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Another Point of View on Proportional Navigation

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Proportional navigation is one of the most popular and one of the most used of the guidance laws. But the way it is studied is always the same: the acceleration needed to reach a known target is derived or analyzed. This way of studying guidance laws is called "the direct problem" by the authors. On the contrary, the problem considered here is to find, from the knowledge of a part of the trajectory of a maneuvering object, the target of this object. The authors call this way of studying guidance laws "the inverse problem".

Keywords: Proportional Navigation; Guidance Laws Modelling; Guidande Laws Analysis

I. INTRODUCTION

The point of view which is used in most of the publications concerning guidance laws is the following. A known maneuvering object is guided towards its known target, and the problem consists in deriving or in analyzing the acceleration which is required so that the object reaches its target ([1], [5], [6],[7], [11], [12], [14], [15], [17], [18]. [19],[20],[21], [22]). This problem could be called the *direct problem.* In this paper, on the contrary, what could be called the *inverse problem* is treated. For, it is here considered a known maneuvering object guided towards an unknown target. The problem consists in answering the following question. Knowing the trajectory of the maneuvering object M on a time interval which does not include the time at which M will reach its target, it is asked how to determine the target in the space? Therefore, it is obvious that the classical approach of guidance is not sufficient because, in this approach, the target is known whereas it is not known in the problem that is considered here. This is why it is necessary to derive a new approach to study the guidance laws. Our subject is limited to the plane proportional navigation guidance scheme because proportional navigation laws are the most popular and the most used ([22]). But, since there exists many definitions of proportional navigation laws, it as been chosen to work with the synthetic definition proposed in references [2], [9] and [10].

First the definition proposed in the previous three references will be analyzed in order to find some parameters which are characteristic of proportional navigation trajectories. This study will allow the determination of an analysis model of proportional navigation trajectories whatever the point from which the observation is performed. Then, this analysis model will be used to find the non maneuvering target T of a maneuvering object M guided by a plane pure proportional navigation.

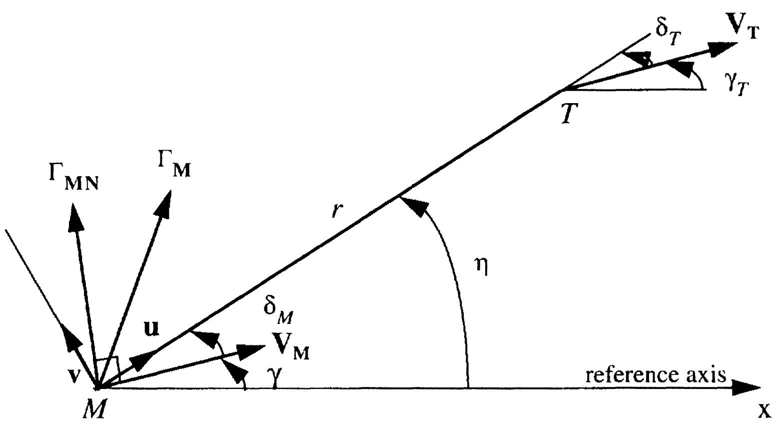

FIGURE 1 General Plane Pursuit Geometry

II. TOWARDS A NEW APPROACH TO STUDY PROPORTIONAL NAVIGATION TRAJECTORIES

II.1 Definitions and notations

It is proposed here to find the target T of a maneuvering object guided by proportional navigation as it has been defined in references [2], [9] and [10]. So it is first necessary to give this definition. In order to do so, the parameters defined in figure 1 are used. In this figure $\mathbf{\Gamma_M}$ is the acceleration vector of M, $\mathbf{\Gamma_{MN}}$ is the normal acceleration vector of M i.e. the projection of $\mathbf{\Gamma_M}$ on to the normal to the trajectory and $\mathbf{V_M}$ is the velocity vector of M. $\mathbf{V_T}$ is the velocity vector of the target. η is the angle between a reference axis and the line of sight *(MT)*. δ_M is the angle between the vector $\mathbf{V_M}$ and the line of sight. δ_T is the angle between the vector $\mathbf{V_T}$ and the line of sight. γ is the angle between the reference axis and the vector $\mathbf{V_M}$. γ_T is the angle between the reference axis and the vector $\mathbf{V_T}$. Since it is considered here that the target is non maneuvering, the reference axis can be chosen colinear to $\mathbf{V_T}$. Therefore γ_T is equal to zero and:

$$\eta = \delta_T. \tag{1}$$

u is the unit reference vector of the line of sight (*MT*) and **v** is the unit vector derived from the derivative with respect to time of **u**. Consequently, it comes:

$$\dot{\mathbf{u}} = \dot{\eta}\mathbf{v}, \tag{2}$$

$$\dot{\mathbf{v}} = -(\dot{\eta})\mathbf{u}, \tag{3}$$

$$\ddot{\mathbf{u}} = -(\dot{\eta})^2\mathbf{u} + \ddot{\eta}\mathbf{v}. \tag{4}$$

In order to write a general proportional navigation law definition the vector $\mathbf{U^2}$:

$$\mathbf{U}^2 = \begin{bmatrix} r \\ \dot{r} \\ \eta \\ \dot{\eta} \end{bmatrix}, \tag{5}$$

is required as well as the functions $F^2_{a,b}$ *(*$\mathbf{U^2}$*)* and $T^2_{F_{a,b}}$ *(*$\mathbf{U^2}$*)* defined by:

$$F^2_{a,b}(\mathbf{U}^2) = b(t)r(\dot{\eta})^2\mathbf{u} - a(t)\dot{r}\dot{\eta}\mathbf{v}, \tag{6}$$

$$T^2_{F_{a,b}}\left(\mathbf{U^2}\right) = F^2_{a,b}\left(\mathbf{U^2}\right) + \left(\ddot{r} - r(\dot{\eta})^2\right)\mathbf{u} + (r\ddot{\eta} + 2\dot{r}\dot{\eta})\mathbf{v}. \tag{7}$$

REMARK 1 *a(t) and b(t) are called guidance functions.*

DEFINITION 1 Plane proportional navigation is a guidance law a model of which is given by:

$$\left\{\begin{array}{c} \left(T^3_{F_{a,b}}\left(\mathbf{U}^2\right)\right) = 0 \\ C_i\left(F^2_{a,b}\right) \\ I_i\left(\mathbf{U}^2(t_0)\right) \end{array}\right. , \tag{8}$$

with $C_i(F^2_{a,b})$ a set of constraints on the function $F^2_{a,b}(U^2)$ so that the set $I_i(\mathbf{U}^2(t_0))$ of initial conditions allowing capture of the target is not empty. Each $C_i(F^2_{a,b})$ generates a class of plane proportional navigation law called "class of plane proportional navigation of type i".

REMARK 2 The equality $(T^3_{F_{a,b}}(U^2)) = 0$ means that the total acceleration of the object is equal to $F^2_{a,b}(U^2)$ with *a(t)*and *b(t)* two given guidance functions.

This definition will be very useful to derive a method to analyze a proportional navigation trajectory from any observation point of the plane. But, first, it is necessary to define some vocabulary. For, as the target is not known, the trajectory may be observed from any polar frame the origin of which is not necessary the real target *T* of the maneuvering object *M*. Therefore a specific vocabulary must be defined in order to avoid possible confusions.

DEFINITION 2 An observation point *O* is any point of the plane from which the trajectory of the maneuvering object is observed and analyzed.

DEFINITION 3 The target *T* is the only observation point so that:

$$\text{there exists } t_f < \infty / OM(t) = 0 \text{ for all } t > t_f. \tag{9}$$

The notations used from an observation point are the same as the one used in the figure 1 but an index is now added: each variable studied in the polar frame the origin of which is *T* (respectively *O*) will be indexed with a "*T*" (respectively with an "*O*").

EXAMPLE In figure 1, "*r*" refers to the range *MT* between the object *M* and the origin *T* of the polar frame. Now this range will be denoted by

"r_T". When the trajectory is studied from an observation point O, origin of the polar frame, this range will be denoted by "r_O".

Some variables do not depend on the origin of the polar frame. This is the case for the moduli of velocity vectors; they don't have an index.

