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Controller Design for Bilinear Systems with Parametric Uncertainties*

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This paper studies the problem of robust control of a class of uncertain bilinear continuous-time systems. The class of uncertain systems is described by a state space model with time-varying norm-bounded parameter uncertainty in the state equation. We address the problem of robust H_∞ control in which both robust stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainties. Both state feedback and output feedback controllers are designed. It has been shown that the above problems can be recast into H_∞ syntheses for related bilinear systems without parameter uncertainty, which can be solved via a Riccati inequality approach. Two examples are given to show the potential of the proposed technique.

Keywords: Bilinear systems; H_∞ control; Parameter uncertainty; Riccati inequality

1 INTRODUCTION

One of the most important requirements for a control system is the so-called *robustness*. In the past decades, the design of control systems that can handle model uncertainties has been one of the most

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challenging problems and received considerable attention from control engineers and scientists. A number of approaches to the problem of robust control design for uncertain dynamical systems have been proposed, for example, robust stabilization, sensitive minimization and H_∞ control, see [3,5,15,36] and references therein.

Since the pioneering work on the so-called H_∞ optimal control theory [36], there has been a dramatic progress in H_∞ control theory in the past few years. Both the cases of continuous-time and discrete-time systems have been intensively studied. The essential idea of H_∞ control is to design a controller to optimize the closed-loop system performance for the worst exogenous input. The goal in H_∞ control is to design a controller such that the H_∞ norm of the transfer function from the disturbance input to the controlled output is minimized. Many familiar robust control problems can be recast as an H_∞ control problem. It was shown in [14,19] that the state feedback H_∞ control of linear systems can be solved in terms of an algebraic Riccati equation. In the seminal paper [6], the state–space solution to the output feedback H_∞ control problem was developed. Similar to the linear quadratic Gaussian control problem, the output feedback H_∞ control can be solved in terms of two Riccati equations. Recently, the problem of nonlinear H_∞ control has been intensively investigated, see, e.g., [2,12,13,16,17,22,23]. A solution is presented in [12] to the problem of disturbance attenuation via a measurement feedback with internal stability for an affine nonlinear system, which is related to the existence of solutions of a pair of Hamilton–Jacobi inequalities [13]. Also, [17] has solved the H_∞ control problems for a class of nonlinear systems via a convex optimization technique. Moreover, it has been proved that the existence of a continuous, local viscosity supersolution of the Hamilton–Jacobi–Isaacs equation corresponding to the H_∞ control problem is sufficient for its solvability [31]. However, to the best of authors' knowledge, the design of robust controllers for uncertain bilinear continuous-time systems has not been fully investigated.

Bilinear systems comprise perhaps the simplest class of nonlinear systems which have a lot of practical applications in various fields (see, e.g., [18,32] and reference therein). The problem of dynamics of heat exchanger with controlled flow is studied in [4] while [7] tackled the problem of reduced order bilinear models for distillation columns. The application of stabilization of bilinear control systems to nonconservative problems in elasticity is considered in [30]. Communicative bilinear systems are investigated in the work of [33]. Recently, the issue of control of hydraulic multi-motor systems based on bilinearization is discussed in [11]. Note that all the above practical examples can be modelled as nearly linear systems but containing the interconnections between the states $x(t)$ and control inputs $u(t)$, which is critical to the characterization of the underlying systems and can not be neglected. Actually, this interconnection plays an important role to describe the real phenomenon in some situations. It should also be noted that, in practice, it is almost always impossible to get an exact mathematical model of a dynamical system due to the complexity of the system, the difficulty of measuring various parameters, environmental noises, uncertain and/or time-varying parameters, etc. Indeed, the model of the system to be dealt with almost always contains some type of uncertainty. There are two main categories of uncertainty. The first one arises from neglected high-frequency dynamics, such as actuator and sensor dynamics, or structural modes. The second category of uncertainty results from unknown and/or time-varying real parameters of the system and can be thought of as low-frequency modelling errors. The motivation to consider norm-bounded time-varying parameter uncertainty in control systems of this paper stems from two facts: (i) the assumption of boundedness of the uncertainty in real systems is reasonable and most of uncertainties satisfy this condition; (ii) the norm-bounded uncertainty has been widely used by scientists and engineers, see, for example, the references in Remark 2.1.

In this paper, we extend the control design methodology proposed in [28] to handle the problem of robust control of a class of uncertain bilinear continuous-time systems. The class of uncertain systems is described by a state space model with norm-bounded parameter uncertainty in the state equation. We address the problem of robust H_∞ control in which both robust stability and a prescribed H_∞ performance are required to be achieved irrespective of the uncertainties. Both state feedback and output feedback control problems will be investigated. Our results show that the above problems can be recast into H_∞ syntheses for related bilinear continuous-time systems without parameter uncertainty. Therefore, the control results on bilinear systems (see, e.g., [28]) can be used to obtain a solution to the problem of robust H_∞ control of uncertain bilinear continuous-time systems.

NOTATION *Throughout this paper the superscript* "T" *denotes matrix transposition and the notation* $X \geq Y$ (*respectively*, $X > Y$) *where* X *and* Y *are symmetric matrices, means that* $X - Y$ *is positive semi-definite* (*respectively, positive definite*). I *is the identity matrix with suitable dimension.* $L_2[0,\infty)$ *stands for the space of square integrable vector functions over the interval* $[0,\infty)$. $\|\cdot\|$ *will refer to the Euclidean vector norm whereas* $\|\cdot\|_{[0,\infty)}$ *denotes the* $L_2[0,\infty)$*-norm over* $[0,\infty)$.

2 PROBLEM FORMULATION AND PRELIMINARIES

In this section, first, we formulate the problem we shall investigate, and second, we design state feedback and output feedback controllers, respectively, for the nominal systems.

2.1 Systems Description

Consider the following time-varying uncertain bilinear continuous-time system:

$$(\Sigma):\quad \dot{x}(t) = [A + \Delta A(t)]x(t) + B_1 w_1(t) + [B_2 + Bx(t)]u(t), \quad \forall t \in [0,\infty), \; x(0) = 0, \tag{2.1}$$

$$z(t) = C_1 x(t) + D_{12}u(t), \quad \forall t \in [0,\infty), \tag{2.2}$$

$$y(t) = C_2 x(t) + D_{21}w_2(t), \quad \forall t \in [0,\infty), \tag{2.3}$$

where $x \in \Re^n$ is the system state, $u \in \Re^\ell$ is the control input, w_1 is the disturbance input, $w_2 \in \Re^q$ is the measurement noise, w_1 and w_2 belong to $L_2[0,\infty)$, $y \in \Re^m$ is the output measurement, $z \in \Re^r$ is controlled output, A, B, B_1, B_2, C_1, C_2, D_{12} and D_{21} are known real constant matrices of appropriate dimensions that describe the nominal system, $\Delta A(t)$ is a real time-varying matrix representing norm-bounded parameter uncertainty.

