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Optimal Boundary Control of Distributed Systems Involving Dynamic Boundary Conditions

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In this paper we consider Lagrange type control problem for systems involving dynamic boundary conditions that is, with boundary operators containing time derivatives. Assuming the existence of optimal controls, B-evolutions theory is used to present necessary conditions of optimality. The result is illustrated by an example from heat transfer problem and also an algorithm for computing optimal controls is presented.

Keywords: B-evolution systems; Semi-linear systems; Dynamic boundary control problem; Semigroup; Generating and closed pair of operators; Optimal control; Lagrange problem

1 MOTIVATION

Many physical systems, with dynamic boundary conditions, have applications in multi-phase problems in physics and engineering [3,4,9,11]. These include heat transfer and Navier–Stokes equations.

For motivation let us consider a heat transfer problem arising in nuclear reactor. Let Ω be a bounded open subset of R^3 modeling the interior of an annular tube of finite length with smooth boundary which consists of two parts: Γ_1, the inner boundary, and Γ_2, the

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outer boundary. The coolant (e.g. heavy water) in the annular region Ω receives heat energy from the heat produced by nuclear reaction inside the fuel rods surrounded by the boundary layer Γ_1. The corresponding control system model can be described as follows:

$$\begin{cases} (\partial/\partial t)T(t,\xi) = \operatorname{div}(k(\xi)\nabla T) + v\cdot\nabla T, \quad \xi\in\Omega,\ t>0, \\ T(t,\xi)|_{\Gamma_2} = h(t,\xi), \quad \xi\in\Gamma_2,\ t\geq 0, \\ (\partial/\partial t)(T(t,\xi)|_{\Gamma_1}) \\ \quad = \Delta_{\Gamma_1}T(t,\xi) - \beta D_\nu(T(t,\xi)|_{\Gamma_1}) + g(\xi, T(t,\xi)|_{\Gamma_1}, u(t,\xi)), \quad t>0, \\ T(0,\xi) = T_0(\xi), \quad \xi\in\Omega, \\ T(0,\xi)|_{\Gamma_1} = T_1(\xi), \quad \xi\in\Gamma_1. \end{cases} \tag{1.1}$$

The quantity T denotes the space–time temperature distribution in the interior of the domain, $k:\overline{\Omega}\mapsto R^+$, represents the thermal conductivity which satisfies

$$k(\xi) = \begin{cases} 0 & \text{on } \Gamma_2, \\ K & \text{on } \Omega, \\ \beta & \text{on } \Gamma_1. \end{cases}$$

The constant, $\beta(>0)$ represents the thermal conductivity of the material that constitutes the boundary layer Γ_1. The quantity $v=v(t,\xi)\in R^3$ denotes the transport velocity of the fluid in Ω and $u=u(t,\xi)$ (the control) temperature of the outer surface Γ_1 of the fuel rod (due to nuclear reaction). The Laplace–Beltrami operator Δ_{Γ_1} represents the rate at which thermal energy is transferred within Γ_1 and D_ν denotes the outward normal derivative. The function g represents a convective heat transfer and is given by $g=\alpha(T_{\mathrm{r}} - T|_{\Gamma_1})$, where the parameter α is the heat transfer coefficient due to convection and T_{r} denotes the surface temperature of the fuel rod which is the control u.

For this example the integrand may be taken as

$$l(T(t),u(t)) \equiv \int_\Omega \left|T(t,\xi) - T^{\mathrm{d}}(t,\xi)\right|^2 \mathrm{d}\xi + \int_{\Gamma_1} |u(t,\xi)|^2 \mathrm{d}\xi$$

and the cost functional to be minimized subject to the system (1.1) is given by

$$J(u) = \int_I l(T(t), u(t))\, \mathrm{d}t.$$

That is, we like to keep the temperature distribution at some preassigned value T^{d}.

The necessary conditions of optimality for the above example will be presented in Section 5 once the general theory has been developed.

The abstract mathematical model for the example (1.1) can be written in two Banach spaces as follows:

$$(P_f) \quad \begin{cases} (\mathrm{d}/\mathrm{d}t)Bx(t) = A(t)x(t) + f(Bx(t), u(t)), & t \in I \\ (Bx)(0) = z_0, \end{cases}$$

where A and B are linear unbounded operators between two Banach spaces, f is a map with values in a Banach space and u is a suitable function representing the control actions taking values in another Banach space.

In addition to covering all classical boundary and distributed control problems, the system (P_f) includes a new set of such problems in which the boundary conditions are determined also by an evolution equation (see example above).

The control problems of systems governed by B-evolutions has not been studied before. Control theory for classical cases ($X = Y$, $B = I$ identity operator) and their abstract versions have been studied extensively in the literature (see Fattorini [5], Li and Yong [6], Lions [7], Ahmed and Teo [10] and many others, and the references therein).

In [6], the author gives necessary conditions of optimality for a semilinear control problem in terms of a semigroup of operators which in general is not available and hence cannot be used to construct an algorithm for computing optimal controls.

In this paper we consider a class of semilinear problems governed by B-evolutions as described above by the system (P_f). We derive necessary conditions of optimality in terms of the available data, that is in terms of the given operators A and B and the cost integrand l.

2 BASIC RESULTS FOR B-EVOLUTION

For any pair of Banach spaces X, Y, respectively with norms, $\|\cdot\|_X$, $\|\cdot\|_Y$, $\mathcal{L}(X,Y)$ and $\mathcal{L}_{ub}(X,Y)$ will denote the class of bounded and unbounded linear operators from X to Y respectively. Let $I=[0,T]$, $T<\infty$, and for $1\leq p<\infty$, $L^P(I,X)$, endowed with its usual p-norm, $\|\cdot\|_p$ denote the class of all pth power Lebesgue–Bochner integrable functions on I with values in X.

As the theory of B-evolutions has been developed only during the past few years, we shall list, for the sake of completeness, the basic definitions and properties of B-evolutions, as introduced in [2,9].

Let B be a linear operator with domain $D(B)\subset X$ and range $R(B)\subset Y$.

DEFINITION 2.1 A family $\{S(t)\colon t>0\}$ of bounded linear operators defined on Y is called a B-evolution if

$$S(t)(Y)\subset D(B), \quad \text{for all } t>0$$

and

$$S(t+s)=S(s)BS(t), \quad \text{for all } s,t>0.$$

From the definition, it follows that the family $\{E(t)\colon t\geq 0\}$ of linear operators in Y, defined by

$$E(t)=BS(t), \quad \text{for all } t>0,$$

satisfies the semigroup property

$$E(t+s)=E(s)E(t), \quad \text{for all } s,t\geq 0.$$

$E(t)$ is called the associed semigroup. The B-evolution $S(t)$ is called strongly continuous if $E(t)$ is a semigroup of class C_0.

DEFINITION 2.2 A strongly continuous, uniformly bounded B-evolution is said to be of type L if

$$P_B(\lambda)y \triangleq \int_0^{\infty} \exp(-\lambda t)S(t)y \,\mathrm{d}t \in D(B),$$

for all $y \in Y$ and a complex number λ, with $\operatorname{Re} \lambda > 0$, and

$$BP_B(\lambda)y = \int_0^{\infty} \exp(-\lambda t) BS(t) y \, \mathrm{d}t.$$

DEFINITION 2.3 A B-evolution $S(t)$ of type L is called holomorphic if the associed semigroup $E(t)$ is holomorphic.