II.2 Methodology to solve the inverse problem

The process used in references [2], [9] and [10] to model proportional navigation guidance laws is the following. After having constrained the acceleration vector of M to have the form defined by equality (6), properties of the guidance functions are derived so that $r(t)$ belongs to a class of functions which become null. Moreover, it is also shown in these references that a trajectory generated by a proportional navigation guidance law is completely defined by the vector $\mathbf{U}^2(t)$ on the interval $t \in [t_0, t_f]$ (t_0 is the initial time of the pursuit and t_f the final time at which the maneuvering object reaches its target) and by the given guidance functions. The trajectory is then completely defined by a sixth order vector. In fact, this number can be reduced. For, as it is well known, whatever the point from which the observation is realized:

$$\dot{r} = -||\mathbf{V_M}|| \cos \delta_M + ||\mathbf{V_C}|| \cos \eta, \tag{10}$$

$$\text{and } r\dot{\eta} = ||\mathbf{V_M}|| \sin \delta_M - ||\mathbf{V_C}|| \sin \eta. \tag{11}$$

In a synthetic way the trajectory is therefore completely defined by the vector $\mathbf{U^2{}_{aT}}$ equal to:

$$\mathbf{U}^2_{\mathbf{a}T}(t) = \begin{bmatrix} r_T(t) \\ \eta_T(t) \\ \delta_{MT}(t) \\ a_T(t) \\ b_T(t) \end{bmatrix}. \tag{12}$$

In the following, the authors use this result to determine if an observation point is the target or not. For a given proportional navigation guidance law, each component of the vector $\mathbf{U^2{}_{aT}}$ is going to be studied. The trajectory will then be analysed from any observation point O. In order to perform this analysis, two guidance functions called *instantaneous guidance functions* are assigned to the observation point. These instantaneous guidance functions are denoted by $a_O(t)$ and $b_O(t)$. They adapt the kinematic reality of each trajectory to the hypothesis of proportional navigation trajectory i.e. they allow the acceleration of the object to be written in the form of (6). A vector $\mathbf{U^2}_{\mathbf{aO}}$ defined by:

$$\mathbf{U}^2_{\mathbf{aO}}(t) = \begin{bmatrix} r_O(t) \\ \eta_O(t) \\ \delta_{MO}(t) \\ a_O(t) \\ b_O(t) \end{bmatrix}, \tag{13}$$

is then assigned to the observation point O. The following point of the method consists in studying the behavior of each component of $\mathbf{U^2}_{\mathbf{aO}}$ and to compare it to the behavior of the corresponding component of $\mathbf{U^2}_{\mathbf{aT}}$. The aim of this comparison is to exhibit one or several specific behavior(s) which characterize(s) the target of the maneuvering object.

This method is now going to be used to characterize the target of a maneuvering object guided by a plane pure proportional navigation. When this guidance law is used, the modulus of the velocity vector of the object is taken as constant. Moreover, in order to simplify the problem, it is, by now, supposed that the velocity of the target is null. Therefore, the problem consists in finding the non moving target of a maneuvering object guided by a plane pure proportional navigation. Using this hypothesis, equations (10) and (11) become:

$$\dot{r} = -||\mathbf{V_M}|| \cos \delta_M, \tag{14}$$

$$r\dot{\eta} = ||\mathbf{V_M}|| \sin \delta_M. \tag{15}$$

In order to solve this problem the pursuit geometry and the notations described in figure 2 are used.

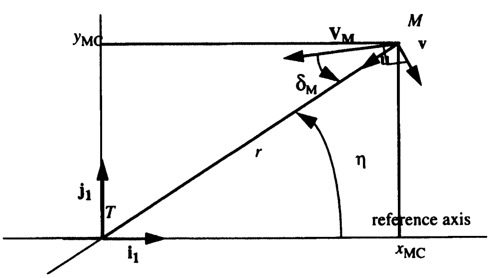

FIGURE 2 Geometry of the pursuit in the (T, u, v) polar frame

III. APPLICATION TO A PLANE PURE PROPORTIONAL NAVIGATION GUIDANCE

III.1 Definitions and notations

When the object is guided by a pure proportional navigation law, the acceleration vector of the object is always perpendicular to the velocity vector of the object. This is an immediate consequence of the constancy of the modulus of V_M. Moreover this acceleration vector is equal to ([1], [13], [22]):

$$\Gamma_{\mathbf{M}} = A\|\mathbf{V_M}\|\dot{\eta}\mathbf{N_M}, \tag{16}$$

with A a constant called the *proportional navigation constant* and $\mathbf{N_M}$ the unit normal vector to the trajectory. It can be shown that this definition is equivalent to constrain:

$$\dot{\delta}_M = (-\mu)\,\dot{\eta}, \tag{17}$$

with μ a constant called the *proportional navigation coefficient* ([13]). The relation between A and μ is then:

$$A = \mu + 1 \tag{18}$$

In these conditions it can be shown that the guidance functions are equal to ([9]):

$$b(t) = \frac{(\mu+1)\|\mathbf{V_M}\|\sin\delta_M}{\|\mathbf{V_M}\|\sin\delta_M - \|\mathbf{V_C}\|\sin\eta} \text{ if } r \neq 0 \text{ and } \dot{\eta} \neq 0. \tag{19}$$

$$a(t) = \frac{-(\mu+1)\|\mathbf{V_M}\|\cos\delta_M}{-\|\mathbf{V_M}\|\cos\delta_M + \|\mathbf{V_C}\|\cos\eta} \text{ if } \dot{r} \neq 0. \tag{20}$$

These two relations show that the studies of the two guidance functions are not necessary in the case of a plane pure proportional navigation guidance scheme. In order to apply the previous method to find the target of the maneuvering object it is enough to study the temporal behavior of the components of the two vectors $\mathbf{X^2}_{\mathbf{aT}}$ and $\mathbf{X^2}_{\mathbf{aO}}$ defined by:

$$\mathbf{X}^2_{\mathbf{aT}}(t) = \begin{bmatrix} r_T(t) \\ \eta_T(t) \\ \delta_{MT}(t) \\ \mu_T(t) \end{bmatrix}, \tag{21}$$

$$\mathbf{X}^2_{\mathbf{aO}}(t) = \begin{bmatrix} r_O(t) \\ \eta_O(t) \\ \delta_{MO}(t) \\ \mu_O(t) \end{bmatrix}, \tag{22}$$

with the component μ_O *(t)* defined by:

$$\dot{\delta}_{MO}(t) = -\mu_O(t)\dot{\eta}_O(t), \tag{23}$$

μ_O *(t)* is called the *instantaneous proportional navigation coefficient.*

III.2 Relation between the instantaneous proportional navigation coefficient and the proportional navigation coefficient

PROPOSAL 1 *At any point where r_T and sin δ_{MO} are not null, the instantaneous proportional navigation coefficient is linked to the components of the vectors $\mathbf{X^2}_{\mathbf{aO}}$ and $\mathbf{X^2}_{\mathbf{aT}}$ by:*

$$\mu_O(t) = \frac{(\mu_T + 1) r_O(t) \sin \delta_{MT}(t)}{r_T(t) \sin \delta_{MO}(t)} - 1. \tag{24}$$

Proof In the polar frame the origin of which is *T,* the curvature ρ_{MT} to the trajectory at point *M* is equal to:

$$\rho_{MT}(t) = \frac{\dot{\eta}_T(t) - \dot{\delta}_{MT}(t)}{||\mathbf{V_M}||} = \frac{\dot{\gamma}(t)}{||\mathbf{V_M}||}. \tag{25}$$

In the same way, this curvature is equal to:

$$\rho_{MO}(t) = \frac{\dot{\eta}_O(t) - \dot{\delta}_{MO}(t)}{||\mathbf{V_M}||}, \tag{26}$$

in the polar frame the origin of which is *O*. Moreover, from basic geometric reasoning it can be shown that whatever the observation point *O*:

$$\dot{\eta}_T(t) - \dot{\delta}_{MT}(t) = \dot{\eta}_O(t) - \dot{\delta}_{MO}(t). \tag{27}$$

Therefore, it can be written that:

$$\frac{\dot{\eta}_O(t) - \dot{\delta}_{MO}(t)}{||\mathbf{V_M}||} = \frac{\dot{\eta}_T(t) - \dot{\delta}_{MT}(t)}{||\mathbf{V_M}||}. \tag{28}$$

By introducing definition (17) of pure proportional navigation and definition (23) of the instantaneous proportional navigation coefficient, equation (28) becomes:

$$(\mu_O + 1)\dot{\eta}_O(t) = (\mu_T(t) + 1)\dot{\eta}_T(t). \tag{29}$$

$\dot{\eta}_T$ and $\dot{\eta}_O$ then being replaced by their expressions deriving from (15), we finally have:

$$\frac{(\mu_T+1)\sin\delta_{MT}(t)}{r_T(t)} = \frac{(\mu_O+1)\sin\delta_{MO}(t)}{r_O(t)} = \rho_{MT} = \rho_{MO}. \qquad (30)$$

III.3 Study of the components of $\mathbf{X}_{\mathbf{aT}}$

The aim of this section is to define, as functions of the initial conditions of the trajectory, the temporal behaviors of the components of $\mathbf{X}_{\mathbf{aT}}$. The proof of the properties stated in this paragraph are based on the results demonstrated by Guelman in his well known publication [13]. In this article, Guelman shows that if the proportional navigation coefficient is greater than one then the phase plane *(r, η)* in relation to the set of differential equations (10), (11) and (17) can be divided into two types of successive sectors called *normal sector of type 1⁺* and *normal sector of type 1⁻*. In each normal sector $\dot{r}$ has the same sign: negative in a normal sector of type 1^- and positive in a normal sector of type 1^+. Each normal sector contains one and only one straight line $\eta_T = \eta_i^n = constant$ on which $\dot{\eta}_T$ is equal to zero. Such a straight line is called a *critic direction.*

If a pure proportional navigation trajectory starts in a normal sector of type 1^-, then the trajectory stays in this sector and capture occurs on the corresponding critic direction. If it starts on a critic direction in a normal sector of type 1^+ then it stays on this critic direction capture will never occur. Finally, if it starts in a normal sector of type 1^+, then it leaves this sector to enter a normal sector of type 1^- where capture occurs once more on the corresponding critic direction.