It should be pointed out that if the matrix B in (2.1)–(2.3) has full column rank which can ensure $Bx \neq 0$ for any $x \neq 0$, system (2.1)–(2.3) will be strictly bilinear which is our interest in this paper. If $Bx = 0$, system (2.1)–(2.3) will reduce to linear system which has been studied intensively in the literature, for example, [6].

The admissible parameter uncertainty is of the form

$$\Delta A(t) = HF(t)E \tag{2.4}$$

where $E \in \Re^{j \times n}$ and $H \in \Re^{n \times i}$ are known real constant matrices, and $F(t) \in \Re^{i \times j}$ is an unknown time-varying matrix satisfying

$$\|F(t)\| \leq 1, \quad \forall t \geq 0 \tag{2.5}$$

with the elements of $F(t)$ being Lebesgue measurable.

Remark 2.1 The matrix $F(t)$ contains the uncertain parameter in the state matrix of the system (Σ) and it is allowed to be state dependent, as long as (2.5) is satisfied along all possible state trajectories. The matrices E and H specify how the uncertain parameters in F affect the nominal matrices of the system (Σ). Observe that the unit overbound for $F(t)$ does not cause any loss of generality. Indeed, $F(t)$ can always be normalized, in the sense of (2.5), by appropriately choosing the matrices E and H. Note that the parameter uncertainty structure as in (2.4)–(2.5) has been widely used in the problems of robust control and filtering of uncertain systems (see, e.g., [8,15,24–27,34] and the references therein) and many practical systems possess parameter uncertainties which can be either exactly modelled, or overbounded by (2.4)–(2.5).

In this paper, we will investigate the design of a feedback controller $(\mathcal{G})$ for (2.1)–(2.3) that reduces z uniformly for any $w = [w_1^{\mathrm{T}}, w_2^{\mathrm{T}}]^{\mathrm{T}}$ in the sense that given a scalar $\gamma > 0$, the closed-loop system of (2.1)–(2.3) with the controller $(\mathcal{G})$ satisfies

$$\|z\|^2_{[0,\infty)} < \gamma^2 \|w\|^2_{[0,\infty)} \tag{2.6}$$

for any nonzero $w \in L_2[0,\infty)$ and for all admissible uncertainties. In this situation, the closed-loop system of (2.1)–(2.3) with $(\mathcal{G})$ is said to have a *robust* H_∞ *performance* γ *over the horizon* $[0,\infty)$.

The robust H_∞ control problem we address in this paper is as follows: *Given a scalar* $\gamma > 0$, *design a controller* $\mathcal{G}$ *based on the system states*, $x(t)$, *if they are available*, *or output measurements*, $y(t)$, *such that*: the closed-loop system of (2.1)–(2.3) with $\mathcal{G}$ is robustly stable and has a robust H_∞ performance γ over $[0,\infty)$.

In the above, '*robustly stable*' means that the closed-loop system is globally uniformly asymptotically stable about the origin for all admissible parameter uncertainties.

We shall make the following assumption for the system of (Σ):

ASSUMPTION 2.1 (a) $D_{12}^{\mathrm{T}}C_1 = 0$, (b) $D_{12}^{\mathrm{T}}D_{12} = I$, (c) $B_1 D_{21}^{\mathrm{T}} = 0$, (d) $D_{21}D_{21}^{\mathrm{T}} = I$.

Remark 2.2 Note that Assumptions 2.1(a)–(c) cause no loss in generality, and are adopted only for the sake of technical simplification. They are also standard assumptions in H_∞ and LQG control and can be easily removed, see, e.g. [1,9,37]. Assumption 2.1(d) means that the robust control problem is "nonsingular". Also, it was implicitly assumed in system (Σ) that the controlled output is disturbance free.

ASSUMPTION 2.2 (i) (A, B_1) *is stabilizable and* (C_1, A) *is detectable.* (ii) (A, B_2) *is stabilizable and* (C_2, A) *is detectable.*

Remark 2.3 Note that, as in Assumption 2.1, Assumption 2.2(i) is made for technical reason, which simplifies the theorem statements and proofs in next section. It should be noted that Assumption 2.2(ii) is sufficient for the existence of a stabilizing controller for the nominal system of (Σ).

2.2 H_∞ Controllers Design

In this sub-section, we will present solutions to the problems of H_∞ control for the nominal bilinear systems of (2.1)–(2.3). We shall formulate the results in state feedback and output feedback cases, respectively.

Firstly, we assume the perfect state $x(t)$ is available for feedback. Let us consider the following Hamilton–Jacobi–Bellman (HJB) inequality

$$\begin{aligned} H(x, w, u) = & V_x(x)[Ax + B_1 w + (B_2 + Bx)u] \\ & + \tfrac{1}{2}x^{\mathrm{T}}C_1^{\mathrm{T}}C_1 x + \tfrac{1}{2}u^{\mathrm{T}}u - \tfrac{1}{2}\gamma^2 w_1^{\mathrm{T}} w_1 \leq 0 \end{aligned} \tag{2.7}$$

where $V_x(x) = \partial V(x)/\partial x$, $V(x) = \frac{1}{2}x^{\mathrm{T}}Px$ and P is a positive definite matrix to be chosen.

From (2.7), it is easy to show that the suboptimal control law which minimizes $H(x, w, u)$ is given by

$$u = -(B_2 + Bx)^{\mathrm{T}} Px. \tag{2.8}$$

Similarly, the worst case $\sup_{w_1 \in L_2[0,\infty)} H(x, w, u)$ occurs when

$$w_1 = \gamma^{-2} B_1^{\mathrm{T}} Px. \tag{2.9}$$

Substituting (2.8) and (2.9) into (2.7), one has

$$\begin{aligned} H(x,w,u) = x^{\mathrm{T}} PAx + \tfrac{1}{2} x^{\mathrm{T}} C_1^{\mathrm{T}} C_1 x - \tfrac{1}{2} x^{\mathrm{T}} P(B_2 + Bx)(B_2 + Bx)^{\mathrm{T}} Px \\ + \gamma^{-2} x^{\mathrm{T}} B_1 B_1^{\mathrm{T}} Px \leq 0. \end{aligned} \tag{2.10}$$

It has been shown in [28] that the solution of (2.10), P, satisfies the following Hamiltonian matrix:

$$H_\infty = \begin{bmatrix} A & -B_2 B_2^{\mathrm{T}} + \gamma^{-2} B_1 B_1^{\mathrm{T}} \\ -C_1^{\mathrm{T}} C_1 & -A^{\mathrm{T}} \end{bmatrix}. \tag{2.11}$$

