_pW_q - Z_qW_p)Y + (W_pY_q - W_qY_p)Z]/W^3$$

The components in the other two directions are similar. Hence it follows that

$$\mathbf{r}_p \times \mathbf{r}_q = \frac{1}{W^3}\begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{0} \\ X & Y & Z & W \\ X_p & Y_p & Z_p & W_p \\ X_q & Y_q & Z_q & W_q \end{vmatrix} \tag{6}$$

As in the previous section, matrices are introduced. Define A as follows.

$$A = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{0} & \mathbf{0} \\ X & Y & Z & W & 0 \\ X_u & Y_u & Z_u & W_u & 1 \\ X_v & Y_v & Z_v & W_v & 1 \\ X_w & Y_w & Z_w & W_w & 1 \end{bmatrix}$$

The determinant of this matrix can be formed by expanding by the last column. This involves three cofactors which have a similar form to the determinant in equation (6). It follows that

$$\det(A) = W^3(\mathbf{r}_v \times \mathbf{r}_w + \mathbf{r}_w \times \mathbf{r}_u + \mathbf{r}_u \times \mathbf{r}_v) \tag{7}$$

Define B to be the following matrix

$$B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & q_2 - q_3 & q_3 - q_1 & q_1 - q_2 \\ 0 & 0 & p_3 - p_2 & p_1 - p_3 & p_2 - p_1 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

From equation (2) it is clear that

$$\det(B) = 2\Delta$$

As in the previous section, the product BA is formed. Equation (3) shows that

$$BA = \begin{bmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{0} & \mathbf{0} \\ X & Y & Z & W & 0 \\ X_p\Delta & Y_p\Delta & Z_p\Delta & W_p\Delta & 0 \\ X_q\Delta & Y_q\Delta & Z_q\Delta & W_q\Delta & 0 \\ X_w & Y_w & Z_w & W_w & 1 \end{bmatrix}$$

The determinant of this matrix is that of the upper left 4×4 minor, and equation (6) shows that

$$\det(BA) = 4\Delta^2 W^3(\mathbf{r}_p \times \mathbf{r}_q)$$

Since this is also the product of the individual determinants, it is deduced that

$$\mathbf{n} = \mathbf{r}_p \times \mathbf{r}_q = [(\mathbf{r}_u \times \mathbf{r}_v) + (\mathbf{r}_v \times \mathbf{r}_w) + (\mathbf{r}_w \times \mathbf{r}_u)] / 2\Delta$$

Again this is equation (4) and the derivation is essentially that of the previous section. The significant point is that there is a means for finding this normal using equation (7). This requires the evaluation of certain cofactors of matrix A. These are composed of partial derivatives, with respect to the barycentric coordinates, of the components of the four dimensional vector $\mathbf{R}$. No quotient is involved. Division by W^3 is required to obtain $\mathbf{n}$ in the form given by equation (7). However this is not necessary if only the direction of the normal is needed. In many applications, a unit vector is required and then there is no need to carry out the division before normalisation is performed.

Evaluation of the normal for a particular surface is illustrated in the next section.

5. EXAMPLE

This example is based on a Bézier triangular patch. The equation for the rational form of such a patch of degree d is as follows

$$\mathbf{R}(u, v, w) = \sum_{\substack{0 \le i,j,k \le d \\ i+j+k=d}} \binom{d}{i\ j\ k} \mathbf{A}_{ijk}\, u^i v^j w^k$$

where u, v, w are the barycentric coordinates, $\mathbf{A}_{ijk}$ the control points in $\Re^3$, of which there are $(d + 1)(d + 2) / 2$, and the term in brackets is the trinomial coefficient

$$\binom{d}{i\ j\ k} = \frac{d!}{i!\,j!\,k!}$$

It is possible to form an exact octant of a sphere using this form of surface patch [1]. The degree of the patch is $d = 4$ and table 1 gives the control points for an octant of a unit sphere whose centre is at the origin. The control points have been modified slightly from those given in [1] so that the fourth component of each corner control point is unity. This does not change the surface.