III. 3. 1 Behavior of $r_T(t)$

PROPOSAL 2 *r_T strictly decreases in the time interval $[t_0, t_f[$ if and only if:*

$$\delta_{MT}(t_0) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]. \qquad (31)$$

It strictly increases then strictly decreases on $[t_0, t_f[$ if and only if:

$$\delta_{MT}(t_0) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]. \qquad (32)$$

Proof Equality (14) shows that if $\delta_{MT}(t_0) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$ then $\dot{r}_T(t_0) < 0$. The trajectory therefore starts in a normal sector of type 1^-. According to the study led by Guelman in reference [13], it can immediately be deduced that r_T strictly decreases. In the same way, if $\delta_{MT}(t_0) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$ trajectory starts in a so-called normal sector of type 1^+: function r_T therefore strictly increases before decreasing.
If function r_T strictly decreases in the time interval $[t_0, t_f[$, then initial conditions are such that trajectory starts in a normal sector of type 1^-. It means that $\dot{r}_T$(t0) < 0 and according to equality (14), it is then necessary that $\delta_{MT}(t_0) \in \left[\frac{-\pi}{2}, \frac{\pi}{2}\right]$. In the same way, if function r_T strictly increases before decreasing, then trajectory begins in a normal sector of type 1^+. According to equality (14) it is necessary to have $\delta_{MT}(t_0) \in \left[\frac{\pi}{2}, \frac{3\pi}{2}\right]$.

III. 3. 2 Behavior of $\eta_T(t)$

PROPOSAL 3 *Function* η_T *strictly increases in the time interval* $[t_0, t_f[$ *if and only if:*

$$\delta_{MT}(t_0) \in]0,\pi[. \tag{33}$$

It strictly increases in $[t_0, t_f[$ *if and only if:*

$$\delta_{MT}(t_0) \in]-\pi,0[. \tag{34}$$

It is strictly constant in $[t_0, t_f[$ *if and only if:*

$$\delta_{MT}(t_0) \in \{0,\pi\}. \tag{35}$$

Proof According to the study led by Guelman in [13], η_T is strictly constant if, in the phase plane (r, η) the trajectory of the maneuvering object starts on a critic direction. Considering a trajectory which does not start with $r_T(t_0)$ equal to zero, it is easy to verify using equality (15), that this trajectory starts on a critic direction if and only if $\delta_{MT}(t_0) \in \{0,\pi\}$.

In the phase plane, the orbit corresponding to the trajectory of the object is completely included in the region determined by two successive zeros of function the $\dot{\eta}_T$. Therefore, variations of the function η_T depend on the sign of $\dot{\eta}_T$ at the beginning of the trajectory. According to equality (15), $\dot{\eta}_T > 0$ if and only if $\delta_{MT}(t_0) \in]0,\pi[$ and $\dot{\eta}_T > 0$ if and only if $\delta_{MT}(t_0) \in]-\pi,0[$.

III. 3.3 Behavior of $\delta_{MT}(t)$

PROPOSAL 4 *δ_{MT} strictly decreases in the time interval $[t_0, t_f[$ if and only if:*

$$\delta_{MT}(t_0) \in]0,\pi[. \tag{36}$$

It strictly increases in $[t_0, t_f[$ if and only if:

$$\delta_{MT}(t_0) \in]-\pi,0[. \tag{37}$$

It is strictly constant if and only if:

$$\delta_{MT}(t_0) \in \{0,\pi\}. \tag{38}$$

Proof Only capture trajectories are considered here. According to the studies led by Guelman in reference [13] the proportional navigation coefficient must be greater than 1. As a consequence, equation (17) states that η_T et δ_{MT} have opposite variations.

REMARK 3 As is shown in [13], δ_{MT} is null at capture time.

III.3.4 Behavior of $\mu_T(t)$

By definition of pure proportional navigation law, μ_T is a constant. Moreover, as only capture trajectories are considered, this coefficient will always be greater than 1 in the following lines.

The study of behaviors of the components of $\mathbf{X^2_{aT}}$ is now achieved. The next step is to study the components of $\mathbf{X^2_{aO}}$. This study needs the statements and the demonstrations of some properties of pure proportional navigation trajectories.

III.4 Properties of pure proportional navigation trajectories

In order to simplify the writing of some properties, a pure proportional navigation trajectory generated from initial conditions $r_T(t_0)$, $\eta_T(t_0)$ and $\delta_{MT}(t_0)$ with a proportional navigation coefficient equal to μ_T will be called "$(r_T(t_0), \eta_T(t_0), \delta_{MT}(t_0), \mu_T)$ pure proportional navigation trajectory".

PROPERTY 1 *Consider an initial (r_T (t_0), η_T (t_0), δ_{MT}(t_0), μ_T) pure proportional navigation trajectory. Then each (r_T (t_0) , η_T (t_0) + θ, δ_{MT} (t_0), μ_T) pure proportional navigation trajectory is deduced from the initial one by a rotation of angle θ the center of which is T.*

Proof The right hand side of equality (15) does not depend on η_T.

PROPERTY 2 *Let M_0 be the initial position of M. Consider an initial (r_T(t_0), η_T (t_0), δ_{MT} (t_0), μ_T) pure proportional navigation trajectory. The (r_T (t_0) , η_T (t_0), $-\delta_{MT}$ (t_0), μ_T) pure proportional navigation trajectory is obtained from the initial one by a symmetry in relation to the straight line (M_0T)*

Proof By considering equations (14), (15) it can be shown that:

$$r_T = r_T(t_0)\left(\left|\frac{\sin(\delta_{MT})}{\sin(\delta_{MT}(t_0))}\right|\right)^{\frac{1}{\mu_T}}, \tag{39}$$

$$\dot{\eta}_T = \frac{\|\mathbf{V_M}\|}{r_T(t_0)}(|\sin(\delta_{MT}(t_0))|)^{\frac{1}{\mu_T}}\frac{\sin(\delta_{MT})}{|\sin(\delta_{MT})|^{\frac{1}{\mu_T}}}. \tag{40}$$

According to these two last equations and according to equation (17), $\dot{\eta}_T$ and $\dot{\delta}_{MT}$ are odd functions of the variable δ_{MT} whereas $\dot{r}_T$ is an even function of the variable δ_{MT}. Consequently, at each time, polar rays of each trajectory described in the statement of property 2 are symmetric in relation to the straight line M_0T and the range is the same.

PROPERTY 3 *Consider a (r_T (t_0) , η_T (t_0) , δ_{MT}(t_0) μ_T) pure proportional navigation trajectory. Then the variation of the angle η_T between initial time and final time of pursuit is equal to:*

$$\eta_T(t_f) - \eta_T(t_0) = \frac{\delta_{MT}(t_0)}{\mu_T}. \tag{41}$$

Proof As it has been shown by Guelman in [13] when only finite final acceleration is considered, each side of equation (15) is null at the final time. Then, using equation (17) it can be shown that:

$$\eta_T(t_f) \in \left\{\left(\eta_T(t_0) + \frac{\delta_{MT}(t_0) - l\pi}{\mu_T}\right) \; with \; (l \in \mathbf{Z})\right\}. \tag{42}$$

In this last expression it must be kept in mind that, as only finite final acceleration is considered, the proportional navigation coefficient is greater than one. Moreover, function δ_{MT} is always so that:

$$\delta_{MT} \in [-\pi, \pi]. \tag{43}$$

Consequently, $\eta_T(t_f)$ belongs to the angular sector:

$$\eta_T(t_0) + \frac{\delta_{MT}(t_0) - \pi}{\mu_T} < \eta_T(t_f) < \eta_T(t_0) + \frac{\delta_{MT}(t_0)}{\mu_T} \text{ if } \delta_{MT}(t) \in [0, \pi], \tag{44}$$

$$\eta_T(t_0) + \frac{\delta_{MT}(t_0)}{\mu_T} < \eta_T(t_f) < \eta_T(t_0) + \frac{\delta_{MT}(t_0) + \pi}{\mu_T} \text{ if } \delta_{MT}(t) \in [-\pi, 0], \tag{45}$$

the limits of which are two successive zeros of $\dot{\eta}_T$. It is known ([9], [13]), that the sign of $\dot{\eta}_T$ does not change between initial time and final time. So considering the sign of $\dot{\eta}_T$ it can easily be shown that:

$$\eta_T(t_f) = \eta_T(t_0) + \frac{\delta_{MT}(t_0)}{\mu_T}. \tag{46}$$

PROPERTY 4 *($r_T(t_0)$, $\eta_T(t_0)$, $\delta_{MT}(t_0)$, μ_T) pure proportional navigation trajectories which are such that:*

$$\frac{-\pi}{4} < \frac{\delta_{MT}(t_0)}{\mu_T} < \frac{\pi}{4}, \tag{47}$$

are included in an angular sector the center of which is T and the angular width of which is equal to $\frac{\pi}{2}$.