However, in order to guarantee P satisfies (2.10) as well, we also need (see [28])

$$C_1^{\mathrm{T}} C_1 - PB_2 B_2^{\mathrm{T}} P \geq 0, \quad \text{if } BxB_2^{\mathrm{T}} \neq 0. \tag{2.12}$$

The Hamiltonian matrix (2.11) is associated with the following algebraic Riccati inequality:

$$A^{\mathrm{T}} P + PA - P(B_2 B_2^{\mathrm{T}} - \gamma^{-2} B_1 B_1^{\mathrm{T}})P + C_1^{\mathrm{T}} C_1 < 0. \tag{2.13}$$

Summarizing the above, we have the following result on H_∞ state feedback control of system (2.1)–(2.2):

THEOREM 2.1 *Consider the system* (2.1)–(2.2) (*with* $\Delta A(t) \equiv 0$) *satisfying Assumptions* 2.1 *and* 2.2. *Then, given a* $\gamma > 0$, *there exists a state feedback controller such that the closed-loop system is stable and*

$$\|z\|^2_{[0,\infty)} < \gamma^2 \|w\|^2_{[0,\infty)}$$

for any nonzero $w \in L_2[0,\infty)$, *if and only if there exists a matrix* $P = P^{\mathrm{T}} > 0$ *satisfying* (2.13) *and* (2.12). *Moreover, a suitable state feedback controller is given by*

$$u(t) = -(B_2 + Bx(t))^{\mathrm{T}} Px(t). \tag{2.14}$$

Next, we consider the case when only the output measurement $y(t)$ is available. In order to provide a state estimate, the following observer designed based on system (2.1)–(2.3) is adopted:

$$\frac{\mathrm{d}}{\mathrm{d}t}\hat{x}(t) = A\hat{x}(t) + [B_2 + B\hat{x}(t)]u(t) + K[y(t) - C_2\hat{x}(t)], \quad \hat{x}(0) = 0 \tag{2.15}$$

where the filter gain K is a constant matrix to be designed. Based on the state estimate $\hat{x}(t)$ in (2.15), we are looking for the output feedback

$$u = u(\hat{x}), \quad w_1 = w_1(\hat{x}), \quad w_2 = w_2(\hat{x}) \tag{2.16}$$

such that the following HJB inequality is satisfied:

$$\begin{aligned} H(x,\hat{x},w_1,w_2,u) = & \frac{\partial V(x,\hat{x})}{\partial x}\dot{x} + \frac{\partial V(x,\hat{x})}{\partial \hat{x}}\dot{\hat{x}} + \frac{1}{2}x^{\mathrm{T}}C_1^{\mathrm{T}}C_1 x \\ & + \frac{1}{2}u^{\mathrm{T}}(\hat{x})u(\hat{x}) - \frac{1}{2}\gamma^{-2}w_1^{\mathrm{T}}(\hat{x})w_1(\hat{x}) \\ & - \frac{1}{2}\gamma^{-2}w_2^{\mathrm{T}}(\hat{x})w_2(\hat{x}) \le 0 \end{aligned} \tag{2.17}$$

where $V(x,\hat{x})$ is a Lyapunov function candidate for the closed-loop system of (2.1)–(2.3) with (2.16).

Similar to Theorem 2.1, the worst case $\sup_{w_1,w_2 \in L_2[0,\infty)} H(x,\hat{x},w_1,w_2,u)$ occurs when

$$w_1 = \gamma^{-2}B_1^{\mathrm{T}}\left(\frac{\partial V(x,\hat{x})}{\partial x}\right)^{\mathrm{T}}, \quad w_2 = \gamma^{-2}K^{\mathrm{T}}\left(\frac{\partial V(x,\hat{x})}{\partial \hat{x}}\right)^{\mathrm{T}}. \tag{2.18}$$

Taking (2.1)–(2.3) into account, together with (2.15) and (2.18), one has from (2.17)

$$\begin{aligned} H(x,\hat{x},w_1,w_2,u) =& \frac{\partial V(x,\hat{x})}{\partial x}[Ax+(B_2+Bx)u(\hat{x})] \\ &+\frac{\partial V(x,\hat{x})}{\partial \hat{x}}[A\hat{x}+(B_2+B\hat{x})u(\hat{x})+K(C_2x-C_2\hat{x})] \\ &+\frac{1}{2}x^{\mathrm{T}}C_1^{\mathrm{T}}C_1x+\frac{1}{2}u^{\mathrm{T}}(\hat{x})u(\hat{x}) \\ &+\frac{1}{2}\gamma^{-2}\frac{\partial V(x,\hat{x})}{\partial x}B_1B_1^{\mathrm{T}}\left(\frac{\partial V(x,\hat{x})}{\partial x}\right)^{\mathrm{T}} \\ &+\frac{1}{2}\gamma^{-2}\frac{\partial V(x,\hat{x})}{\partial \hat{x}}KK^{\mathrm{T}}\left(\frac{\partial V(x,\hat{x})}{\partial \hat{x}}\right)^{\mathrm{T}} \leq 0. \end{aligned} \tag{2.19}$$

Using the same argument as in [28], one can assume that $V(x,\hat{x})$ has the following form:

$$V(x,\hat{x}) = \tfrac{1}{2}x^{\mathrm{T}}Px+\tfrac{1}{2}\gamma^{-2}(x-\hat{x})^{\mathrm{T}}Z^{-1}(x-\hat{x}) \tag{2.20}$$

where

$$Z = Q(I-\gamma^{-2}PQ)^{-1}$$

and P and Q are positive definite matrices to be chosen such that $PQ<\gamma^2 I$. Now, one has from (2.20) that

$$\frac{\partial V(x,\hat{x})}{\partial x} = x^{\mathrm{T}}P+\gamma^2(x-\hat{x})^{\mathrm{T}}Z^{-1}, \quad \frac{\partial V(x,\hat{x})}{\partial \hat{x}} = -\gamma^2(x-\hat{x})^{\mathrm{T}}Z^{-1}. \tag{2.21}$$

By Theorem 2.1, it is natural to consider the controller $u(\cdot)$ being the following form:

$$u(\hat{x}) = -(B_2+B\hat{x})^{\mathrm{T}}P\hat{x}. \tag{2.22}$$

By setting $x=\hat{x}$ and considering (2.21) and (2.22), one has from (2.19) that

$$\begin{aligned} H(x,\hat{x},w_1,w_2,u) = x^{\mathrm{T}}PAx+\tfrac{1}{2}x^{\mathrm{T}}C_1^{\mathrm{T}}C_1x-\tfrac{1}{2}x^{\mathrm{T}}P(B_2+Bx) \\ \times (B_2+Bx)^{\mathrm{T}}Px+\tfrac{1}{2}\gamma^{-2}x^{\mathrm{T}}B_1B_1^{\mathrm{T}}Px \leq 0. \end{aligned} \tag{2.23}$$