DEFINITION 2.4 The infinitesimal generator A of B-evolution $S(t)$ is defined by

$$D(A) = \{x \in D(B)\colon\ Ax = \lim_{h \to 0} h^{-1}(BS(h) - B)x \text{ exist}\}.$$

It is clear from the last definition that $D(A) \subset D(B)$.

Remark 2.1 The infinitesimal generator A of B-evolution is not necessarily closed or densely defined.

Some useful results due to Sauer [9] are given by the following lemmas:

LEMMA 2.1 ([9], Theorem 2.1, p. 289) *Let $S(t)$ be a strongly continuous B-evolution. Then*

(a) *for $x \in D(A)$, $S(t)Bx \in D(A)$, and*

$$AS(t)Bx = BS(t)Ax = (\mathrm{d}/\mathrm{d}t)BS(t)Bx, \quad \textit{for } t > 0;$$

(b) *if A_Y is the infinitesimal generator of $E(t)$ then $x \in D(A)$ if and only if $Bx \in D(A_Y)$ and for such x*

$$Ax = A_Y Bx;$$

(c) *$B(D(A))$ is dense in Y;*
(d) *for $y \in Y$, the mapping $t \to S(t)y$ is right continuous.*

LEMMA 2.2 ([9], Theorem 2.3, p. 290) *Let $S(t)$, $t > 0$, be a strongly continuous uniformly bounded B-evolution. Then*

(a) *for each $y \in Y$, $t \to S(t)y$, $t > 0$, is strongly continuous with values in X;*
(b) *there exists an operator $C \in \mathcal{L}(Y, X)$ such that $Cy = \lim_{t \to 0^+} S(t)y$ for each $y \in Y$; and $S(t)y = CE(t)y$, $t > 0$;*
(c) *C restricted to the range of B is the right inverse of B.*

LEMMA 2.3 ([9], Theorem 5.1, p. 296) *The pair* $\langle A_0, B_0\rangle$, *where* A_0 *and* B_0 *are suitable restrictions of* A *and* B *respectively, is the generating pair of a B-evolution* $S(t)$, $t>0$, *of type L if and only if*:

(a) B_0 *has a bounded inverse on its range* $R(B_0)\subset Y$;
(b) $A_0B_0^{-1}$ *generates a uniformly bounded* C_0*-semigroup* $E(t)$, *for* $t>0$, *in* Y;
(c) *the bounded linear operator* C *which is the strong limit of* $C_n\equiv(B_0-(1/n)A_0)^{-1}$ *is invertible on* $U_{t>0}E(t)(Y)+R(B_0)$.

In case $E(t)$, $t>0$, is holomorphic semigroup in Y or B_0 is closeable the last condition is superfluous. In this case the pair $\langle A_0, B_0\rangle$ coincides with the pair $\langle A, B\rangle$.

The following assumptions will be used:

Assumptions (I) The domain $D(A(t))=D(A)$ is independent of t.

(II) There exists a number $\epsilon>0$ such that for all λ in $\Theta\equiv\{\lambda\in C: \lambda\neq 0,\ -(\epsilon+\pi/2)<\arg\lambda<\epsilon+\pi/2\}$ the resolvents $(\lambda B-A(t))^{-1}$ and $B(\lambda B-A(t))^{-1}$ are strongly continuous in t, with respect to the norm topologies of $\mathcal{L}(Y,X)$ and $\mathcal{L}(Y)$ respectively. The continuity in t is uniform in λ on every compact subset of Θ.

(III) There exist positive constants M and N such that for $\lambda\in\Theta$ and $t\in[0,T]$

$$\|(\lambda B-A(t))^{-1}\|_{\mathcal{L}(Y,X)}\leq M/|\lambda|,$$
$$\|B(\lambda B-A(t))^{-1}\|_{\mathcal{L}(Y)}\leq N/|\lambda|.$$

(IV) B is injective and has a bounded inverse on its range $R(B)$.

(V) There exists a constant $\tilde{N}>0$ such that

$$\|(A(\tau)B^{-1})^{-1}\|_{\mathcal{L}(Y)}\leq\tilde{N},\quad \tau\in[0,T].$$

(VI) There exists a constant $K>0$ such that, for $t,s,\tau\in[0,T]$, $0<\alpha<1$,

$$\|(A(t)-A(s))A^{-1}(\tau)\|_{\mathcal{L}(Y)}\leq K|t-s|^{\alpha}.$$

A characterization of a generating pair of operators is given by the following lemma.

LEMMA 2.4 ([2]) *Under the assumptions* (I)–(VI) *the pair* $\langle A(\cdot), B\rangle$ *generates a holomorphic B-evolution V of type L with values* $\{V(t,s), t>s\geq 0\}\in\mathcal{L}(Y,X)$ *satisfying the following properties:*

(P1) $V(t,s)$ *is continuous on* $0\leq s<t\leq T$ *in the strong operator topology of* $\mathcal{L}(Y,X)$.

(P2) $E(t,s)\equiv BV(t,s)$ *is uniformly bounded in* $\mathcal{L}(Y)$, *i.e. there exists a constant* $C_T>0$ *such that*

$$\|BV(t,s)\|_{\mathcal{L}(Y)}\leq C_T, \quad \textit{for } 0\leq s<t\leq T.$$

(P3) $BV(t,s)$ *is continuously differentiable in the strong topology of* $\mathcal{L}(Y)$ *on* $[0,T)$ *and*

$$\begin{cases}(\partial/\partial t)BV(t,s) = A(t)V(t,s),\\ (\partial/\partial s)BV(t,s) = -BV(t,s)A(s)B^{-1}.\end{cases}$$

(P4) BV *satisfy the following evolution property:*

$$\begin{cases}BV(t,s)BV(s,\tau) = BV(t,\tau), \quad \textit{for } 0\leq\tau<s\leq T,\\ BV(t,t) = I.\end{cases}$$

3 PREPARATORY RESULTS

For the development of necessary conditions of optimality, we need some preliminary results.

Here we consider the case of holomorphic B-evolution of type L. In this case the operators A and B are not necessarily closed but the pair $\langle A(t), B\rangle$ from $D(A)\cap D(B)\subset X$ to $Y\times Y$ is closed. It follows from Lemma 2.1, that the Cauchy problem

$$\begin{cases}(\mathrm{d}/\mathrm{d}t)Bx = A(t)x,\\ s-\lim_{t\to 0^+}(Bx) = z_0\end{cases}$$

has a unique solution given by

$$x(t) = V(t,0)z_0, \quad t>0. \tag{3.1}$$

As a consequence of Lemma 2.1, it follows that $A(t)B^{-1}$ with domain $B(D(A)) \subset Y$ is the infinitesimal generator of the transition operator $E(t,s) \equiv BV(t,s)$, $t > s \geq 0$. In this case the system (P_f) can be written as

$$\begin{cases} \dot{z} = A(t)B^{-1}z + f(z,u), \\ z(0) = z_0 \end{cases} \tag{3.2}$$

and call $x(t) \equiv Cz(t)$, $t \geq 0$, as the generalized solution of the system (P_f) where z is the mild solution of Eq. (3.2) and $C \in \mathcal{L}(Y,X)$ is the operator given by Lemma 2.2.