Proof This property is a direct consequence of the introduction of inequality (47) in equality (41).

Properties 1 to 4 have immediate consequences. According to property 1, it is not necessary to consider all initial conditions for the η_T component. As all ($r_T(t_0)$, $\eta_T(t_0) + \theta$, $\delta_{MT}(t_0)$, μ_T) trajectories can be deduced from the ($r_T(t_0)$, $\eta_T(t_0)$, $\delta_{MT}(t_0)$, μ_T) trajectory by a simple rotation, it is enough to realize our study for a particular value of $\eta_T(t_0)$. Therefore, without any loss of generality, the following value will now be considered:

$$\eta_T(t_0) = \frac{\pi}{4} \tag{48}$$

It has been chosen because it allows simpler proof writing. Moreover, as ($r_T(t_0)$, $\eta_T(t_0)$, $-\delta_{MT}(t_0)$, μ_T) trajectories can be deduced from ($r_T(t_0)$, $\eta_T(t_0)$, $\delta_{MT}(t_0)$, μ_T) trajectories by a simple symmetry in relation to an axis, it is enough to realize the study for the following initial conditions:

$$\delta_{MT}(t_0) \in [0, \pi]. \tag{49}$$

In fact, the following study will only be made for:

$$\delta_{MT}(t_0) \in [0, \frac{\pi}{4}]. \tag{50}$$

The reason for this choice will be explained after the statement of the theorem which allows one to characterize the target of the maneuvering object.

Now, the notation "$\left(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in \left[0, \frac{\pi}{4}\right], \mu_T \geq 1\right)$ pure proportional navigation trajectory", refers to a trajectory the initial conditions of which satisfy constraints (48), (50) and

$$\mu_T \geq 1, \tag{51}$$

PROPERTY 5 *The slope of the tangent to a* $\left(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in 0, \frac{\pi}{4}, \mu_T \geq 1\right)$ *pure proportional navigation trajectory is* a *positive strictly increasing function if:*

$$\delta_{MT}(t_0) \neq 0. \tag{52}$$

The slope is constant if:

$$\delta_{MT}(t_0) = 0. \tag{53}$$

Proof: Let $\alpha(t)$ be the slope of the tangent at time t. According to figure 2, this slope is equal to:

$$\alpha(t) = \tan(\eta_T(t) - \delta_{MT}(t)). \tag{54}$$

According to the result of proposal 3, as $\delta_{MT}(t_0) \in [0, \frac{\pi}{4}]$, function $\eta_T(t)$ strictly increases with time if $\delta_{MT}(t_0) \neq 0$. According to the result of proposal 4, function δ_{MC} then strictly decreases. Function $(\eta_T - \delta_{MT})$ therefore strictly increases if $\delta_{MT}(t_0) \neq 0$. It is even possible to determine the initial value and the final value of this difference. The initial value is naturally equal to:

$$\eta_T(t_0) - \delta_{MT}(t_0). \tag{55}$$

According to equation (41) and remark 2 the final value is equal to:

$$\eta_T(t_0) + \frac{\delta_{MT}(t_0)}{\mu_T}. \tag{56}$$

Considering now constraints (48), (50), (51) on the initial conditions, it is clear that:

$$0 \leq \eta_T(t) - \delta_{MT}(t) \leq \frac{\pi}{2}. \tag{57}$$

The slope of the tangents to the trajectory being strictly increasing in the interval $]0, \frac{\pi}{2}[$, the slope $\alpha(t)$ is then a positive strictly increasing function if $\delta_{MT}(t_0) \neq 0$. If $\delta_{MT}(t_0) = 0$, then according to proposals 3 and 4, functions $\delta_{MT}(t)$ and $\eta_T(t)$ are constant. The slope is therefore constant and positive.

REMARK 4 By symmetry in relation to the axis (M_0C), the slope of the tangent at each point of a $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in [-\frac{\pi}{4}, 0], \mu_T \geq 1)$ pure proportional navigation trajectory always decreases.

PROPERTY 6 $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) = 0, \mu_T \geq 1)$ *pure proportional navigation trajectory is a straight line.*

Proof If $\delta_{MT}(t_0) = 0$, then according to proposals 3 and 4, functions $\delta_{MT}(t)$ and $\eta_T(t)$ are constant.

The frame $(C, \mathbf{i_1}, \mathbf{j_1})$ defined in figure 2 is now considered. In the following lines, x_{MT} and y_{MT} respectively refer to the abscissa in relation to $(C, \mathbf{i_1})$ and the ordinate in relation to $(C, \mathbf{j_i})$ of M in $(C, \mathbf{i_1}, \mathbf{j_1})$. These cartesian coordinates are linked to the polar coordinates by the well known relations:

$$x_{MT} = r_T \cos(\eta_T), \tag{58}$$

$$y_{MT} = r_T \sin(\eta_T). \tag{59}$$

Moreover, at each point where $\dot{x}_{MT}$ and $\dot{y}_{MT}$ are not infinite at the same time and are not equal to zero at the same time, the slope of the tangent to the trajectory is equal to:

$$\alpha(t) = \frac{\dot{y}_{MT}}{\dot{x}_{MT}}. \tag{60}$$

PROPOSAL 5 *Consider a* $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in \,]0, \frac{\pi}{4}], \mu_T \geq 1)$ *pure proportional navigation trajectory. Along this trajectory, functions* $\dot{x}_{MT}$ *and* $\dot{y}_{MT}$ *such that:*

$$\dot{x}_{MT} < 0 \quad \text{if} \quad t \in [t_0, t_f[, \tag{61}$$

$$\dot{y}_{MT} < 0 \quad \text{if} \quad t \in \,]t_0, t_f], \tag{62}$$

$$\dot{x}_{MT}(t_f) \leq 0, \tag{63}$$

$$\dot{y}_{MT}(t_0) \leq 0. \tag{64}$$

Functions x_{MT} *and* y_{MT} *therefore strictly decrease in the time interval* $[t_0, t_f]$.

Proof According to proposition 5, the slope of the trajectory is positive. Functions $\dot{x}_{MT}$ and $\dot{y}_{MT}$ then have the same sign. Moreover, as $\delta_{MT}(t_0) \in \,]0, \frac{\pi}{4}]$, function r_T strictly decreases according to the result of proposal 2. Consequently, $\dot{x}_{MT}$ and $\dot{y}_{MT}$ are negative and cannot be null at the same time (this can easily be proved by writing the derivative of $(r_T)^2$). Moreover, as functions r_T and $\dot{r}_T$ are always finite, $\dot{x}_{MT}$ and $\dot{y}_{MT}$ are finite too since functions r_C and η_C as well as their derivatives are continuous finite. Equality (60) is then always valid.

As $\delta_{MT}(t_0) \neq 0$ the slope is a positive strictly increasing function. Therefore, function $\dot{y}_{MT}$ may only be null at the initial time. In the same way, $\dot{x}_{MT}$ may only be null at final time. For, as has already be shown, functions $\dot{x}_{MT}$ and $\dot{y}_{MT}$ are always finite. Under these conditions, as functions $\dot{x}_{MT}$ et $\dot{y}_{MT}$ cannot be null at the same time, if $\dot{x}_{MT}$ is equal to zero then the slope is infinite. In this case equation (54) implies that:

$$\eta_T(t) - \delta_{MT}(t) = \frac{\pi}{2}. \tag{65}$$

But according to inequalities (57), the value of $\eta_T(t) - \delta_{MT}(t)$ given in (65) is the maximum value of this strictly increasing function. It then may only be reached at final time.

