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Linear-Quadratic Optimization and Some General Hypotheses on Optimal Control

L.I. ROZONOER*

Institute of Control Sciences, Moscow, Russia

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Necessary and sufficient conditions for existence of optimal control for all initial data are proved for *LQ*-optimization problem. If these conditions are fulfilled, necessary and sufficient conditions of optimality are formulated. Basing on the results, some general hypotheses on optimal control in terms of Pontryagin's maximum condition and Bellman's equation are proposed.

Keywords: Optimal control; *LQ*-optimization; Pontryagin's maximum principle; Bellman's equation

1 INTRODUCTION

The *LQ*-optimization is a well-developed part of the theory of optimal control. Results related to the *LQ*-problem are well-known and usually are included into textbooks (for example, see [1]). However, usually only positive definite quadratic forms of phase coordinates are considered. A generalization to non-positive forms leads to some new non-trivial problems, among which conditions of existence of an optimal control have to be considered. Besides, usually an analysis of the problem in terms of Riccati's equation is provided for finding out only sufficient (but not necessary) conditions of optimality. It is turned out that

* Address for correspondence: 61 Moraine Street, #4, Belmont, MA 02478, USA. E-mail: lev@ma.ultranet.com.

the existence of a solution of the Riccati's equation yields a necessary condition of an optimal control existence for each initial datum. We prove that the existence of a solution of the Riccati's equation is a necessary and sufficient condition of the existence of an optimal control for all initial data.

In addition, there are some auxiliary problems, connected with LQ-problem, which can be interesting not only for the optimal control theory (among them connections between boundary problems and Riccati's equation). On the basis of our results for LQ-optimization, we make some general suppositions about optimal control, the main of which is a hypothesis on the necessary and sufficient conditions for existence of optimal control for all initial data.

2 LQ-OPTIMIZATION

Linear systems and square functionals are considered. The systems are described by the equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = A(t)z + B(t)w + \phi(t), \quad z(t_0) = z^0. \tag{2.1}$$

Here $z(t)$ is a phase-coordinate vector, $w(t)$ a control vector, $\phi(t)$ a fixed vector-function, $A(t)$ and $B(t)$ are matrices of corresponding dimensions, z^0 is an initial-state vector. We will suppose that the system (2.1) is a controllable one.

A functional that defines the aim of control has the form:

$$J\{w(\cdot)\} = \int_{t_0}^{T} \left[\tfrac{1}{2}\langle z, Rz\rangle + \tfrac{1}{2}\langle w, Pw\rangle\right] \mathrm{d}t, \tag{2.2}$$

where R and P are matrices; it is supposed that the matrix P is positive definite; symbol $\langle\, ,\rangle$ stand for the scalar product of two vectors. Matrix R is not necessary positive definite or zero. A case of nonpositive and nonzero matrix R is interesting from a mathematical point of view for a general statement of the optimization problem. As to the control theory, this case appears in a dynamic game approach to H^{∞}-optimal control (see [2]).

The LQ-optimization problem is: minimize functional (2.2) under the condition (2.1), that is, find such a $w^*(\cdot)$ that

$$J\{w^*(\cdot)\} \leq J\{w(\cdot)\} \tag{2.3}$$

for all $w(\cdot)$.

The following questions are studied in this section of the paper: (1) conditions under which solutions of the problem exist; (2) necessary conditions; (3) sufficient conditions.

Note that a solution of a LQ-optimization problem can exist for some initial value z^0 and not exist for others. We will consider conditions under which a solution exists with any initial value.

Following statements present the main results.

THEOREM 1 *An optimal control under any z^0 in the problem of minimization of the functional*

$$J\{w(\cdot)\} = \int_{t_0}^{T} \left[\tfrac{1}{2}\langle z, Rz\rangle + \tfrac{1}{2}\langle w, Pw\rangle\right] \mathrm{d}t$$

for the system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = A(t)z + B(t)w + \phi(t), \quad z(t_0) = z^0$$

exists iff there exists a solution of the Riccati's equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} + SA + A'S + SGS = R, \quad S(T) = 0, \; t \in (t_0, T),$$

where $G = BP^{-1}B'$. If this condition is fulfilled then an optimal control, $w^*(t)$, is determined by the unique solution of the equations

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + Gp + \phi, \; \frac{\mathrm{d}p}{\mathrm{d}t} = Rz - A'p, \quad z(t_0) = z^0, \; p(T) = 0,$$

by the formula $w^*(t) = P^{-1}B'p(t)$.

For the homogeneous case ($\phi(t) \equiv 0$), it is possible to substitute $p(t) = S(t)x(t)$.

THEOREM 2 *An optimal control for any z^0 in the problem of minimization of the functional*

$$J\{w(\cdot)\} = \int_{t_0}^{T} \left[\tfrac{1}{2}\langle z, Rz\rangle + \tfrac{1}{2}\langle w, Pw\rangle\right] \mathrm{d}t$$

for the system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = A(t)z + B(t)w + \phi(t), \quad z(t_0) = z^0$$

exists iff there exists only the trivial solution of the boundary value problem:

$$\frac{\mathrm{d}z^\wedge}{\mathrm{d}t} = Az^\wedge + \Gamma p^\wedge, \quad \frac{\mathrm{d}p^\wedge}{\mathrm{d}t} = Rz^\wedge - A'p^\wedge,$$
$$z^\wedge(\tau) = 0, \; p^\wedge(T) = 0, \; \Gamma = BP^{-1}B'$$

for all $\tau \in [t_0, T)$. If this condition is fulfilled then the optimal control is determined by the unique solution of the boundary value problem:

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + \Gamma p + \phi, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = Rz - A'p, \; z(t_0) = z^0, \; p(T) = 0,$$

by the formula $w^*(t) = P^{-1}B'p(t)$.