Thus in the case of holomorphic B-evolution the problem (P_f) is related to the classical problem (3.2). In the homogeneous case (i.e. $f \equiv 0$) the solution of the problem (P_0) is given by $x(t) = B^{-1}z(t)$ since $z(t) = E(t,0)z_0 \in R(B)$ and $C|_{R(B)} = B^{-1}$. In the nonhomogeneous case, $z(t)$ may not be in the range of the operator B and hence the definition of generalized solution $x(t) = Cz(t)$ makes sense.

For simplicity of notation we have written $s - \lim_{t \to 0^+}(Bx) = Bx(t)|_{t=0} = z(0)$.

In order to study the control problem given by the system (3.2) we introduce the class of admissible controls as follows:

Admissible controls Let Λ be a closed, bounded and convex subset of U. For admissible controls, we choose the set

$$\mathcal{U}_{\mathrm{ad}} \equiv \{u\text{: strongly measurable and } u(t) \in \Lambda, \text{ a.e.}\}.$$

Occasionally, we use the notation $x(u)$ to denote the solution, of the system (P_f) corresponding to $u \in \mathcal{U}_{\mathrm{ad}}$.

In the following lemma we present an *a priori* bound and existence result.

LEMMA 3.1 *Suppose the following assumptions hold*:

(b1) *The pair* $\langle A(t), B\rangle$ *is the generating pair of a holomorphic* B*-evolution* V *of type* L.

(b2) $f: Y \times U \to Y$ *is a map such that* f *is locally Lipshitz in* Y *i.e. for each* $0 < r < \infty$, *and* $z_0 \in Y$, *there exists a positive constant*

$K_r \equiv K_r(z_0)$ such that

$$\|f(z_1, u) - f(z_2, u)\|_Y \leq K_r \|z_1 - z_2\|_Y$$

for all $z_1, z_2 \in B_r(z_0) \equiv \{z \in Y\colon \|z(t) - z_0\|_Y \leq r\}$ and $u \in \Lambda$ and satisfies the growth condition

$$\|f(z, u)\|_Y \leq K(1 + \|z\|_Y),$$

for some $K > 0$, $z \in Y$, $u \in \Lambda$. Then

(i) *there exist finite positive numbers $\tilde{M}$ such that:*

$$\sup\{\|x(u)(t)\|_X,\ t \in I,\ u \in \mathcal{U}_{\mathrm{ad}}\} \leq \tilde{M};$$

(ii) *for each $z_0 \in Y$, the problem (P_f) has a unique generalized solution $x \in C(I, X)$ and this is given by $x = Cz$ where z is the solution of the integral equation*

$$z(t) = E(t,0)z_0 + \int_0^t E(t,s) f(z(s), u(s))\, \mathrm{d}s, \quad t \in I. \tag{3.3}$$

Proof By virtue of Lemma 2.4, $A(t)B^{-1}$ is the infinitesimal generator of uniformly bounded holomorphic evolution operator $E(t, s)$ in Y, $0 \leq s < t \leq T$. Using the "variation of constants" formula we obtain the following integral equation:

$$z(t) = E(t,0)z_0 + \int_0^t E(t,s) f(z(s), u(s))\, \mathrm{d}s, \quad t \in I. \tag{3.4}$$

Since the controls are contained in a bounded set and f satisfies the growth condition (assumption (b2)), using Gronwall's inequality and the integral equation above one can easily verify that there exists an $M > 0$ such that

$$\sup\{\|z(t)\|_Y,\ t \in I\} \leq M, \tag{3.5}$$

where $M \equiv M(\|z_0\|_Y, T)$ is a positive constant dependent on the parameters shown. Thus the map $u \mapsto z$ from $\mathcal{U}_{\mathrm{ad}}$ to $L^\infty(I; Y)$ is bounded. Since C is bounded, there exists a constant $\tilde{M} > 0$ such that

$$\sup\{\|x(t)\|_X,\ t \in I\} \leq \tilde{M}. \tag{3.6}$$

That is, the map $u \mapsto x$ from $\mathcal{U}_{\text{ad}}$ to $L^\infty(I; X)$ is bounded. This justifies the conclusion (i).

To prove (ii), let $z \in C(I, Y)$ satisfying $z(0) = z_0$ and for some $0 < r < \infty$ $z(t) \in B_r(z_0)$ for all $t \in I$. Define the operator Q by

$$(Qz)(t) = E(t,0)z_0 + \int_0^t E(t,s)f(z(s),u(s))\,\mathrm{d}s, \quad \text{for } t \in I. \tag{3.7}$$

Using the strong continuity of $E(t,s)$ on $\Delta \equiv \{0 \leq s \leq t \leq T\}$ in $\mathcal{L}(Y)$, the assumption (b2), the estimate (P2) of Lemma 2.4 and the above integral equation, one can show that $(Qz)(t) \in B_r(z_0)$. Further, $t \mapsto (Qz)(t)$ is continuous Y-valued function on I. Define

$$\Sigma_r \equiv \left\{ z \in C([0,\sigma], Y)\colon\ z(0) = z_0 \text{ and } \sup_{t \in [0,\sigma]} \|z(t) - z_0\|_Y \leq r \right\}.$$

The set Σ_r, furnished with the natural metric topology

$$\rho(z_1, z_2) \equiv \sup_{t \in [0,\sigma]} \|z_1(t) - z_2(t)\|_Y, \quad z_1, z_2 \in \Sigma_r,$$

is a complete metric space and Q maps Σ_r to Σ_r.

Under the property (P2) and assumption (b2), one can show that Q is a contraction in Σ_r and hence from the Banach fixed point theorem it follows that Q has a unique fixed point $z \in \Sigma_r$. Since $C \in \mathcal{L}(Y, X)$, it follows that the Cauchy problem (P_f) has unique generalized solution $x = Cz(t) \in C(I; X)$. This completes the proof of the lemma.

Remark 3.1 By virtue of the *a priori* bound, all solutions $\{x(u)\}$ of the system (P_f) lie in a closed ball $B_{\tilde{M}}(X) \equiv \{\xi \in X\colon \|\xi\| \leq \tilde{M}\}$.

For the necessary conditions of optimality we shall introduce the following assumptions:

(A1) assumptions of Lemma 3.1 hold,

(A2) $f : Y \times U \to Y$ is Frechet differentiable with respect to $z \in Y$ and $u \in U$ with respective Frechet derivatives $f_1 : Y \times U \to \mathcal{L}(Y)$,

$f_2 : Y \times U \to \mathcal{L}(U, Y)$ being continuous and bounded on bounded subsets of $Y \times U$.

Let z^0 and $z(u^\epsilon)$ be the solution of the system (3.2) corresponding to control u^0 and u^ϵ respectively and $x^0 = Cz^0$ and $x(u^\epsilon) = Cz(u^\epsilon)$ be the generalized solution of (P_f) corresponding to control u^0 and u^ϵ respectively. Then if $u^\epsilon \equiv u^0 + \epsilon(u - u^0) \to u^0$, the solution $x(u^\epsilon) \to x^0$ in $C(I; X)$ and $z(u^\epsilon) \to z^0$ in $C(I; Y)$.