PROPERTY 7 *Each* $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0)\ ]0, \frac{\pi}{4}], \mu_T \geq 1)$ *pure proportional navigation trajectory is concave in the* $(C, \mathbf{i}_1, \mathbf{j}_1)$ *frame.*

Proof Change in the concavity occurs at a point if and only if curvature is equal to zero at this point. According to equation (30) of the expression of the curvature, it only appears at final time when $\dot{\eta}_T = 0$. Concavity is then unchanged along the trajectory. According to proposal 5, the slope of the tangent to the trajectory is a strictly increasing function. Therefore the slope of the curve y_{MT} (x_{MT}) strictly decreases when x_{MT} increases because $\dot{x}_{MT}$ and $\dot{y}_{MT}$ are strictly negative on $]t_0, t_f[$. The trajectory is then concave.

REMARK 5 By symmetry in relation to the axis M_0T, $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]\frac{-\pi}{4}, 0], \mu_T \geq 1)$ pure proportional navigation trajectories are convex in the frame $(C, \mathbf{i_1}, \mathbf{j_1})$.

Two other fundamental properties are going to be stated and proven. First, it is necessary to define some particular straight lines and subspaces. These straight lines and subspaces are drawn in figure 3 and are drawn in the following. Consider a $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]0, \frac{\pi}{4}], \mu_T \geq 1)$ pure proportional navigation trajectory in $(C, \mathbf{i_1}, \mathbf{j_1})$. D_{M_0} is the straight line the slope of which is equal to $\alpha(t_0)$ and which includes the point M_0 (the initial velocity vector of the object is then a reference vector of this straight line). D_C is the straight line the slope of which is α (t_f) and which includes the target point T (the final velocity vector of the object is then a reference vector of this straight line). The intersection point between these two straight lines is called I. In the frame $(C, \mathbf{i_1}, \mathbf{j_1})$ its coordinates are (x_I, y_I). The following straight lines and subspaces are then defined: (see after)

$D_{M_0}^{x \geq x_{MT}(t_0)}$ the half straight line defined from D_{M_0} and the abscissa of which are greater or equal to $x_{MT}(t_0)$.

$D_{M_0}^{x \leq x_I}$ the half straight line defined from D_{M_0} and the abscissa of which are smaller or equal to x_I.

$D_C^{x \leq 0}$ the half straight line defined from D_C and the abscissa of which are smaller or equal to zero.

Finally, $D_C^{x \geq x_I}$ is the half straight line defined from D_C and the abscissa of which are greater or equal to x_I.

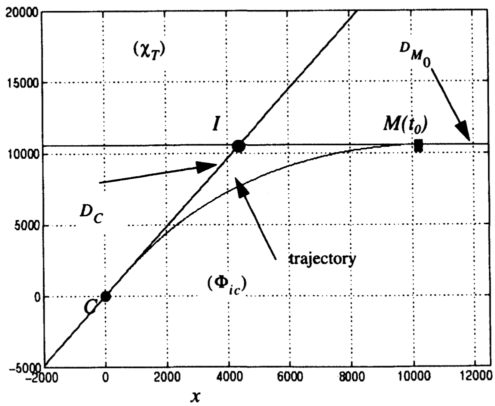

FIGURE 3 Sectorisation of the plane in relation to the trajectory

$(\Im_T)$ is then the subspace the frontier of which are the $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]0, \frac{\pi}{4}], \mu_T \geq 1)$ pure proportional navigation trajectory, the straight line D_{M_0} and the straight line D_C. This subspace has the following properties:

PROPERTY 8 *Consider a* $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]0, \frac{\pi}{4}], \mu_T \geq 1)$ *pure proportional navigation trajectory. The set of the intersection points between two tangents to the trajectory generates* $(\Im_T)$.

Proof Let us consider two positions $M(t_1)$ and $M(t_2)$ *(with* $t_2 > t_1$*)* of the object on its trajectory. In the frame $(C, \mathbf{i_1}, \mathbf{j_1})$, these points have respectively the coordinates $(x_{MT}(t_1), y_{MT}(t_1))$ and $(x_{MT}(t_2), y_{MT}(t_2))$. According to the result stated in proposal 5, these coordinates are such that:

$$x_{MT}(t_1) > x_{MT}(t_2), \tag{66}$$

$$y_{MT}(t_1) > y_{MT}(t_2). \tag{67}$$

Let $J_{t_1t_2}$ be the intersection point of the two tangents to the trajectory at the points $M(t_1)$ and $M(t_2)$. Its coordinates in the frame $(C, \mathbf{i_1}, \mathbf{j_1})$ are $(x_J(t_1, t_2), y_J(t_1, t_2))$. According to property 5, the slope of the tangents to the trajectory are positive strictly increasing functions of time, so it is easy to show that:

$$x_{MT}(t_2) < x_J(t_1,t_2) < x_{MT}(t_1), \tag{68}$$

$$y_{MT}(t_2) < y_J(t_1,t_2) < y_{MT}(t_1). \tag{69}$$

Therefore, the set of the intersection points between two tangents to the trajectory is necessarily inside the rectangle the diagonal of which is $M(t_0)T$ and which includes the trajectory and the point I. The space generated by the intersection points between two tangents is then included in this rectangle which also includes the space $(\Im_T)$. As the trajectory is concave and inequalities (66) and (67) are satisfied for each couple (t_1, t_2), intersection can only occur above the trajectory. Suppose that an intersection between two tangents occurs outside the space $(\Im_T)$. Then according to inequalities (68) and (69), there exists one of the considered tangents the slope of which is smaller than the one of D_{M_0} or greater than the one of D_C. But, according to proposal 5, this result is impossible. The set of the intersection points between two tangents is then included inside $(\Im_T)$.

By working now on the slope of possible straight lines issued from a point of $(\Im_T)$, it is easy to show that each point of this space lies into two tangents to the trajectory. Property 8 is thus demonstrated.

Now consider the space (Φ_{ic}) defined as follows. A point P the coordinates of which are (x, y) in the frame $(C, \mathbf{i_1}, \mathbf{j_1})$ belongs to (Φ_{ic}) if and only if:

$$P \text{ is under the half straight line } D_C^{x \leq 0}, \tag{70}$$

$$P \text{ is under trajectory when } 0 \leq x \leq x_{MT}(t_0), \tag{71}$$

$$P \text{ is under the half straight line } D_{M_0}^{x \geq x_{MT}(t_0)}. \tag{72}$$

(Φ_{ec}), is then defined as the complementary subspace of (Φ_{ic}) in the plane. Finally, (χ_T) is the subspace of (Φ_{ec}) defined as follows. A point P the coordinates of which are (x, y) in the frame $(C, \mathbf{i_1}, \mathbf{j_1})$ belongs to the subspace (χ_T) if and only if:

$$P \text{ is above the half straight line } D_{M_0}^{x \leq x_I}, \tag{73}$$

$$P \text{ is above the half straight line } D_C^{x \geq x_I}. \tag{74}$$

PROPERTY 9 *Let us consider a* $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in \left]0, \frac{\pi}{4}\right], \mu_T \geq 1)$ *pure proportional navigation trajectory. The tangents to the trajectory generate a subspace equal to* $((\Phi_{ec})) - (\chi_T))$.

Proof According to initial and final values (55) and (56) of function $(\eta_T - \delta_{MT})$ and according to expression (54) of the slope of a tangent to the trajectory, that slope increases from the initial value:

$$\tan(\eta_T(t_0) - \delta_{MT}(t_0)), \tag{75}$$

to the final value:

$$\tan\left(\eta_T(t_0) + \frac{\delta_{MT}(t_0)}{\mu_T}\right). \tag{76}$$

As tangents does not intersect the trajectory (see proof of property 8) and as functions $\dot{x}_{MT}$ and $\dot{y}_{MT}$ are strictly negative, the space generated by the tangents is then equal to the intersection of the subspace (Φ_{ec}) and of the subspace equal to the union of the two half planes containing the trajectory and the frontier of which are the straight lines D_{M_0} and D_C. This tangent space is then equal to the space $((\Phi_{ec}) - (\chi_T))$.

III.5 Behavior of the components of $X_{aO}(t)$

III.5. 1 Introduction

The aim of this study is to determine the temporal behavior of the components of $\mathbf{X}^2_{\mathbf{aO}}$ in relation to the position of the observation point O. Once this study is performed, the work will consist in finding specific behaviors characterizing the target point. That is why the studies of components which are not characteristic are not necessary. This is the case of the first component of $\mathbf{X}^2_{\mathbf{aO}}$: the range between O and M. For, since functions x_{MT} and y_{MT} are strictly decreasing, for each point O the abscissa of which is negative in the frame (C, $\mathbf{i_1}$, $\mathbf{j_1}$), the range between that point and the object strictly decreases with time for any $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]0, \frac{\pi}{4}], \mu_T \geq 1)$ pure proportional navigation trajectory. Therefore, decrease of r_O is not characteristic of the target point. In fact, when the target is not known only the final value of this function allows the characterization of the target: this result is not very interesting.