Consider what happens at the corner of the patch where $u = 1$ and $v = w = 0$. Here

$$\mathbf{r} = \mathbf{A}_{400}$$

$$\mathbf{r}_u = 4\mathbf{A}_{400}$$

$$\mathbf{r}_v = 4\mathbf{A}_{310}$$

$$\mathbf{r}_w = 4\mathbf{A}_{301}$$

and so equations (4) and (7) show that the normal direction at this point is

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} & \mathbf{0} & \mathbf{0} \\ 1 & 0 & 0 & 1 & 0 \\ 4 & 0 & 0 & 4 & 1 \\ \frac{1}{2}(3\sqrt{2}+\sqrt{6}) & \sqrt{2} & 0 & \frac{1}{2}(3\sqrt{2}+\sqrt{6}) & 1 \\ \frac{1}{2}(3\sqrt{2}+\sqrt{6}) & 0 & \sqrt{2} & \frac{1}{2}(3\sqrt{2}+\sqrt{6}) & 1 \end{vmatrix}$$

Subtracting the fourth column from the first shows that the normal is certainly in the $\mathbf{i}$ direction. Further evaluation shows that $\mathbf{n} = 2\mathbf{i}$.

Figure 1 shows various views of the octant. Also shown is the normal at various regularly spaced points. These have been calculated using the determinant approach (in fact using Cramer's rule for each evaluation). The division by W^3 in equation (7) has been carried out and the normals are drawn at one tenth of their length. Visually from the figure, it is clear that the normals are all in the radial direction as would be expected. This can be confirmed by listing the unit normals and comparing their components with those of the corresponding points on the surface. (The computer program used gave agreement to at least five decimal places.)

6. CONCLUSIONS

Surface normals have been investigated for parametric surface patches defined over a triangular region of the plane. Barycentric coordinates can be used for the parameterisa-

Table I Control points for an octant of a sphere

	X	Y	Z	W
$\mathbf{A}_{400}$	1	0	0	1
$\mathbf{A}_{310}$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$	$\frac{1}{4}\sqrt{2}$	0	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$
$\mathbf{A}_{220}$	$\frac{1}{6}(2+\sqrt{3})$	$\frac{1}{6}(2+\sqrt{3})$	0	$\frac{1}{6}(2+\sqrt{3})$
$\mathbf{A}_{130}$	$\frac{1}{4}\sqrt{2}$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$	0	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$
$\mathbf{A}_{040}$	0	1	0	1
$\mathbf{A}_{301}$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$	0	$\frac{1}{4}\sqrt{2}$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$
$\mathbf{A}_{211}$	$\frac{1}{12}(2+2\sqrt{3}+\sqrt{6})$	$\frac{1}{24}(2+\sqrt{2}+2\sqrt{3}+\sqrt{6})$	$\frac{1}{24}(2+\sqrt{2}+2\sqrt{3}+\sqrt{6})$	$\frac{1}{12}(2+2\sqrt{3}+\sqrt{6})$
$\mathbf{A}_{121}$	$\frac{1}{24}(2+\sqrt{2}+2\sqrt{3}+\sqrt{6})$	$\frac{1}{12}(2+2\sqrt{3}+\sqrt{6})$	$\frac{1}{24}(2+\sqrt{2}+2\sqrt{3}+\sqrt{6})$	$\frac{1}{12}(2+2\sqrt{3}+\sqrt{6})$
$\mathbf{A}_{031}$	0	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$	$\frac{1}{4}\sqrt{2}$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$
$\mathbf{A}_{202}$	$\frac{1}{6}(2+\sqrt{3})$	0	$\frac{1}{6}(2+\sqrt{3})$	$\frac{1}{6}(3+\sqrt{3})$
$\mathbf{A}_{112}$	$\frac{1}{24}(2+\sqrt{2}+2\sqrt{3}+\sqrt{6})$	$\frac{1}{24}(2+\sqrt{2}+2\sqrt{3}+\sqrt{6})$	$\frac{1}{12}(2+2\sqrt{3}+\sqrt{6})$	$\frac{1}{12}(2+2\sqrt{3}+\sqrt{6})$
$\mathbf{A}_{022}$	0	$\frac{1}{6}(2+\sqrt{3})$	$\frac{1}{6}(2+\sqrt{3})$	$\frac{1}{6}(3+\sqrt{3})$
$\mathbf{A}_{103}$	$\frac{1}{4}\sqrt{2}$	0	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$
$\mathbf{A}_{013}$	0	$\frac{1}{4}\sqrt{2}$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$	$\frac{1}{8}(3\sqrt{2}+\sqrt{6})$
$\mathbf{A}_{004}$	0	0	1	1