First define

$$F_1^\epsilon(s) \equiv \int_0^1 f_1\big(z^0(s) + \theta(z(u^\epsilon)(s) - z^0(s)), u^0(s)\big)\, \mathrm{d}\theta, \quad s \in I,$$

and

$$F_2^\epsilon(s) \equiv \int_0^1 f_2\big(z^0(s), u^0 + \theta(u^\epsilon(s) - u^0(s))\big)\, \mathrm{d}\theta, \quad s \in I.$$

Clearly under the assumption (A2) and the *a priori* bound (see Remark 3.1), there exists a constant $\tilde{C} > 0$ such that

$$\sup\{\|F_1^\epsilon(t)\|_{\mathcal{L}(Y)}, \|F_2^\epsilon(t)\|_{\mathcal{L}(U;\,Y)}, t \in I\} \leq \tilde{C}.$$

Further, as $\epsilon \to 0$, the operators $F_1^\epsilon(t)$ and $F_2^\epsilon(t)$ converge to $F_1^0(t)$ and $F_2^0(t) \equiv f_2(z^0(t), u^0)$ in the uniform operator topology for almost all $t \in I$. Thus $t \mapsto F_1^\epsilon(t)$ and $t \mapsto F_2^\epsilon(t)$ are uniformly measurable operator valued functions taking values from $\mathcal{L}(Y)$ and $\mathcal{L}(U; Y)$ respectively.

Let Y^* denote the dual of the Banach space Y and Y_w^* the space Y^* endowed with the w^* topology and $C(I; Y_w^*)$ the topological space of w^*-continuous Y^*-valued functions defined on the interval $I = [0, T]$. Let $\langle \cdot, \cdot \rangle_{Y^*, Y}$ denote the duality pairing between Y^* and Y. Let A^* denote the adjoint of the operator A. For the study of control problem we shall need the following Cauchy problem called the adjoint equation:

$$\begin{cases} (\mathrm{d}/\mathrm{d}t)\psi(t) + (A(t)B^{-1} + F_1^0)^*\psi(t) = -g(t), \quad \text{a.e.}, \\ \psi(T) = 0 \in Y^*. \end{cases} \tag{3.8}$$

LEMMA 3.2 *Let $g \in L^1(I, Y^*)$ and $F_1^0(t)$ the operator as defined above. Then the adjoint problem* (3.8) *has a unique mild solution $\psi \in C(I; Y_w^*)$, which satisfies Eq.* (3.8) *also in the weak sense.*

Proof Let $E^*(t,s)$, $0 \leq s \leq t < \infty$, denote the adjoint of the operator $E(t,s)$ and write (3.8) as a Volterra integral equation

$$\psi(t) = \int_t^T E^*(\theta,t)g(\theta)\,\mathrm{d}\theta + \int_t^T E^*(\theta,t)(F_1^0)^*(\theta)\psi(\theta)\,\mathrm{d}\theta. \tag{3.9}$$

Define the operator $\tilde{Q}$ by

$$\begin{aligned}(\tilde{Q}\psi)(t) &= \int_t^T E^*(\theta,t)g(\theta)\,\mathrm{d}\theta + \int_t^T E^*(\theta,t)(F_1^0)^*(\theta)\psi(\theta)\,\mathrm{d}\theta \\ &\equiv h_1(t) + h_2(t).\end{aligned} \tag{3.10}$$

We show that $\tilde{Q}$ has a fixed point in $L^\infty(I; Y^*)$ and any such solution is actually w^*-continuous. First we show that $\tilde{Q}$ maps $L^\infty(I; Y^*)$ to $L^\infty(I; Y^*)$.

Since the operator $E(t,\theta)$, $0 \leq \theta \leq t \leq T$, is bounded and $g \in L^1(I; Y^*)$ it is easy to verify that $h_1 \in L^\infty(I; Y^*)$. Since the evolution operator $E(\theta,t)$ and the operator $F_1^0(t)$, $0 \leq t \leq \theta \leq T$, are bounded and uniformly measurable, their adjoints are also bounded and measurable. Hence the integral in (3.9) is well defined in the Bochner sense and moreover for $\psi \in L^\infty(I; Y^*)$, $h_2 \in L^\infty(I; Y^*)$. Hence $\tilde{Q}$ maps $L^\infty(I; Y^*)$ into itself.

Since the operators $E^*(\theta,t)$ and $(F_1^0)^*(\theta)$ are bounded on $0 \leq t \leq \theta \leq T$, there exists a constant $\tilde{K} > 0$ so that

$$\left\|(\tilde{Q}\psi_1 - \tilde{Q}\psi_2)(t)\right\|_{Y^*} \leq \tilde{K}\int_t^T \|(\psi_1 - \psi_2)(\theta)\|_{Y^*}\,\mathrm{d}\theta \tag{3.11}$$

for all $\psi_1, \psi_2 \in L^\infty(I; Y^*)$ and $t \in I$. Substituting (3.11) into itself, at the nth iteration, we obtain

$$\left\|(\tilde{Q}^n\psi_1 - \tilde{Q}^n\psi_2)(t)\right\|_{Y^*} \leq (\tilde{K}T)^n/n! \int_t^T \|(\psi_1 - \psi_2)(\theta)\|_{Y^*}\,\mathrm{d}\theta.$$

Taking the supremum norm in the last inequality we obtain

$$d(\tilde{Q}^n\psi_1, \tilde{Q}^n\psi_2) \leq \alpha_n d(\psi_1, \psi_2),$$

where

$$d(y_1, y_2) \equiv \operatorname{ess\,sup}\{\|y_1(t) - y_2(t)\|_{Y^*}\ t \in I\}$$

and

$$\alpha_n = (\tilde{K}T)^n T/n!.$$

Then one can choose a positive integer n_0 such that $\alpha_{n_0} < 1$. Hence, for $n > n_0$, $\tilde{Q}^n$ is a contraction on $L^\infty(I; Y^*)$. It follows from Banach fixed point theorem that $\tilde{Q}^n$, and hence $\tilde{Q}$, has a unique fixed point in $L^\infty(I; Y^*)$ and therefore Eq. (3.9) has a unique mild solution $\psi \in L^\infty(I; Y^*)$.