By the result of [28], the positive definite matrix P satisfies the following Hamiltonian matrix:

$$H_\infty = \begin{bmatrix} A & -B_2B_2^\mathrm{T} + \gamma^{-2}B_1B_1^\mathrm{T} \\ -C_1^\mathrm{T}C_1 & -A^\mathrm{T} \end{bmatrix}$$

and

$$C_1^\mathrm{T}C_1 - PB_2B_2^\mathrm{T}P \geq 0, \quad \text{if } BxB_2^\mathrm{T} \neq 0.$$

Similarly, by setting $\hat{x} = 0$ in (2.21) and considering (2.22), from (2.19) it results that

$$\begin{aligned} &H(x, \hat{x}, w_1, w_2, u) \\ &= \gamma^2 x^\mathrm{T} Q^{-1} Ax + \tfrac{1}{2} x^\mathrm{T} C_1^\mathrm{T} C_1 x + \frac{1}{2}\gamma^2 x^\mathrm{T} Q^{-1} B_1 B_1^\mathrm{T} Q^{-1} x - \frac{1}{2}\gamma^2 x^\mathrm{T} C_2^\mathrm{T} C_2 x \\ &\quad + \frac{1}{2}\gamma^{-2}\left[\gamma^2 C_2 x + K^\mathrm{T}\left(\frac{\partial V}{\partial \hat{x}}\right)^\mathrm{T}\right]^\mathrm{T}\left[\gamma^2 C_2 x + K^\mathrm{T}\left(\frac{\partial V}{\partial \hat{x}}\right)^\mathrm{T}\right] \leq 0. \end{aligned} \tag{2.24}$$

It can be easily shown that the optimal controller which minimizes $H(x, \hat{x}, w_1, w_2, u)$ in (2.24) is given by (2.22) and (2.15) with

$$K^\mathrm{T}\left(\frac{\partial V}{\partial \hat{x}}\right)^\mathrm{T} = -\gamma^2 C_2 x$$

which implies that

$$K = ZC_2^\mathrm{T} = Q(I - \gamma^{-2}PQ)^{-1}C_2^\mathrm{T}.$$

Hence, the solution of (2.24), Q, satisfies the following Hamiltonian matrix:

$$J_\infty = \begin{bmatrix} A^\mathrm{T} & \gamma^{-2}C_1^\mathrm{T}C_1 - C_2^\mathrm{T}C_2 \\ -B_1B_1^\mathrm{T} & -A \end{bmatrix}$$

which means that Q is a positive definite solution of the following algebraic Riccati inequality:

$$AQ + QA^\mathrm{T} - Q(C_2^\mathrm{T}C_2 - \gamma^{-2}C_1^\mathrm{T}C_1)Q + B_1B_1^\mathrm{T} < 0. \tag{2.25}$$

All the above analysis leads to

THEOREM 2.2 *Consider the system of* (2.1)–(2.3) (*with* $\Delta A(t) \equiv 0$) *satisfying Assumptions* 2.1 *and* 2.2. *Then, given a* $\gamma > 0$, *there exists an output feedback controller such that the closed-loop system is stable and*

$$\|z\|^2_{[0,\infty)} < \gamma^2 \|w\|^2_{[0,\infty)}$$

for any nonzero $w \in L_2[0,\infty)$, *if and only if the following conditions hold:*

(i) *there exists a matrix* $P = P^{\mathrm{T}} > 0$ *satisfying* (2.13);
(ii) *there exists a matrix* $Q = Q^{\mathrm{T}} > 0$ *satisfying* (2.25);
(iii) $PQ < \gamma^2 I$;
(iv) *if* $Bx(t)B_2^{\mathrm{T}} \neq 0$, *then* (2.12) *is true.*

Moreover, a suitable output feedback controller can be chosen as

$$u(t) = -(B_2 + B\hat{x}(t))^{\mathrm{T}} P\hat{x}(t), \tag{2.26}$$

$$\begin{aligned}\dot{\hat{x}} &= A\hat{x}(t) - [B_2 + B\hat{x}(t)][B_2 + B\hat{x}(t)]^{\mathrm{T}} P\hat{x}(t) \\ &\quad + Q(I - \gamma^{-2}PQ)^{-1} C_2^{\mathrm{T}} [y(t) - C_2\hat{x}(t)].\end{aligned} \tag{2.27}$$

3 MAIN RESULTS

This section deals with the H_∞ synthesis for system (2.1)–(2.3) in order to achieve a robust H_∞ performance in the presence of parameter uncertainty. We will derive results for the H_∞ control problems by both state feedback and output feedback.

Before stating our main results, let us recall the following lemma.

LEMMA 3.1 ([10]) *Consider the system*

$$\begin{aligned}\dot{x}(t) &= [A + \Delta A(t)]x(t) + [B_w + \Delta B_w(t)]w(t) \\ &\quad + [B_u + \Delta B_u(t)]u(t), \\ z(t) &= [C_z + \Delta C_z(t)]x(t) + [D_{zw} + \Delta D_{zw}(t)]w(t) \\ &\quad + [D_{zu} + \Delta D_{zu}(t)]u(t), \\ y(t) &= [C_y + \Delta C_y(t)]x(t) + [D_{yw} + \Delta D_{yw}(t)]w(t) \\ &\quad + [D_{yu} + \Delta D_{yu}(t)]u(t),\end{aligned}$$

where $u(t)$ is the control input, $w(t)$ is the disturbance input, $y(t)$ is the measured output and $z(t)$ is the controlled output, with uncertainties

$$\begin{bmatrix} \Delta A(t) & \Delta B_w(t) & \Delta B_u(t) \\ \Delta C_z(t) & \Delta D_{zw}(t) & \Delta D_{zu}(t) \\ \Delta C_y(t) & \Delta D_{yw}(t) & \Delta D_{yu}(t) \end{bmatrix} = \begin{bmatrix} H_x \\ H_z \\ H_y \end{bmatrix} \Delta(t) [E_x \quad E_w \quad E_u],$$

$$\Delta^{\mathrm{T}}(t)\Delta(t) \leq I.$$

This system is stabilizable with disturbance attenuation $\gamma > 0$ by a linear output feedback control if and only if there exists a $\lambda > 0$ such that the uncertainty-free system

$$\begin{aligned} \dot{x} &= Ax(t) + [B_w \quad \gamma\lambda H_x] \begin{bmatrix} w(t) \\ \hat{w}(t) \end{bmatrix} + B_u u(t), \\ \begin{bmatrix} z(t) \\ \hat{z}(t) \end{bmatrix} &= \begin{bmatrix} C_z \\ \lambda^{-1} E_x \end{bmatrix} x(t) + \begin{bmatrix} D_{zw} & \gamma\lambda H_z \\ \lambda^{-1} E_w & 0 \end{bmatrix} \begin{bmatrix} w(t) \\ \hat{w}(t) \end{bmatrix} + \begin{bmatrix} D_{zu} \\ \lambda^{-1} E_u \end{bmatrix} u(t), \\ y(t) &= C_y(t)x(t) + [D_{yw} \quad \gamma\lambda H_y] \begin{bmatrix} w(t) \\ \hat{w}(t) \end{bmatrix} + D_{yu} u(t), \end{aligned} \tag{3.1}$$

with $u(t)$ the control input, $[w^{\mathrm{T}}(t) \;\; \hat{w}^{\mathrm{T}}(t)]^{\mathrm{T}}$ the disturbance input, $y(t)$ the measured output and $[z^{\mathrm{T}}(t) \;\; \hat{z}^{\mathrm{T}}(t)]^{\mathrm{T}}$ the controlled output, is stabilizable with disturbance attenuation γ via an output feedback control.