III.5.2 Behavior of $\eta_O(t)$

The remark just made for r_O could also be made for η_O. This is because it is also easy to verify that the same temporal behavior as the one for η_T could be found for an infinite set of observation points. Nevertheless, the study of the third component which is much more interesting needs some results about the behaviours of η_T.

PROPOSAL 6 *Consider a* (r_T (t_0), $\frac{\pi}{4}$, δ_{MT} $(t_0) \in]\ 0, \frac{\pi}{4}]$, $\mu_T \geq 1$) *pure proportional navigation trajectory observed from a point O the coordinates of which are* (x_O, y_O) *in the frame* $(C, \mathbf{i_1}, \mathbf{j_1})$. *Temporal variations of angle* η_O *(t) are the following:*

if $O \in (\Phi_{ic})$ *then* η_O *has the same variations as* η_T,

if $O \in (\chi_T)$ *then* η_O *has opposite variations to those of* η_T,

if $O \in (\Im_T)$ *then* η_O *has two changes in its monotonicity: its monotonicity being the same as the one of* η_T *at initial time,*

In other cases, η_O *has only one change of monotonicity: if O is such that* $x_O > x_I$, *then* η_O *has, at the initial time, an opposite monotonicity to that of* η_T *; otherwise, the monotonicity is the same as at the initial time.*

Proof $M(t)$ is the position of M on its trajectory at time t. Its coordinates in the frame $(C, \mathbf{i_1}, \mathbf{j_1})$ are $x_{MT}(t)$ and $y_{MT}(t)$. These coordinates are linked to the angle η_O by the relation:

$$\tan(\eta_O(t)) = \frac{y_{MT}(t) - y_O}{x_{MT}(t) - x_O}. \tag{77}$$

The sign of function $\dot{\eta}_O$ is then the sign of:

$$\dot{y}_{MT}(x_{MT} - x_O) - \dot{x}_{MT}(y_{MT} - y_O), \tag{78}$$

which is null when:

$$\frac{\dot{y}_{MT}(t)}{\dot{x}_{MT}(t)} = \frac{(y_{MT}(t) - y_O)}{(x_{MT}(t) - x_O)}. \tag{79}$$

Equality (79) means that the study of the sign of (78) and then the sign of $\dot{\eta}_O$ is equivalent to comparing the value of the slope of the tangent to the trajectory with the value of the slope of the straight line $OM(t)$.This first result allows one to assert that if the observation point O is outside the space generated by tangents to the trajectory then (78) cannot be null. This is the case of the observation point O which belong to the subspaces (Φ_{ic}) and (χ_T). If, on the contrary, this point O belongs to the space generated by

tangents to the trajectory, then function (78) becomes null once or twice: it depends if O belongs or not to the space generated by the intersection of two tangents to the trajectory i.e. $(\Im_T)$. In order to find the successive monotonicities it is enough to compare, at the initial time for instance, the slope of the tangent with the slope of the straight line $OM(t_0)$. The variations of η_O are represented in table 1 with respect to the position of O.
The result stated in proposal 6 shows that variations of function $\eta_O(t)$ are not characteristic of the target since for all the points of the subspace (Φ_{ic}) the behavior of $\eta_O(t)$ is the same as the one for $\eta_T(t)$. Nevertheless, this study is going to be used to perform one of more importance: the study of $\delta_{MO}(t)$

TABLE I Variation table of η_O in function of position of

position of O	**Temporal Variations**
$O = C$	↗
$O \in (\Phi_{ic})$	↗
$O \in (\chi_T)$	↘
$O \in (\Im_T)$	↗ ↘ ↗
other and $x_O > x_I$	↘ ↗
other and $x_O < x_I$	↗ ↘

III.5.3 Behavior of $\delta_{MO}(t)$

PROPOSAL 7 *Consider a* $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]0, \frac{\pi}{4}], \mu_T \geq 1)$ *pure proportional navigation trajectory observed from a point O the coordinates of which are* (x_O, y_O) *in the frame* (C, i_1, j_1). *In each time interval on which* η_O *increases,* δ_{MO} *decreases.*

Proof This result is an immediate consequence of equality (28) and of the fact that function $\eta_T - \delta_{MT}$ increases.

PROPOSAL 8 *Let us consider* a $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]\,0, \frac{\pi}{4}], \mu_T \geq 1)$ *pure proportional navigation trajectory observed from a point O the coor-*

dinates of which are (x_O, y_O) *in the frame* $(C, \mathbf{i_1}, \mathbf{j_1})$. *If O does not belong to the straight line* D_{M_0}, *then on a time interval including the final time,* η_O *and* δ_{MO} *have the same monotonicity.*

Proof At the final time, it has been seen that $\dot{\eta}_T$ and $\dot{\delta}_{MT}$ are equal to zero. This proposal is then an immediate consequence of equality (28).

PROPOSAL 9 *Consider a* $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in\ ]\,0, \frac{\pi}{4}\,], \mu_T \geq 1\,)$ *pure proportional navigation trajectory observed from a point O the coordinates of which are* (x_O, y_O) *in the frame* $(C, \mathbf{i_1}, \mathbf{j_1})$:

if $O \in (\Phi_{ic})$ *then* $\delta_{MO}(t)$ *is positive at initial time and never becomes null.*

if $O \in (\chi_T)$ *then* $\delta_{MO}(t)$ *is negative at initial time, never becomes null and strictly decreases in* $[t_0, t_f]$.

if $O \in (\mathfrak{I}_T)$ *then* $\delta_{MO}(t)$ *is positive at initial time. It becomes null only once and take after that only once the value* $(-\pi)$*: function* δ_{MO} *strictly decreases between these two values.*

if O belongs to the space generated by tangents to the trajectory but not to the space $(\mathfrak{I}_T)$ *and if O is so that* $x_O < x_I$, *then* $\delta_{MO}(t)$ *is positive at initial time, becomes null only once and strictly decreases after it has become negative. On the contrary, if* $x_O > x_I$ *then* $\delta_{MO}(t)$ *is always negative, and never becomes equal to zero. It takes the value* $(-\pi)$ *only once and strictly decreases after that moment.*

Proof In all cases, the sign at initial time is obtained by comparing the slope of the tangent to the slope of the straight line $OM(t_0)$.

By definition of the angle $\delta_{MO}(t)$, that angle can only be null when O is on a tangent to the trajectory. Therefore, for all the points of the spaces (Φ_{ic}) and (χ_T), this angle cannot be null. If on the contrary, this point belongs to the space generated by tangents, $\delta_{MO}(t)$ becomes null or equal to $(-\pi)$ only once when O does not belong to the space $(\mathfrak{I}_T)$ (The value 0 and $(-\pi)$ depends on the fact that vector $\mathbf{V_M}$ is directed towards or outwards O). That angle takes the successive value 0 and $(-\pi)$ if on the contrary O belong to the space $(\mathfrak{I}_T)$. Taking into account inequalities (68) and (69), it is easy to show that $\delta_{MO}(t)$ is first equal to zero and then equal to $(-\pi)$.

Demonstration of the variations stated in proposal 9 is an immediate consequence of both proposal 7 and the study of the behavior of η_O.

PROPOSAL 10 *Let us consider a* $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0) \in]0, \frac{\pi}{4}], \mu_T \geq 1)$ *pure proportional navigation trajectory observed from a point O the coordinates of which are* (x_O, y_O) in *the frame* $(C, \mathbf{i_1}, \mathbf{j_1})$. *If point O belongs to the straight line* D_C *and if* $x_0 < 0$ *then* $\delta_{MO}(t)$ *is positive and:*

$$\delta_{MO}(t_f) = 0. \tag{80}$$

Proof The straight line D_{M_0} is the last tangent to the trajectory. When $x_0 < 0$, vector $\mathbf{V_M}$ is always directed towards O which then belongs to the space generated by tangents to the trajectory but not to $(\Im_T)$. Equality (80) is then realized at final time.

Moreover, the slope of the tangent at any point of a trajectory is inferior to the one of the straight line D_C (because the slope strictly increases). $\delta_{MO}(t)$ observed from a point O so that $x_0 < 0$ is then positive.

The behaviors of $\delta_{MO}(t)$ shown during this study are drawn in figures 4 (a) à 10 (a) for several observation points. As can be seen in figures 4 (a) to 7 (a) the behaviour of $\delta_{MO}(t)$ for a point O located on the straight line D_C, is the same as that for the target: $\delta_{MO}(t)$ strictly decreases towards zero at final time. For other points the behaviors are different and can be used to remove some observation points of the set of possible targets. For instance, if a null value is observed whereas the range is not null, the corresponding observation point cannot be the target. As can be seen this result is interesting and can be used to find the target. That is what has been done in references [3], [4] and [8]. But there exists a behavior which is more characteristic.