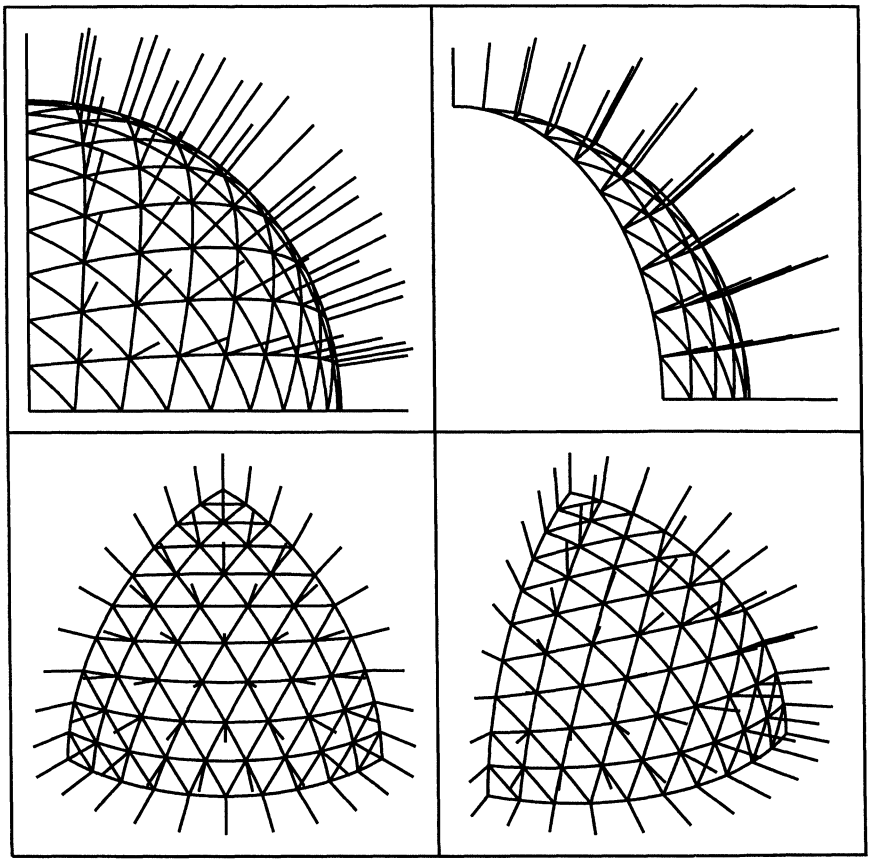

Figure 1 Surface normals on an octant of a sphere.

tion. A formula for the surface normal in terms of partial derivatives with respect to these coordinates has been obtained. This exhibits symmetry between the coordinates.

The same idea has been considered for rational surfaces. Here the required division by the additional fourth coordinate can cause difficulties. A means for evaluating normals has been obtained. This requires the evaluation of certain cofactors of a 5×5 matrix, whose entries are simply partial derivatives with respect to the barycentric coordinates. The only division required is right at the end of the evaluation and, as this only represents a rescaling of the normal, it can be omitted.

The evaluation has been illustrated by means of an example based on the octant of a sphere.

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