Remark 3.1 Lemma 3.1 establishes that the problem of output feedback H_∞ control of uncertainty systems can be converted to a standard H_∞ control problem for systems without parameter uncertainty. Note that the latter can be solved by existing results, (see, e.g., [20]).

We also recall a matrix inequality which will be needed in the proof of our main results.

LEMMA 3.2 ([15]) *Let A, E, F and H be real matrices of appropriate dimensions. Then, for any scalar $\varepsilon > 0$ and for all matrices F satisfying $\|F\| \leq 1$,*

$$HFE + E^{\mathrm{T}} F^{\mathrm{T}} H^{\mathrm{T}} \leq \frac{1}{\varepsilon} HH^{\mathrm{T}} + \varepsilon E^{\mathrm{T}} E.$$

In connection with the problem of robust H_∞ control for system (2.1)–(2.3), motivated by Lemma 3.1, we introduce the following auxiliary system:

$$
\begin{aligned}
(\Sigma^{a}):\quad \dot{x}(t) &= Ax(t) + \begin{bmatrix}\varepsilon^{-1}H & \gamma^{-1}B_1\end{bmatrix}\hat{w}_1(t) \\
&\quad + [B_2 + Bx(t)]u(t), \quad x(0) = 0, &(3.2)\\
\hat{z}(t) &= \begin{bmatrix}\varepsilon E \\ C_1\end{bmatrix}x(t) + \begin{bmatrix}0 \\ D_{12}\end{bmatrix}u(t), &(3.3)\\
y(t) &= C_2x(t) + \begin{bmatrix}0 & \gamma^{-1}D_{21}\end{bmatrix}\hat{w}_2(t) &(3.4)
\end{aligned}
$$

where $x(t) \in \Re^n$ is the state, $\hat{w}_1(t) \in \Re^{p+i}$ is the disturbance input, $\hat{z} \in \Re^{r+j+n}$ is the controlled output, $y \in \Re^m$ is the output measurement, $\hat{w}_2 \in \Re^{q+j}$ is the measurement noise, $\gamma > 0$ is the prescribed robust H_∞ performance we wish to achieve for system (2.1)–(2.3), $\varepsilon > 0$ is a scalar to be chosen (like the parameter λ in Lemma 3.1), A, B, B_1, B_2, C_1, C_2, D_{12}, D_{21}, E and H are the same as in system (2.1)–(2.3).

The motivation for introducing the auxiliary system (3.2)–(3.4) for system (2.1)–(2.3) is that if we are able to solve the H_∞ control problem for the uncertainty-free system (3.2)–(3.4) by Theorem 2.1, then we may solve the same problem for uncertain system (2.1)–(2.3) via Lemma 3.1.

3.1 State Feedback

We now present a solution to the robust H_∞ state feedback control problem for system (2.1)–(2.2).

THEOREM 3.1 *Consider the system* (2.1)–(2.2) *satisfying Assumptions* 2.1 *and* 2.2. *If there exists a state feedback controller,* $\mathcal{G}$ *of the form* (2.14), *such that the closed-loop system,* (Σ_{cl}) *of* (3.2)–(3.3) *with* $\mathcal{G}$ *is globally uniformly asymptotically stable and*

$$\|\hat{z}\|^2_{[0,\infty)} < \|\hat{w}_1\|^2_{[0,\infty)},$$

then the closed-loop system of (2.1)–(2.2) *with* $\mathcal{G}$ *is robustly stable and has a robust* H_∞ *performance* γ *over the horizon* $[0,\infty)$.

Proof By Theorem 2.1, the closed-loop system, (Σ_{cl}), of (3.2)–(3.3) with $\mathcal{G}$ is globally uniformly asymptotically stable and

$$\|\hat{z}\|^2_{[0,\infty)} < \|\hat{w}_1\|^2_{[0,\infty)},$$

implies that there exists a matrix $P = P^{\mathrm{T}} > 0$, such that

$$\begin{aligned} &A^{\mathrm{T}}P + PA - P(B_2B_2^{\mathrm{T}} - \gamma^{-2}B_1B_1^{\mathrm{T}})P + \varepsilon^{-2}PHH^{\mathrm{T}}P \\ &\quad + \varepsilon^2 E^{\mathrm{T}}E + C_1^{\mathrm{T}}C_1 < 0; \end{aligned} \tag{3.5}$$

and (2.12) is true.

Next, in view of Lemma 3.2, it results from (3.5) that

$$\begin{aligned} &(A + HF(t)E)^{\mathrm{T}}P + P(A + HF(t)E) - P(B_2B_2^{\mathrm{T}}) - \gamma^{-2}B_1B_1^{\mathrm{T}})P \\ &\quad + C_1^{\mathrm{T}}C_1 < 0, \end{aligned} \tag{3.6}$$

for any admissible parameter uncertainty $F(t)$ satisfying (2.4) $\forall t > 0$.

Combining the inequality (3.6) with (2.12), applying Theorem 2.1 again, we conclude that the closed-loop system of (2.1)–(2.2) with $\mathcal{G}$ is robustly stable and has a robust H_∞ performance γ over the horizon $[0, \infty)$.

3.2 Output Feedback

A solution to the robust H_∞ output feedback control problem is provided by the next theorem.

THEOREM 3.2 *Consider the system* (2.1)–(2.3) *satisfying Assumptions* 2.1 *and* 2.2. *If there exists an output feedback controller,* $\mathcal{G}$ *of the form* (2.26)–(2.27), *such that the closed-loop system,* (Σ_{cl}), *of* (3.2)–(3.4) *with* $\mathcal{G}$ *is globally uniformly asymptotically stable and*

$$\|\hat{z}\|^2_{[0,\infty)} < \|\hat{w}\|^2_{[0,\infty)},$$

where $\hat{w} = [\hat{w}_1^{\mathrm{T}}, w_2^{\mathrm{T}}]^{\mathrm{T}}$, *then the closed-loop system of* (2.1)–(2.3) *with* $\mathcal{G}$ *is robustly stable and has a robust* H_∞ *performance* γ *over the horizon* $[0, \infty)$.