III.5.4 Behavior of $\mu_O(t)$

THEOREM 1 Let us consider a $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0), \mu_T)$ pure proportional navigation trajectory observed from an observation point O the coordinates of which are (x_O, y_O) in the frame $(C, \mathbf{i_1}, \mathbf{j_1})$. Function μ_O is constant in any time interval if and only if O is equal to T.

Proof By definition of the proportional navigation coefficient, if O is equal to T then μ_O is constant and equal to:

$$\mu_O = \mu_T. \tag{81}$$

Now consider an observation point O which is not the target. Suppose that there exists a time interval $[t_1, t_2] \subseteq [t_0, t_f]$ in which $\mu_O(t)$ is constant. From point O, equality (14) and (15) are written:

$$\dot{r}_O(t) = -\|\mathbf{V_M}\| \cos \delta_{MO}(t), \tag{82}$$

$$r_O(t)\dot{\eta}_O(t) = \|\mathbf{V_M}\| \sin \delta_{MO}(t). \tag{83}$$

By differentiating equality (29), we have:

$$(\mu_T + 1)\ddot{\eta}_T(t) = (\mu_O(t) + 1)\ddot{\eta}_O(t). \tag{84}$$

And by differentiating (83):

$$\ddot{\eta}_O(t) = \frac{(\mu_O(t) - 1)\dot{r}_O(t)\dot{\eta}_O(t)}{r_O(t)}. \tag{85}$$

Written at point T, equality (85) becomes:

$$\ddot{\eta}_T(t) = \frac{(\mu_T - 1)\dot{r}_T(t)\dot{\eta}_T(t)}{r_T(t)}. \tag{86}$$

By introducing (85) and (86) as well as (29) into equality (84), it is clear that in the interval $[t_1, t_2] \subseteq [t_0, t_f]$

$$\frac{(\mu_O(t) - 1)\dot{r}_O(t)}{r_O(t)} = \frac{(\mu_T - 1)\dot{r}_T(t)}{r_T(t)}. \tag{87}$$

By integrating this last equation between t_1 and $t \in [t_1, t_2]$ we have:

$$r_O(t) = r_O(t_1)\left(\frac{r_T(t)}{r_T(t_1)}\right)^{\left(\frac{\mu_T - 1}{\mu_O - 1}\right)} \quad \text{for every } t \in [t_1, t_2]. \tag{88}$$

Let us now call d_{OT} the range between the target point T and the observation point O, general results of geometry in triangle *(M(t)OT)* allow one to write:

$$\begin{aligned} r_O^2(t) = r_T^2(t) + d_{OT}^2 - 2r_T(t)d_{OT}\cos(\eta_O(t) - \eta_T(t)) \\ \text{if } (\eta_O(t) - \eta_T(t)) \in [-\frac{\pi}{2}, \frac{\pi}{2}], \end{aligned} \tag{89}$$

and

$$r_O^2(t) = r_T^2(t) + d_{OT}^2 + 2r_T(t)d_{OT}\cos(\eta_O(t) - \eta_T(t)) \text{ else.} \tag{90}$$

Relation (88) expresses the fact that there exists an infinity of triangles such that the relation between the length of two sides does not depend upon the third side and does not depend upon the angles of the triangles. This conclusion is in contradiction to the results (89) and (90) except when O and T represent the same point.

REMARK 6 *Theorem 1 is a solution to the initial characterization problem.*

REMARK 7 *The result expressed in this theorem is more general than those expressed for other components. In particular, no hypotheses are made on the initial value of $\delta_{MO}(t_0)$. That is why it was unnecessary to realize the study of the other components in the general case. For, according to this study, behaviors of components $r_O(t)$, $\eta_O(t)$ and $\delta_{MO}(t)$ are not characteristic of the target. Thus, they are not characteristic for any value of $\delta_{MO}(t_0)$.*

$\mu_O(t)$ being the solution to our characterization problem, it is interesting to give some of its properties.

PROPOSAL 11 *Let us consider a $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0), \mu_T)$ pure proportional navigation trajectory observed from a point O which does not belong to the straight line D_C. At the final time μ_O is equal to:*

$$\mu_O = -1. \tag{91}$$

Proof By using equality (83), equality (24) becomes:

$$\mu_O(t) = \frac{(\mu_T + 1) r_O(t) \dot{\eta}_T(t)}{\|\mathbf{V_M}\| \sin \delta_{MO}(t)} - 1. \tag{92}$$

But, at final time, $\dot{\eta}_T$ is equal to zero whereas functions r_O and $\sin\delta_{MO}$ are not null if O is not on the straight line D_C.

PROPOSAL 12 *Let us consider a $(r_T(t_0), \frac{\pi}{4}, \delta_{MT}(t_0), \mu_T)$ pure proportional navigation trajectory observed from a point O which belongs to the space generated by tangents. Then, function μ_O has two discontinuities if O belongs to the space $(\Im_T)$ and only one otherwise. Moreover it diverges towards infinity around discontinuities.*

Proof It has already been seen that the curvature to the trajectory is equal to zero only at final time. For every point belonging to the space generated by tangents to the trajectory, function δ_{MO} becomes null (modulo π) twice if O belongs to the space $(\Im_T)$ and once if it belongs to the space generated by tangents but not to $(\Im_T)$. According to equality (30), μ_O must become infinite so that curvature remains finite. Moreover, at that moment, as the sign of sin $\delta_{MO}(t)$ changes, then the sign of μ_O must also change.

Examples of behaviors of the instantaneous proportional navigation coefficient are drawn in figures 4(b) to 10(b), for points O belonging to each of the subspaces introduced in this study. A more detailed study is available in [9].

IV. CONCLUSION

In this paper the authors propose a method to analyze a plane proportional navigation trajectory of a maneuvering object when it is not observed from the target point. This method has required the definition of new functions called the instantaneous guidance functions which adapt the kinematic reality of the trajectory to the hypothesis of proportional navigation trajectory. The characterization of the target needs the study of the temporal behaviors of the kinematic parameters of the maneuvering object and of the instantaneous guidance functions.

This method has then been used to characterize the target of a maneuvering object guided by a pure proportional navigation trajectory. In order to do so an instantaneous proportional navigation coefficient has been introduced. The main result of this study is that only the target has an instantaneous proportional navigation coefficient which is constant in any time interval included in the time interval during which the maneuvering object describes its trajectory. By analyzing the temporal behavior of this function it is then possible to find if an observation point is or is not the target of the maneuvering object. Using this characterization Duflos in [9] has derived an algorithm which allows one to find the unknown target of the maneuvering object in the plane. An example of a possible application of this study is its incorporation into the general reasoning of reference [16].

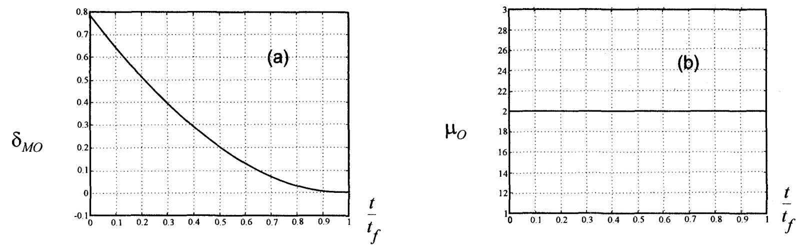

FIGURE 5 Example of temporal variations of δ_{MO} (a) and μ_O (b) when O = C

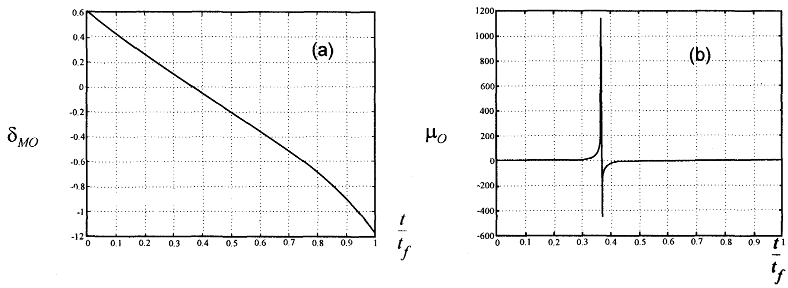

FIGURE 6 Example of temporal variations of δ_{MO} (a) and μ_O (b) when O belongs to the space generated by tangents ($x_0 < x_1$) but not to ($\mathfrak{I}_T$)