We show that $\psi \in C(I; Y_w^*)$ and that it is also a weak solution. Taking $\eta \in Y$, it follows from (3.9) that

$$\begin{aligned}\langle \psi(t), \eta\rangle_{Y^*, Y} &= \int_t^T \langle g(\theta), E(\theta, t)\eta\rangle_{Y, Y^*}\,\mathrm{d}\theta \\ &\quad + \int_t^T \langle \psi(\theta), F_1^0(\theta)E(\theta, t)\eta\rangle_{Y^*, Y}\,\mathrm{d}\theta \\ &\equiv I_1(t) + I_2(t). \end{aligned} \tag{3.12}$$

Since $t \mapsto E(\theta, t)$ is strongly continuous in $\mathcal{L}(Y)$ on $[0, \theta]$ and $g \in L^1(I; Y^*)$, it follows that $t \mapsto I_1(t)$ is continuous on I. Further, since $E(\theta, t)$ is strongly continuous in $\mathcal{L}(Y)$, for $\psi \in L^\infty(I; Y^*)$, $t \mapsto I_2(t)$ is also continuous on I. Thus $t \mapsto \psi \in C(I; Y_w^*)$. Replacing η by z for $z \in B(D(A))$, it follows from (3.12) that

$$\begin{aligned}\langle \psi(t), z\rangle_{Y^*, Y} &= \int_t^T \langle g(\theta), E(\theta, t)z\rangle_{Y^*, Y}\,\mathrm{d}\theta \\ &\quad + \int_t^T \langle (F_1^0)^*(\theta)\psi(\theta), E(\theta, t)z\rangle_{Y^*, Y}\,\mathrm{d}\theta.\end{aligned}$$

Using the differentiation property (P3) (see Section 3) it is not difficult to verify that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t), z\rangle_{Y^*,Y} + \langle\psi(t), F_1^0(t)z\rangle_{Y^*,Y} \\
&\quad + \int_t^T \langle E^*(\theta,t)(F_1^0)^*(\theta)\psi(\theta) + E^*(\theta,t)g(\theta), A(t)B^{-1}z\rangle_{Y^*,Y}\,\mathrm{d}\theta \\
&\quad = -\langle g(t), z\rangle_{Y^*,Y}.
\end{aligned}
$$

Thus, for all $z \in B(D(A))$,

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\langle\psi(t), z\rangle_{Y^*,Y} + \langle\psi(t), F_1^0(t)z\rangle_{Y^*,Y} + \langle\psi(t), A(t)B^{-1}z\rangle_{Y^*,Y} \\
&\quad = -\langle g(t), \xi\rangle_{X^*,X} \qquad (3.13)
\end{aligned}
$$

for almost all $t \in I$. Clearly by (3.9) $\psi(T) = 0$. Thus, ψ as defined above, solves the problem (3.8) also in the weak sense.

4 NECESSARY CONDITIONS OF OPTIMALITY

In this section we present our main results on the necessary conditions of optimality for the following Lagrange problem:

(P) find $u^0 \in \mathcal{U}_{\mathrm{ad}}$ such that $J(z^0, u^0) \le J(z, u)$, for all $u \in \mathcal{U}_{\mathrm{ad}}$

where $J(z, u) \equiv \int_I l(Cz, u)\,\mathrm{d}t$.

Here z denotes the solution of the system (3.2) corresponding to the control $u \in \mathcal{U}_{\mathrm{ad}}$.

In what follows we shall assume that the optimal control problem (P) has a solution, that is there exists an admissible state–control pair (z^0, u^0) such that

$$
J(z^0, u^0) \le J(z, u), \quad \text{for all } u \in \mathcal{U}_{\mathrm{ad}}. \tag{4.1}
$$

We consider two cases: the cases where the cost integrand l is Frechet differentiable and merely continuous in the control variable.

Case A: l Frechet Differentiable in Control Variable

THEOREM 4.1 *Let $(z^0, u^0) \in C(I, Y) \times L^\infty(I, U)$ be any state–control pair associated to system* (3.2) *and suppose the following conditions hold*:

(d1) *$l : I \times Y \times X \times U \to R$ such that $l(w, \zeta)$ is continuously Frechet differentiable in w and ζ with Frechet derivatives denoted by l_1, and l_2 respectively. Further $C^* l_1^0 \in L^1(I; Y^*)$ and $l_2^0 \in L^1(I; U^*)$ along the pair (z^0, u^0).*

(d2) *f satisfies assumptions* (b2) *of Lemma* 3.1 *and* (A2).

Then, in order that (z^0, u^0) be the optimal pair, it is necessary that there exists $\psi \in C(I; Y_w^*)$ so that the triple (z^0, u^0, ψ) satisfies the following equations and inequalities:

(1) $(\mathrm{d}/\mathrm{d}t)z^0 = A(t)B^{-1}z^0 + f(z^0, u^0)$, $z^0 = z(0)$.

(2) $(\mathrm{d}/\mathrm{d}t)\psi(t) + (A(t)B^{-1})^*\psi + (F_1(t))^*\psi = -C^* l_1^0(t)$, $\psi(T) = 0$.

(3) $\int_I \langle l_2^0 + F_2^0 \psi, (u - u^0)\rangle_{U^*, U}\, \mathrm{d}t \geq 0$, *for all $u \in \mathcal{U}_{\mathrm{ad}}$, where $l_i^0(t) \equiv l_i(z^0(t), Cz^0(t), u^0(t))$, for $i = 1, 2$.*

Proof Let (z^0, u^0) be the optimal pair for the problem (3.2). By convexity of $\mathcal{U}_{\mathrm{ad}}$, for $u \in \mathcal{U}_{\mathrm{ad}}$, $u^\epsilon \equiv u^0 + \epsilon(u - u^0) \in \mathcal{U}_{\mathrm{ad}}$, for $0 \leq \epsilon \leq 1$. According to Lemma 3.1, the state Eq. (3.2) has a unique mild solution z^ϵ corresponding to the control u^ϵ and by definition of optimality (4.1) we have

$$\int_I l(Cz^\epsilon(t), u^\epsilon(t))\, \mathrm{d}t - \int_I l(Cz^0(t), u^0(t))\, \mathrm{d}t \geq 0. \tag{4.2}$$

Define $y^\epsilon = (z - z^0)/\epsilon$. Note that y^ϵ satisfies the integral equation

$$y^\epsilon(t) = \int_0^t E(t,s) F_1^\epsilon(s) y^\epsilon(s)\, \mathrm{d}s + \int_0^t E(t,s) F_2^\epsilon(s)(u(s) - u^0(s))\, \mathrm{d}s. \tag{4.3}$$

By virtue of assumption (A2) and once more applying dominated convergence theorem one can justify taking ϵ to zero in the above equation to obtain

$$y(t) = \int_0^t E(t,s) F_1^0(s) y(s)\, \mathrm{d}s + \int_0^t E(t,s) F_2^0(s)(u(s) - u^0(s))\, \mathrm{d}s. \tag{4.4}$$

Since a linear Volterra integral equation has a unique solution (see Theorem 2.4.3 of [10]) y is a mild solution of equation

$$\begin{cases} (\mathrm{d}/\mathrm{d}t)y(t) = (AB^{-1} + F_1^0)(t)y + F_2^0(t)(u(t) - u^0(t)), \\ (y)(0) = 0. \end{cases} \tag{4.5}$$

Note that y is the Gateaux differential of z in the direction $u - u^0$.

By use of the hypothesis (d1), and some elementary computations, one obtains from (4.2) the following inequality:

$$\int_I \langle C^* l_1^0, z\rangle_{Y^*,Y}\,\mathrm{d}t + \int_I \langle l_2^0, u - u^0\rangle_{U^*,U}\,\mathrm{d}t \geq 0. \tag{4.6}$$

By virtue of (d1), $C^* l_1^0 \in L_1(I, Y^*)$, and hence, by Lemma 3.2, the adjoint equation

$$\begin{cases} (\mathrm{d}/\mathrm{d}t)\psi(t) + (A(t)B^{-1})^*\psi + (F_1^0)^*(t)\psi = -C^* l_1^0, \\ \psi(T) = 0 \end{cases} \tag{4.7}$$

has a unique weak solution $\psi \in C(I, Y_w^*)$.