Proof Applying Theorem 2.2, the closed-loop system, (Σ_{cl}), of (3.2)–(3.4) with $\mathcal{G}$ is globally uniformly asymptotically stable and

$$\|\hat{z}\|^2_{[0,\infty)} < \|\hat{w}\|^2_{[0,\infty)},$$

implies that there exist matrices $P = P^{\mathrm{T}} > 0$ and $Q = Q^{\mathrm{T}} > 0$, such that

$$\begin{aligned} &A^{\mathrm{T}}P + PA - P(B_2B_2^{\mathrm{T}} - \gamma^{-2}B_1B_1^{\mathrm{T}})P \\ &\quad + \varepsilon^{-2}PHH^{\mathrm{T}}P + \varepsilon^2E^{\mathrm{T}}E + C_1^{\mathrm{T}}C_1 < 0, \end{aligned} \tag{3.7}$$

$$\begin{aligned} &AQ + QA^{\mathrm{T}} - Q(C_2^{\mathrm{T}}C_2 - C_1^{\mathrm{T}}C_1)Q + \varepsilon^2QE^{\mathrm{T}}EQ \\ &\quad + \varepsilon^{-2}HH^{\mathrm{T}} + \gamma^{-2}B_1B_1^{\mathrm{T}} < 0, \end{aligned} \tag{3.8}$$

$$PQ < \gamma^2 I, \tag{3.9}$$

and (2.12) is true.

Now, consider Lemma 3.2 with (3.7) and (3.8), respectively; we have that

$$\begin{aligned} &(A + HF(t)E)^{\mathrm{T}}P + P(A + HF(t)E) \\ &\quad - P(B_2B_2^{\mathrm{T}} - \gamma^{-2}B_1B_1^{\mathrm{T}})P + C_1^{\mathrm{T}}C_1 < 0, \end{aligned} \tag{3.10}$$

$$\begin{aligned} &(A + HF(t)E)Q + Q(A + HF(t)E)^{\mathrm{T}} \\ &\quad - Q(C_2^{\mathrm{T}}C_2 - C_1^{\mathrm{T}}C_1)Q + \gamma^{-2}B_1B_1^{\mathrm{T}} < 0 \end{aligned} \tag{3.11}$$

for any admissible parameter uncertainty $F(t)$ satisfying (2.4) $\forall t \geq 0$.

Therefore, considering (3.10), (3.11), (3.9) and (2.12), the desired result can be established by applying Theorem 2.2 to the closed-loop system of (2.1)–(2.3) with $\mathcal{G}$.

Remark 3.2 Theorems 3.1 and 3.2 show that the problems of both state feedback and output feedback robust H_∞ control for the uncertain system (2.1)–(2.3) can be solved in terms of H_∞ syntheses for related bilinear systems without parameter uncertainty. Hence, the techniques for standard H_∞ bilinear control of Theorems 2.1 and 2.2 and some existing results (see, e.g., [28,29]) can be used to solve the above robust bilinear synthesis problems in terms of Riccati inequalities, like linear continuous-time case with one extra condition (2.12).

Motivated by the more recent results of H_∞ output feedback control for bilinear systems [21], we may consider that the controller is of the form

$$\dot{\xi}(t) = \eta_1(\xi) + \eta_2(\xi)u + \eta_3(\xi)y, \tag{3.12}$$

$$u(t) = \theta(\xi), \tag{3.13}$$

where $\eta_1(\xi)$, $\eta_2(\xi)$, $\eta_3(\xi)$ and $\theta(\xi)$ are sufficiently smooth functions to be chosen, with $\eta_1(0) = 0$ and $\theta(0) = 0$.

THEOREM 3.3 *Consider the system* (2.1)–(2.3) *satisfying Assumptions* 2.1 *and* 2.2. *Suppose there exist matrices* $P > 0$ *and* $Q > 0$ *such that for all* $(x, \xi) \neq 0$, *the following inequalities hold:*

$$\begin{aligned} &P(A + \gamma^{-2}QC_1^{\mathrm{T}}C_1) + (A + \gamma^{-2}QC_1^{\mathrm{T}}C_1)^{\mathrm{T}}P - P\big[\bar{B}(\xi)\bar{B}^{\mathrm{T}}(\xi) \\ &\quad - \gamma^{-2}QC_2^{\mathrm{T}}C_2Q\big]P + C_1^{\mathrm{T}}C_1 + \varepsilon^{-2}PHH^{\mathrm{T}}P + \varepsilon^2E^{\mathrm{T}}E < 0, \end{aligned} \tag{3.14}$$

$$\begin{aligned} &QA^{\mathrm{T}} + AQ - Q(C_2^{\mathrm{T}}C_2 - \gamma^{-2}C_1^{\mathrm{T}}C_1)Q + B_1B_1^{\mathrm{T}} \\ &\quad + \varepsilon^{-2}PHH^{\mathrm{T}}P + \varepsilon^2E^{\mathrm{T}}E < 0, \end{aligned} \tag{3.15}$$

$$\begin{aligned} &\xi^{\mathrm{T}}T_1\xi + \gamma^2(x - \xi)^2Q^{-1}\Big[T_2 + Q\{\underline{v}(\xi)M\}^{\mathrm{T}} + \{\underline{v}(\xi)M\}Q\Big] \\ &\quad \times Q^{-1}(x - \xi) < 0, \end{aligned} \tag{3.16}$$

where $\bar{B}(\xi) = B_2 + B\xi$, T_1 *and* T_2 *are the left hand sides of* (3.14) *and* (3.15), *respectively*, $\underline{v}(\xi) = -\bar{B}^{\mathrm{T}}(\xi)P\xi$, M *is a matrix such that* $Bxu = Mux$ *and* $\varepsilon > 0$ *is a scalling parameter.*

Then, the closed-loop system, (Σ_{cl}), *of* (2.1)–(2.3) *with* (3.12)–(3.13) *is internally robust stable in the maximum hyper-ellipsoid*

$$\Omega_1(c_1) = \Big\{(x, \xi) | \xi^{\mathrm{T}}P\xi + \gamma^2(x - \xi)^{\mathrm{T}}Q^{-1}(x - \xi) \leq c_1\Big\}$$

where $c_1 > 0$ *is a constant, and has a robust* H_∞ *performance* γ.

Furthermore, the controller of (3.12)–(3.13) *is given as*

$$\dot{\underline{x}}(t) = A\underline{x} - \bar{B}(\underline{x})\bar{B}^{\mathrm{T}}(\underline{x})P\underline{x} + \gamma^{-2}QC_1^{\mathrm{T}}C_1\underline{x} + QC_2^{\mathrm{T}}(y - C_2\underline{x}),$$
$$(t) = -\bar{B}^{\mathrm{T}}(\underline{x})P\underline{x}.$$

Proof It can be established by using the same technique as in Theorem 3.2, together with Theorem 3.1 in [21].