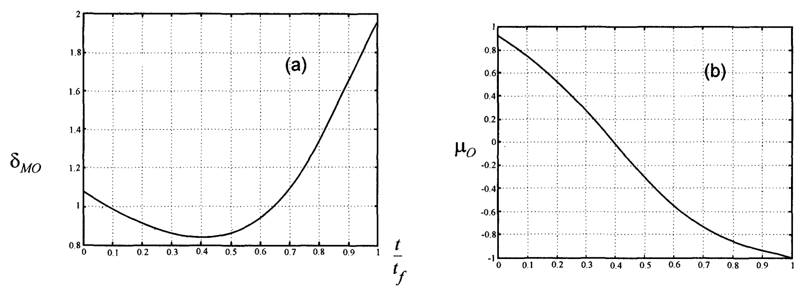

FIGURE 7 Example of temporal variations of δ_{MO} (a) and μ_O (b) when *O* belongs to the (Φ_{ic})

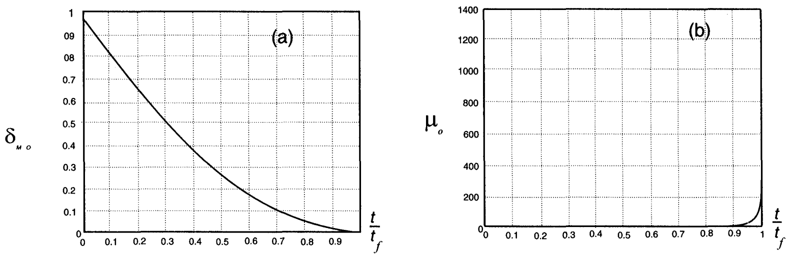

FIGURE 7 Example of temporal variations of δ_{MO} (a) and μ_O when O belongs to the straight line (D_C)

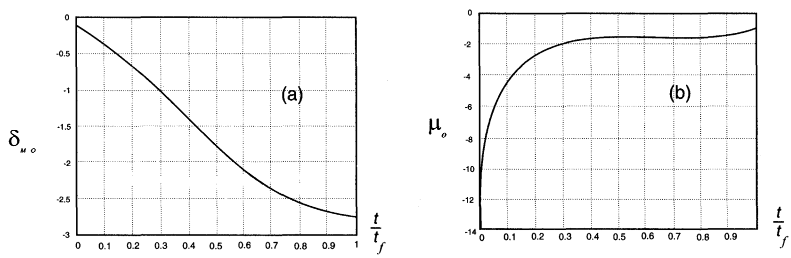

FIGURE 8 Example of temporal variations of δ_{MO} (a) and μ_O when O belongs to (χ_T)

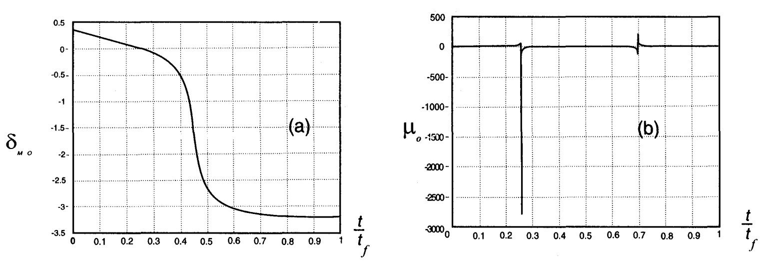

FIGURE 9 Example of temporal variations of δ_{MO} (a) and μ_O when O belongs to $(\Im\sigma_T)$

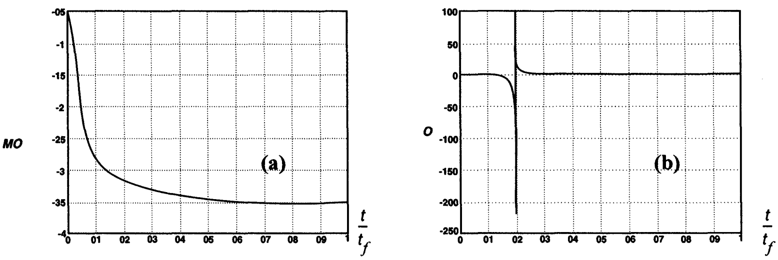

FIGURE 10 Example of temporal variations of δ_{MO} (a) and μ_O (b) when O belongs to the space generated by tangents ($x_0 > x_1$) but not to ($\Im_T$)

References

[1] Becker K., "Closed-Form Solution of Pure Proportional Navigation", *IEEE Transactions on Aerospace and Electronic Systems,* pp. 526–533, vol. 26, n° 3, may 1990.

[2] Borne P., Duflos E., Penel P., Vanheeghe P., "Towards a New Approach to model Guidance laws", Plenary Lectures at the First International Conference On Nonlinear Problems In Aviation and Aerospace, Proceedings to be published.

[3] Boulet V., Druon E, Willaeys D., Borne P., Vanheeghe P., "Target Estimation Using Fuzzy Logic", *Proceedings of IEEE Systems Man and Cybernetics International Conference,* vol. 4, pp. 674–679, Le Touquet, oct. 1993.

[4] Boulet V., Druon E., Duflos E., Willaeys D., Borne P., Vanheeghe P., "Targets Estimation Techniques", *Proceedings of Symposium on Math. Mod.,* vol. 2, pp. 277–278, Vienna, feb. 1994.

[5] Boulet V., Druon E., Duflos E., Borne P., Vanheeghe P., "Optimal Control in Estimation Problems", *Proceedings of IEEE Systems Man and Cybernetics International Conference,* vol. 3, pp. 2658–2662, San Antonio, oct. 1994.

[6] Boulet V., *Contribution à la Determination de Points de Rencontre de Mobiles en Déplacement sous Contraites,* thèse de doctorat, Université de Lille 1, september 1995.

[7] Dhar A., Ghose D., Capture Region for a Realistic TPN Guidance Law" *IEEE Transactions on Aerospace and Electronic Systems,* pp. 995–1003, vol. 29, n° 3, july 1993.

[8] Druon E., *Contribution de la Logique Floue à la Determination de Points de Rencontre entre Mobiles sous Contraintes,* thèse de doctorat, Université de Valenciennes et du Hainaut-Cambrésis, septembre 1995.

[9] Duflos E., *Contribution à la modélisation des lois de guidages: caractérisation d'un point cible prédéfini ou non,* thèse de doctorat, Université de Toulon et du Var, septem bre 95.

[10] Duflos E., Penel P., Druon E., Boulet V., Vanheeghe P., "General 3D Guidance Law Modeling", *Proceedings of IEEE Systems Man and Cybernetics International Conference,* pp Vancouver, oct. 1995.

[11] Ghose D., "True Proportional Navigation with Maneuvering Target", *IEEE Transactions on Aerospace and Electronic Systems,* pp. 229–237, vol. 30, n° 1, jan. 1994.

[12] Ghose D., "On the Generalization Of True Proportional Navigation", *IEEE Transactions on Aerospace and Electronic Systems,* vol. 30, n° 2, april 1994.

[13] Guelman M., "A Qualitative Study of Proportional Navigation", *IEEE Transactions on Aerospace and Electronic Systems,* pp. 637–643, vol 7, n° 4, july 1971.

[14] Guelman M., "Missile acceleration in Proportional Navigation", *IEEE Transactions on Aerospace and Electronic Systems,* pp. 462–463, vol 9, n° 3, may 1973.

[15] Guelman M., "The Closed-Form Solution of True Proportional Navigation", *IEEE Transactions on Aerospace and Electronic Systems,* pp. 472–482, vol. 12, n° 4, july 1976.

[16] Huang P., Kar P., "An Autonomous Optimal Weapon Assigment Algorithm for Ship Self Defense", *Proceedings of IEEE Systems Man and Cybernetics International Conference,* vol. 3, pp. 2545–2549, San Antonio, oct. 1994.

[17] Murtaugh S.A., Criel H.E., "Fundamentals of Proportional Navigation", *IEEE Spectrum,* pp. 75–85, vol.3, dec. 1966.

[18] Nazaroff G.J., "An Optimal Terminal Guidance Law", *IEEE Transaction of Automatic Control,* pp. 407–408, june 1976.

[19] Yang C.D., Yeh F.B., Chen J.H., "The Closed Form Solution of Generalized Proportional Navigation", *Journal of Guidance, Control and Dynamics,* pp. 216–218, vol. 10, n° 2, march-april 1987.

[20] Yang C.D., Hsiao F.B.,Yeh F.B., "Generalized Guidance Law for Homing Missiles", *IEEE Transactions on Aerospace and Electronic Systems,* pp. 197–212, vol. 25, n° 2, march 1989.

[21] Yang C.D., Yang C.C., "Analytical Solution of Generalized 3D Proportional Navigation", Proceedings of the 34th IEEE Conference on Decision and Control, Vol 4, pp.3974–3979, New Orleans, December 13–15 1995.

[22] Zarchan, *Tactical and Strategic Missiles Guidance,* vol 157, Second Edition, AIAA, 1994.