Since the solution $y(t)$ need not belong to $B(D(A))$, following the same technique as in [13], we use the Yosida approximation of the identity, $J_n(t) = nR(n, A(t)B^{-1})$ where $R(\lambda, A(t)B^{-1})$ is the resolvant of the operator AB^{-1} corresponding to $\lambda \in \rho(AB^{-1})$. It is well known (see [1]) that $J_n(t) \to J$ (identity operator in Y) as $n \to \infty$ in the strong operator topology of $\mathcal{L}(Y)$ uniformly with respect to $t \in I$ and for any $z \in Y$, $J_n z \in D(AB^{-1}) = B(D(A))$ for $n \in \rho(AB^{-1})$.

Now we regularize Eq. (4.5) as follows:

$$\begin{cases} (\mathrm{d}/\mathrm{d}t)(y_n) = A(t)B^{-1}y_n + J_n(t)F_1^0(t)y_n + J_n(t)F_2^0(t)(u - u^0), \\ y_n(0) = 0. \end{cases} \tag{4.8}$$

Equation (4.8) has a unique strong solution y_n with $y_n(t) \in B(D(A))$ for almost all $t \in I$ provided $n \in \rho(AB^{-1})$. Since a strong solution is obviously a mild solution, the y_n satisfies the following integral equation:

$$y_n(t) = \int_0^t E(t,s) J_n F_1^0(s) y_n(s)\,\mathrm{d}s + \int_0^t E(t,s) J_n F_2^0(s)\big(u(s) - u^0(s)\big)\,\mathrm{d}s. \tag{4.9}$$

Using Gronwall inequality it is easy to verify that $y_n \to y$ in the usual topology of $C(I, Y)$. Hence it follows from Eq. (4.7) that

$$\begin{aligned}
&\int_I \langle C^* l_1^0, z\rangle_{Y^*,Y}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_I \langle C^* l_1^0, z_n\rangle_{Y^*,Y}\,\mathrm{d}t \\
&= \lim_{n\to\infty} \int_I \left\langle -\frac{\mathrm{d}}{\mathrm{d}t}(\psi) - ((AB^{-1})^* + (F_1^0)^*)\psi, z_n\right\rangle_{Y^*,Y} \mathrm{d}t \\
&= \lim_{n\to\infty} \int_I \left\langle \psi, \frac{\mathrm{d}}{\mathrm{d}t}(z_n) - AB^{-1} z_n - (J_n F_1^0) z_n - F_1^0 z_n + J_n F_1^0 z_n \right\rangle_{Y^*,Y} \mathrm{d}t \\
&= \lim_{n\to\infty} \int_I \langle \psi, J_n F_2^0(u - u^0) - F_1^0 z_n + J_n F_1^0 z_n\rangle_{Y^*,Y}\,\mathrm{d}t \\
&= \int_I \langle \psi, F_2^0(u - u^0)\rangle_{Y^*,Y}\,\mathrm{d}t.
\end{aligned} \tag{4.10}$$

Here we have used the strong convergence of J_n to J and uniform convergence of z_n to z and the following estimate:

$$\left\|J_n F_1^0 z_n - F_1^0 z_n\right\|_Y \le \left\|J_n F_1^0 (z_n - z)\right\|_Y + \left\|J_n(F_1^0 z) - F_1^0 z\right\|_Y + \left\|F_1^0(z - z_n)\right\|_Y.$$

Combining (4.6) and (4.10), we have

$$\int_I \langle (F_2^0)^* \psi + l_2^0, u - u^0\rangle_{U^*,U}\,\mathrm{d}t \ge 0. \tag{4.11}$$

This proves inequality (3) and completes the proof of Theorem 4.1.

Case B: l Merely Continuous in Control Variable

In the above result we assumed that l is Frechet differentiable in the control variable. In case $l(x,u)$ is merely continuous in u and Frechet differentiable in $x=Cz$ and $\Lambda \subset U$ is a closed bounded convex set, we can prove Pontryagin type necessary conditions of optimality using well known Ekland's variational principle. Define

$$\mathcal{M} \equiv \{u : I \to U, \text{ strongly measurable: } u(t) \in \Lambda, \text{ a.e.}\}$$

with the topology induced by the metric

$$\rho(u,v) \equiv \lambda\{t \in I: u(t) \neq v(t)\},$$

where λ denotes the Lebesgue measure. Since Λ is a closed subset of a Banach space, the set $\mathcal{M}$, with the metric ρ as defined above, is a complete metric space.

We need the continuous dependence of solutions on control.

LEMMA 4.1 *Suppose the assumptions* (A1) *and* (A2) *hold and* $\mathcal{U}_{\mathrm{ad}}=\mathcal{M}$. *Then for the semilinear system* (3.2) *the mapping*

$$u \to z(u)$$

is continuous from $\mathcal{M}$ to $C\ (I,Y)$ in the respective metric topologies and further there exists a constant β such that

$$\|z(u)-z(v)\|_{C(I,Y)} \leq \beta\rho(u,v)$$

for all $u, v \in \mathcal{M}$.

Proof Let $z(u)$ and $z(v)$ denote the solutions corresponding to u and v respectively. Let $\sigma \equiv \{t \in I: u(t) \neq v(t)\}$. We have

$$\begin{aligned}
z(t,u)-z(t,v) &= \int_0^t E(t,s)[f(z(s,u),u(s)) - f(z(s,v),v(s))]\,\mathrm{d}s \\
&= \int_0^t E(t,s)\int_0^1 f_1(z(s,v)+\theta(z(s,u)-z(s,v)),u(s))\,\mathrm{d}\theta \\
&\quad \times (z(t,u)-z(t,v))\,\mathrm{d}s \\
&\quad + \int_0^t E(t,s)[f(z(s,v),u(s)) - f(z(s,v),v(s))]\,\mathrm{d}s.
\end{aligned}$$

It follows from our assumptions that there exist constants a, b such that

$$\|z(t,u) - z(t,v)\|_Y \leq a \int_0^t \|z(s,u) - z(s,v)\|_Y \, \mathrm{d}s + b\rho(u,v).$$

Thus the assertion follows from Gronwall inequality.

THEOREM 4.2 *Suppose the assumptions of the Lemma* 4.1 *hold and further* $u \to l(Cz, u)$ *is merely continuous and* $z \to l(Cz, u)$ *is continuously Frechet differentiable with Frechet derivative denoted by* l_1. *Further* $C^* l_1 \in L^1(I; Y^*)$. *Then the optimality conditions* (1)–(3) *of Theorem* 4.1 *hold and* (3) *is replaced by* (3)′:

$$\begin{aligned} (3)': \qquad & l(Cz^0(t), u^0(t)) + \langle \psi(t), f(z^0(t), u^0(t)) \rangle_{Y^*, Y} \\ & \leq l(Cz^0(t), v) + \langle \psi(t), f(z^0(t), v) \rangle_{Y^*, Y} \end{aligned}$$

for all $v \in \Lambda$.