THEOREM 3.4 *Consider the system* (2.1)–(2.3) *satisfying Assumptions* 2.1 *and* 2.2. *Suppose there exist matrices* $P>0$ *and* $Q>0$ *such that for all* $(x,\xi)\neq 0$, *the following inequalities hold:*

$$\begin{aligned}
&PA + A^{\mathrm{T}}P - P\big[\bar{B}(x)\bar{B}^{\mathrm{T}}(x) - \gamma^{-2}B_1B_1^{\mathrm{T}}\big]P + C_1^{\mathrm{T}}C_1 \\
&\quad + \varepsilon^{-2}PHH^{\mathrm{T}}P + \varepsilon^2E^{\mathrm{T}}E < 0, &(3.17)\\
&Q(A+\gamma^{-2}B_1B_1^{\mathrm{T}}P)^{\mathrm{T}} + (A+\gamma^{-2}B_1B_1^{\mathrm{T}}P)Q \\
&\quad - Q\big[C_2^{\mathrm{T}}C_2 - \gamma^{-2}P\bar{B}(x)\bar{B}^{\mathrm{T}}(x)P\big]Q + B_1B_1^{\mathrm{T}} \\
&\quad + \varepsilon^{-2}PHH^{\mathrm{T}}P + \varepsilon^2E^{\mathrm{T}}E < 0, &(3.18)\\
&x^{\mathrm{T}}S_1x + \gamma^2(x-\xi)^2Q^{-1}\Big[S_2 + Q\{\hat{v}(\xi)M\}^{\mathrm{T}} + \{\hat{v}(\xi)MQ\}Q\Big] \\
&\quad \times Q^{-1}(x-\xi) + \|\bar{B}^{\mathrm{T}}(x)Px - \bar{B}^{\mathrm{T}}(\xi)P\xi\|^2 \\
&\quad - \|\bar{B}^{\mathrm{T}}(x)P(x-\xi)\|^2 < 0, &(3.19)
\end{aligned}$$

where $\bar{B}(\xi) = B_2 + B\xi$, $\bar{B}(x) = B_2 + Bx$, S_1 *and* S_2 *are the left hand sides of* (3.17) *and* (3.18), *respectively*, $\hat{v}(\xi) = -\bar{B}^{\mathrm{T}}(\xi)P\xi$.

Then, the closed-loop system, (Σ_{cl}) of (2.1)–(2.3) *with* (3.12)–(3.13) *is internally robust stable in the maximum hyper-ellipsoid*

$$\Omega_2(c_2) = \Big\{(x,\xi)|\xi^{\mathrm{T}}P\xi + \gamma^2(x-\xi)^{\mathrm{T}}Q^{-1}(x-\xi) \leq c_2\Big\}$$

where $c_2>0$ *is a constant, and has a robust* H_∞ *performance* γ.

Furthermore, the controller of (3.12)–(3.13) *is given as*

$$\dot{\hat{x}}(t) = A\hat{x} - \bar{B}(\hat{x})\bar{B}^{\mathrm{T}}(\hat{x})P\hat{x} + \gamma^{-2}B_1B_1^{\mathrm{T}}P\hat{x} + QC_2^{\mathrm{T}}(y - C_2\hat{x}), \quad (3.20)$$

$$u(t) = -\bar{B}^{\mathrm{T}}(\hat{x})P\hat{x}. \quad (3.21)$$

Proof It can be carried out along the same line as in Theorem 3.2, combining with Theorem 3.2 in [21].

Remark 3.3 In Theorem 3.3 (respectively, 3.4), the positive definite solutions P and Q of (3.14)–(3.15) (respectively, (3.17)–(3.18)) are used to construct the controller of (3.12)–(3.13) (respectively, (3.20)–(3.21)), and (3.16) (respectively, (3.19) is used to evaluate the domain Ω_1 (respectively, Ω_2) in which the closed-loop system, (Σ_{cl}), of (3.2)–(3.4) with (3.12)–(3.13) (respectively, (3.20)–(3.21)) is internally robust stable. Note that the Riccati inequalities (3.14)–(3.15) (respectively, (3.17)–(3.18)) correspond to the Riccati inequalities in a linear H_∞ control problem.

4 APPLICATION EXAMPLES

4.1 Paper-making Machine System

This example is from [35] for the nominal system. The dryer section of a paper-making machine is described by the following second order single input bilinear system:

$$\dot{x}(t) = [A + \Delta A(t)]x(t) + [B_2 + Bx(t)]u(t) + B_1 w(t), \tag{4.1}$$

$$z(t) = C_1 x(t) + u(t), \tag{4.2}$$

where

$$A = \begin{bmatrix} -0.046 & 0 \\ -0.7632 & -3.197 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0.986 \\ 0 \end{bmatrix},$$

$$B = \begin{bmatrix} -0.027 & 0 \\ 0 & 0 \end{bmatrix}, \qquad C_1 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} 0 \\ -3.197 \end{bmatrix}, \qquad \Delta A(t) = HF(t)E,$$

$$H = \begin{bmatrix} 0.15 \\ 0.02 \end{bmatrix}, \quad E = [0.06 \quad 0.15], \quad F(t) = \delta(t), \quad |\delta(t)| \leq 1.$$

We can solve the Riccati inequality (3.5) for $\gamma = 1.5125$ and $\varepsilon = 1$; we have that

$$P = \begin{bmatrix} 1.034 & 0.0379 \\ 0.0379 & 0.9986 \end{bmatrix}.$$

It can be verified that the closed-loop nonlinear system, (4.1)–(4.2) with the controller, $u(t) = -(B_2 + Bx(t))^{\mathrm{T}} Px(t)$, is globally uniformly asymptotically stable for $\gamma \geq 1.5125$. Also the condition, $C_1^{\mathrm{T}} C_1 - PB_2 B_2^{\mathrm{T}} P \geq 0$ is satisfied.