Proof Since u^0 is optimal, again by the inequality (4.1), we have

$$\int_I l(Cz(t,u), u) \, \mathrm{d}t - \int_I l(Cz^0, u^0) \, \mathrm{d}t \geq 0, \quad \forall u \in \mathcal{M}.$$

For any measurable set $\sigma \in I$ and $v \in \Lambda$, define

$$u^\sigma(t) \quad \begin{cases} u^0(t), & t \in I \backslash \sigma, \\ v(t), & t \in \sigma. \end{cases}$$

Let z^σ be the solution of the system (3.2) corresponding to u^σ. Then

$$\begin{aligned} & \int_I l(Cz^\sigma, u^\sigma) \, \mathrm{d}t - \int_I l(Cz^0, u^0) \, \mathrm{d}t \\ & = \int_\sigma l(Cz^\sigma, v) \, \mathrm{d}t - \int_\sigma l(Cz^0, u^0) \, \mathrm{d}t + \int_{I \backslash \sigma} [l(Cz^\sigma, u^0) - l(Cz^0, u^0)] \, \mathrm{d}t \geq 0. \end{aligned} \tag{4.12}$$

By virtue of Frechet differentiability of l, we have

$$
\begin{aligned}
\int_{I\setminus\sigma} & \left[l(Cz^{\sigma},u^{0}) - l(Cz^{0},u^{0})\right] \mathrm{d}t \\
&= \int_{I\setminus\sigma} \langle C^{*}l_{1}(Cz^{0},u^{0}), z^{\sigma} - z^{0}\rangle_{Y^{*},Y} \mathrm{d}t + \mathrm{o}(\lambda(\sigma)) \\
&= \int_{I} \langle C^{*}l_{1}(Cz^{0},u^{0}), z^{\sigma} - z^{0}\rangle_{Y^{*},Y} \mathrm{d}t + \mathrm{o}(\lambda(\sigma)), \qquad (4.13)
\end{aligned}
$$

where $\mathrm{o}(\cdot)$ stands for small order of approximation.

Hence expression (4.12) reduces to

$$
\begin{aligned}
\int_{\sigma} l(Cz^{0},u^{0}) \mathrm{d}t \leq \int_{\sigma} l(Cz^{\sigma},v) + \int_{I} \langle C^{*}l_{1}(Cz^{0},u^{0}), z^{\sigma} - z^{0}\rangle_{Y^{*},Y} \mathrm{d}t \\
+ \mathrm{o}(\lambda(\sigma)). \qquad (4.14)
\end{aligned}
$$

Using the adjoint Eq. (2) of Theorem 4.1 and following similar arguments as in that theorem, one can verify that

$$
\begin{aligned}
\int_{I} & \langle C^{*}l_{1}(Cz^{0},u^{0}), z^{\sigma} - z^{0}\rangle_{Y^{*},Y} \mathrm{d}t \\
&= \int_{I} \langle \psi, f(z^{\sigma},u^{\sigma}) - f(z^{0},u^{0}) - F_{1}^{0}(t)(z^{\sigma} - z^{0})\rangle_{Y^{*},Y} \mathrm{d}t \\
&= \int_{I} \langle \psi, f(z^{0}, u^{\sigma}) - f(z^{0},u^{0})\rangle_{Y^{*},Y} \mathrm{d}t + \mathrm{o}(\lambda(\sigma)) \\
&= \int_{\sigma} \langle \psi, f(z^{0}, v)\rangle_{Y^{*},Y} \mathrm{d}t - \int_{\sigma} \langle \psi, f(z^{0},u^{0})\rangle_{Y^{*},Y} \mathrm{d}t + \mathrm{o}(\lambda(\sigma)).
\end{aligned}
$$

Thus the expression (4.14) reduces to

$$
\begin{aligned}
\int_{\sigma} & l(Cz^{0},u^{0}) \mathrm{d}t + \int_{\sigma} \langle \psi, f(z^{0},u^{0})\rangle_{Y^{*},Y} \mathrm{d}t \\
&\leq \int_{\sigma} l(Cz^{\sigma},v) \mathrm{d}t + \int_{\sigma} \langle \psi, f(z^{0},v)\rangle_{Y^{*},Y} \mathrm{d}t + \mathrm{o}(\lambda(\sigma)). \qquad (4.15)
\end{aligned}
$$

Let t be any Lebesgue density point of u^0 and σ any measurable set containing t shrinking to the one point set $\{t\}$ as $\lambda(\sigma) \to 0$. Dividing (4.15) by $\lambda(\sigma)$ and letting it converge to zero, we obtain the inequality $(3)'$. This completes the proof.

5 AN EXAMPLE

In this section, we work out in detail an example (heat transfer) of boundary control problem illustrating the applicability of our results.

Example 1 (Heat transfer) In this example we consider the heat transfer problem as stated in the Motivation section. In order to formulate and treat problem (1.1) as two-space evolution equation, we introduce the following notations: $H^m(\Omega)$ denotes the standard Sobolev space and $\gamma_i\phi \equiv \phi|_{\Gamma i}$ $(i = 1, 2)$ denotes the trace operator. Let L, M denote the formal differential operators

$$L\phi = \operatorname{div}(k\nabla\phi) + v \cdot \nabla\phi,$$
$$M\phi = \triangle_{\Gamma_1}\phi - \beta D_\nu\phi.$$

We take $X \equiv L^2(\Omega)$ and $Y \equiv L^2(\Omega) \times L^2(\Gamma_1)$, with the norm topology on Y given by

$$\|y\|_Y = \left(\|y^1\|^2_{L^2(\Omega)} + \|y^2\|^2_{L^2(\Gamma_i)}\right)^{1/2}, \quad \text{for } y = \{y^1, y^2\} \in Y.$$

The operators A and B are defined as follows:

$$D(A) \equiv \{\phi \in H^2(\Omega): L\phi \in L^2(\Omega), M\phi \in L^2(\Gamma_1) \text{ and } \gamma_2\phi = h\},$$
$$A\phi \equiv \{L\phi, M\phi\}, \quad \phi \in D(A)$$

and

$$D(B) \equiv \{\phi \in H^2(\Omega): \gamma_2\phi = h\},$$
$$B\phi \equiv (\phi, \gamma_1\phi), \quad \phi \in D(B).$$

The range of the operator B is given by

$$R(B) \equiv \{(\psi_1, \psi_2) \in Y: \gamma_1\psi_1 = \psi_2, \text{ a.e.}\}.$$

Defining $x(t) \equiv T(t, \cdot)$, the control $u \in L^2(I; L^2(\Gamma_1))$, $f(y, u(t)) \equiv \{0, \alpha(u - y^2)\}$ and $z_0 \equiv \{T_0, T_1\}$, the heat transfer Eq. (1.1) can be written as an abstract B-evolution in two Banach spaces $\{X, Y\}$ as follows:

$$(Q_f) \quad \begin{cases} (\mathrm{d}/\mathrm{d}t)(Bx)(t) = A(t)\,x(t) + f(Bx(t), u(t)), \\ Bx|_{t=0} = z_0. \end{cases}$$

The integrand $l(v, w) \equiv \int_\Omega |v(\xi)|^2 \mathrm{d}\xi + \int_{\Gamma_1} |w(\eta)|^2 \mathrm{d}\eta$ maps $L^2(\Omega) \times L^2(\Gamma_1)$ to R, for $v \in L^2(\Omega)$, $w \in L^2(\Gamma_1)$ and the cost functional is given by

$$J(u) \equiv \int_I l(x(t), u(t))\,\mathrm{d}t.$$

Following standard procedure as in [9] one can verify that the operators A and B satisfy the following properties:

(i) $R(B)$, range of B, is dense in Y,
(ii) A is closed,
(iii) B is injective and has a bounded inverse on $R(B) \subset Y$.