4.2 Motor Bilinear System

Let us consider a motor system which contains two sets of windings shown in Fig. 1. One set is the armature, and the other is the rotating part of the machine. For the armature, the time domain equation governing the electrical part of the system is

$$\frac{\mathrm{d}i_\mathrm{a}}{\mathrm{d}t} = -\frac{R_\mathrm{a}}{L_\mathrm{a}} i_\mathrm{a} + \frac{1}{L_\mathrm{a}} e_\mathrm{t} - \frac{1}{L_\mathrm{a}} e_\mathrm{b}.$$

In the mechanical portion (Fig. 2), it is general to analyze the system assuming a simple, inertia load on the system, so

$$J\ddot{q} + F\dot{q} + Kq = T_\mathrm{m}.$$

The definitions and values of electric and mechanical parameters are shown in Table I. The flux in the air gap is a function of its armature, which can be applied for the state feedback control law. Assuming that the torque is proportional to the flux in the air gap and armature windings gives

$$T_\mathrm{m} = Ki_\mathrm{a} i_\mathrm{e}.$$

Furthermore, the back emf induced in the armature windings is assumed to be proportional to the angular velocity of the output shaft. Therefore,

$$e_\mathrm{b} = K\dot{q} i_\mathrm{e}.$$

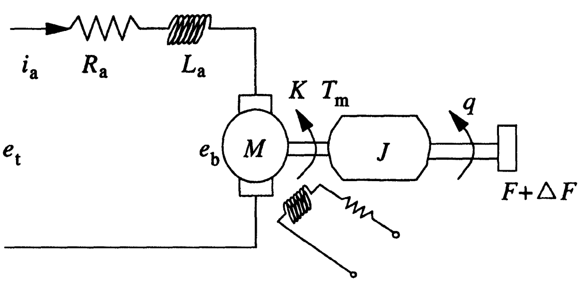

FIGURE 1 The motor system with the bounded uncertain load.

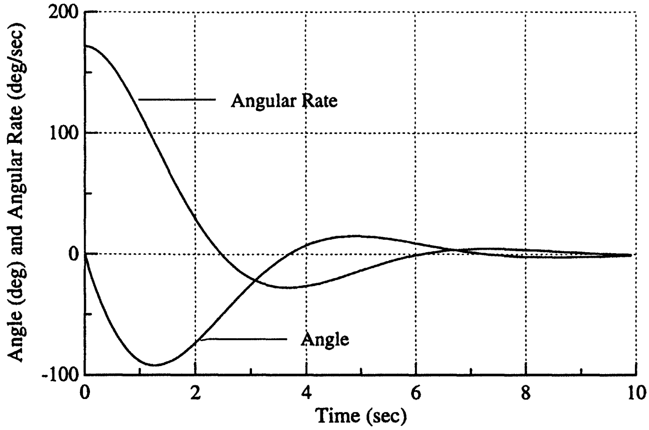

FIGURE 2 The state responses of the mechanical part of the controlled motor system.

TABLE I Coefficients and values for the motor system

e_b	Back emf voltage (V)
e_t	Applied motor voltage (V)
$F=1$	Load (N s/rad)
ΔF ($\|\Delta F\| \leq 0.8$)	Bounded uncertain load (N s/rad)
i_a	Armature current (A)
$J=1$	Moment of inertia of motor (N s^2/rad)
$K_1=1$	Armature constant
$K_2=1$	Torsional spring constant (N/rad)
$L_a=1$	Armature inductance
q	Angle of the link
$R_a=1$	Motor resistance (Ω)
T_m	Torque constant (N)

We assume that the load on the motor is variant as $(F+\Delta F)$, where ΔF is bounded under a certain value. In this paper, $\|\Delta F\| = 0.8F$ is used to find the robust control for the motor bilinear system (Fig. 3). Combining all the formulations and assumptions described on the above and employing the small perturbation theory for the system, the motor control problem can be described as the following state space model:

$$\dot{x}(t) = (A + \Delta A)x(t) + B_1 w(t) + (B_2 + Bx(t))u(t), \tag{4.3}$$
$$z(t) = C_1 x(t) + D_{12} u(t), \tag{4.4}$$

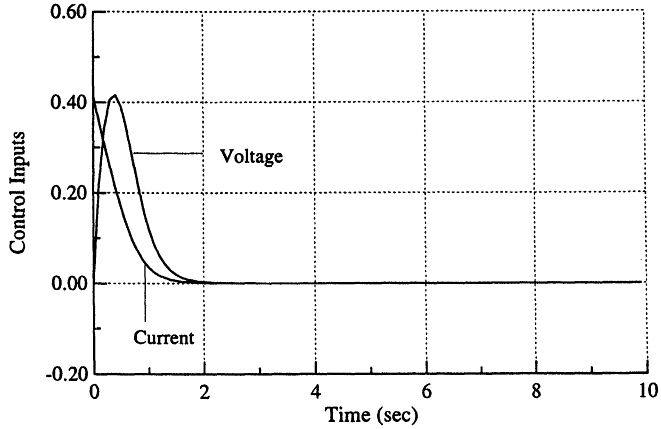

FIGURE 3 Control responses of the controlled motor system.

where

$$A = \begin{bmatrix} -R_a/L_a & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -K/J & -F/J \end{bmatrix}, \qquad B = \begin{bmatrix} -1/L_a & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix},$$

$$B_1 = \begin{bmatrix} -R_a/L_a & 0 \\ 0 & 0 \\ 0 & -F/J \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 & Kx_3 \\ 0 & 0 \\ 0 & Kx_1 \end{bmatrix},$$

$$C_1 = I_{3\times 3}, \qquad D_{12} = I_{2\times 2}.$$

Note that the states are $x = (i_a, q, \dot{q})^{\mathrm{T}}$, w is the disturbance, and the control input is $u = (e_t, i_e)^{\mathrm{T}}$.

We will design a robust controller for the uncertain bilinear system such that the resulting closed-loop system can be robustly stabilized. By using Theorem 3.2, the following algebraic Riccati solution can be obtained:

$$P = \begin{bmatrix} 0.4142 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1.5 \end{bmatrix}.$$

Note that the following values of (ε, E, H) are employed to compute the solution of the Riccati inequality:

$$\varepsilon = 1, \quad E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad H = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.1 \end{bmatrix}.$$

Therefore, the state feedback control can be obtained as

$$u(t) = \begin{bmatrix} 0.41421x_1 \\ -0.91421x_1x_3 \end{bmatrix}. \tag{4.5}$$

Remark 4.1 Note that the asymptotic stability of the closed-loop system (4.3) with the controller (4.5)

$$\dot{x}(t) = \begin{bmatrix} -1.41421x_1 - 0.91421x_1x_3^2 \\ x_2 \\ -x_2 - (1 + \Delta F)x_3 - 0.91421x_1^2x_3 \end{bmatrix} x(t) + Gw(t)$$

for all admissible uncertainties, $\|\Delta F\| \leq 0.8$, can be easily checked by choosing the Lyapunov function

$$V(x) = \tfrac{1}{2}(x_1^2 + x_2^2 + x_3^2).$$

5 CONCLUSIONS

This paper has considered the problems of robust H_∞ synthesis for a class of uncertain bilinear continuous-time systems. The uncertainties we considered comprise norm-bounded parameter uncertainty in the state equation. Both H_∞ control problems via state feedback and output feedback have been studied. It has been shown that controllers for the above problems can be designed by solving H_∞ syntheses for related bilinear continuous-time systems without parameter uncertainty, which is in terms of one (respectively, two) Riccati inequality (respectively, inequalities) for state feedback (respectively, output feedback) control with one constraint. Two examples are presented to illustrate the theoretic results.

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