Thus $AB^{-1} : D(AB^{-1}) \equiv B(D(A)) \equiv R(B) \subset Y \mapsto Y$ is a closed densely defined linear operator.

Following similar procedure as in [9] one can show that for each $\psi \in R(B)$

$$\|(\lambda B - A)B^{-1}\psi\|_Y \geq (\lambda - \omega)\|\psi\|_Y, \quad \text{for } \lambda \in R,\ \lambda > 0,$$
$$\|(\lambda B - A)B^{-1}\psi\|_Y \geq |\lambda|\|\psi\|_Y, \quad \text{for } \lambda \in C.$$

The quantity $\omega(>0)$ is dependent on the L^∞ bound of v and the material constant K and in fact is given by $\omega = (\|v\|_{L^\infty}/K)$. From the above estimates it follows that there exists a constant $M \geq 1$ and $0 < \delta < (\pi/2)$ such that for λ in the sectoriel domain

$$\Sigma_{\omega,\delta} = \{\lambda \in C, \mathrm{Re}\,\lambda > \omega, -(\pi/2 + \delta) < \arg\lambda < (\pi/2 + \delta)\},$$

the operator $(\lambda B - A)B^{-1}$ has a bounded inverse satisfying

$$\|B(\lambda B - A)^{-1}\|_{\mathcal{L}}(Y) \leq M/(\mathrm{Re}\,\lambda - \omega),\ \mathrm{Re}\,\lambda > \omega.$$

Hence AB^{-1} generates a holomorphic semigroup (see [8,12], Theorem 3.2.7, p. 82]). Thus by Lemma 2.4, the pair $\langle A, B\rangle$ is the generating pair of a holomorphic B-evolution of type L.

For the set of controls we define

$$\Lambda \equiv \{\omega \in U \equiv L^2(\Gamma_1)\colon \omega = T_r|_{\Gamma_1} \text{ and } 0 \leq \omega(\xi) \leq \gamma_d\}$$

and for admissible controls take $\mathcal{U}_{\mathrm{ad}} \equiv \{w \in L^\infty(I, U)\colon w(t) \in \Lambda, \text{ a.e.}\}$.

Clearly Λ is a closed bounded convex subset of U. We note that f as defined above is locally Lipschitz, satisfies the growth conditions and maps $Y \times U$ to Y. One can also verify that the integrand l satisfies the hypothesis of Theorem 4.1. Thus all the assumptions of the main theorem are satisfied, and hence the necessary conditions of optimality holds.

6 COMPUTATIONAL ALGORITHM

Based on the necessary conditions of optimality given in Theorem 4.1 (Case A), we can compute the optimal solution of our original problem:

(Q) Find $u^0 \in \mathcal{U}_{\mathrm{ad}}$ such that $J(x^0, u^0) \leq J(x, u)$ for all $u \in \mathcal{U}_{\mathrm{ad}}$, subject to the following system:

$$(P_f) \quad \begin{cases} (\mathrm{d}/\mathrm{d}t)Bx(t) = A(t)x(t) + f(Bx(t), u(t)), & t \in I, \\ (Bx)(0) = z_0, \end{cases}$$

where $J(x, u) \equiv \int_I l(x, u)\, \mathrm{d}t$.

Here $x = Cz$ denotes the generalized solution of the system (P_f) and z the solution of the system (3.2) corresponding to the control $u \in \mathcal{U}_{\mathrm{ad}}$.

The computation of the optimal solution of the problem (Q) can be done by constructing an algorithm for computing the optimal solution of the problem (P) (see Section 4). For this purpose, we require the duality maps.

The map $\nu\colon U^* \mapsto U$ denotes the duality map, that is, for $\xi \in U^*$

$$\nu_1(\xi) \equiv \{\eta \in U\colon \langle \xi, \eta \rangle_{U^*, U} = \|\xi\|_{U^*}^2 = \|\eta\|_U^2\}.$$

For Frechet differentiable l, we can define $D_u J(\cdot) \equiv l_2^0 + F_2^0 \psi$. The inequality (3) of Theorem 4.1 is then equivalent to the following inequality:

$$\langle D_u J, u - u^0 \rangle = \int_I \langle l_2^0 + F_2^0 \psi, u - u^0 \rangle_{U^*,U} \, dt \geq 0$$

for all $u \in \mathcal{U}_{ad}$.

Now the algorithm may be stated as follows:

ALGORITHM

Step 1. Suppose that the nth stage, the control is given by $u^n \in U$.

Step 2. Use u^n to determine $\{z^n, \psi^n\}$ where z^n is the solution of Eq. (3.2) corresponding to u^n and ψ^n is the solution of the adjoint Eq. (2) of Theorem 4.1.

Step 3. Compute $D_u J(u^n)$.

Step 4. Define

$$u^{n+1} \equiv u^n - \epsilon \tilde{u}, \quad \epsilon > 0,$$

choosing ϵ sufficiently small so that

$$\begin{aligned} J(u^{n+1}, z^{n+1}) &\equiv J(u^n, z^n) - \epsilon \langle D_u J(u^n, z^n), \tilde{u}^n \rangle + 0(\epsilon) \\ &= J(u^n, z^n) - \epsilon \| D_u J(u^n, z^n) \|_{U^*}^2 + 0(\epsilon) \leq J(u^n, z^n). \end{aligned}$$

Step 5. Solve the state equation corresponding to u^n and compute $J(u^{n+1}, z^{n+1})$ using the following expression:

$$J(u^{n+1}, z^{n+1}) = \int_I l(Cz^{n+1}(t), u^{n+1}(t)) \, dt.$$

If $|J(u^{n+1}, z^{n+1}) - J(u^n, z^n)| < \delta$ for some preassigned small positive number δ, stop, otherwise go back to Step 2 with new control u^{n+1}.

7 CONCLUSIONS

The obtained results represent an important preliminary step for boundary control of distributed systems involving dynamic boundary conditions. In this work B-evolution concept was used where an important special case of holomorphic B-evolutions has been considered allowing the pair of operators $\{A, B\}$ to be closed.

Assuming the operator B to have an inverse on its range we converted the original problem to a classical one involving the spatial operator AB^{-1} which is not closed, since the operators A and B are not necessarily closed.

With the help of the closed pair of operators, necessary conditions for optimality have been derived for semilinear problems. Also an algorithm for computing optimal controls has been presented.

We note also that control problem for systems involving dynamic boundary has not yet been studied by means of B-evolutions theory or by classical theory. Our results cover a general class of boundary control problems. In fact if the operator B is the identity the problem is reduced to a standard class of control problems involving non-dynamic boundary conditions.

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