The theorem means that Pontryagin's maximum principle gives the optimal control if there exists a unique (for each boundary values) control which obeys the maximum principle.

The rest of this section is devoted to the proof of the theorems.

Along with system (2.1), we will consider a homogeneous system

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = A\xi + B\nu, \quad \xi(t_0) = \xi^0, \tag{2.4}$$

and will prove that the minimization of functional (2.2) under conditions (2.1) is reduced to minimization of a functional

$$J^\wedge\{\nu(\cdot)\} = \int_{t_0}^{T} \left[\tfrac{1}{2}\langle \xi, R\xi\rangle + \tfrac{1}{2}\langle \nu, P\nu\rangle\right] \mathrm{d}t \tag{2.5}$$

under conditions (2.1) with some ξ^0.

LEMMA 1 *For any $w(\cdot), \phi(\cdot), z^0$, there exist such $\nu(\cdot)$ and ξ^0 that the functional* (1.2) *differs from the functional* (2.2) *by a constant depending may be of $\phi(\cdot)$ and z^0, but not of $w(\cdot)$.*

Proof of Lemma 1 Consider the equations

$$\frac{\mathrm{d}\alpha}{\mathrm{d}t} = A\alpha + BP^{-1}B'\beta + \phi, \ \frac{\mathrm{d}\beta}{\mathrm{d}t} = R\alpha - A'\beta,$$
$$\alpha(T) = \beta(T) = 0, \tag{2.6}$$

where α and β are vectors of the same dimension as z, the symbol A' denotes the transpose of matrix A. Define

$$\xi(t) = z(t) - \alpha(t), \quad \nu(t) = w(t) - P^{-1}B'\beta. \tag{2.7}$$

It is easy to check that by virtue of (2.1) and (2.6), $\xi(t)$ from (2.7) is a solution of (2.4), if $\nu(t)$ is taken from (2.7). By the direct substitution of $z(t) = \xi(t) + \alpha(t)$ into (2.2) we have

$$J\{w(\cdot)\} = \int_{t_0}^{T} \left[\tfrac{1}{2}\langle\xi, R\xi\rangle + \tfrac{1}{2}\langle\nu, P\nu\rangle\right] \mathrm{d}t + 2\int_{t_0}^{T} [\langle\xi, R\alpha\rangle + \langle\nu, A'\beta\rangle]\, \mathrm{d}t. \tag{2.8}$$

From (2.4) and (2.6) we have

$$\frac{\mathrm{d}[\langle\xi, \beta\rangle]}{\mathrm{d}t} = \langle\beta, A\nu\rangle + \langle\xi, R\alpha\rangle.$$

By integrating of the last equality we have

$$\int_{t_0}^{T} [\langle\xi, R\alpha\rangle + \langle\nu, A'\beta\rangle]\, \mathrm{d}t = -\xi(0)\beta(0) = -(z^0 - \alpha(0))\beta(0).$$

Therefore, from (2.5) we deduce

$$J\{w(\cdot)\} = J^{\wedge}\{\nu(\cdot)\} - 2(z^0 - \alpha(0))\beta(0).$$

The expression $(z^0 - \alpha(0))\beta(0)$ depends through (2.6) only on z^0 and on $\phi(\cdot)$. The lemma is proved.

Based on the lemma, we can assert that an optimal control for a system (2.1), (2.2) exists for all z^0 if and only if an optimal control exists for a homogeneous system (2.4), (2.5) for all ξ^0.

Consider the following boundary value problem:

$$\frac{d\xi}{dt} = A\xi + \Gamma p, \; \frac{dp}{dt} = R\xi - A'p, \quad \xi(t_0) = \xi^0, \; p(T) = 0, \qquad (2.9)$$

where $G = BP^{-1}B'$, the vector $p(t)$ has the same dimension as $\xi(t)$. The following Lemma 2 is an analog of the well-known Jacobi's necessary condition in the classical calculus of variations.

LEMMA 2 *If an optimal control exists in the problem* (2.1), (2.2) *for each* ξ^0, *then each nontrivial solution of* (2.9), $\xi(t)$, *is not equal to* 0 *in any inner point* $t \in (t_0, T)$.

Proof of Lemma 2 Consider the optimization problem for (2.1), (2.2) with $\xi(t_0) = \xi^0$. Pontryagin's maximum principle yields necessary conditions:

$$\nu(t) = P^{-1}B'p(t), \qquad (2.10)$$

$$\begin{gathered} \frac{d\xi}{dt} = A\xi + \Gamma p, \; \frac{dp}{dt} = R\xi - A'p, \\ \xi(t_0) = \xi^0, \; p(T) = 0, \; \Gamma = BP^{-1}B'. \end{gathered} \qquad (2.11)$$

Thus, the optimal trajectory $\xi(t)$, if it exists, is a solution of the boundary value problem (2.9). Suppose an opposite to the lemma assertion, i.e., that such a $t^\wedge \in (t_0, T)$ exists that $\xi(t^\wedge) = 0$. If $\xi(t)$ were optimal then it would be a nontrivial solution of (2.9). Consider together with $\nu(t)$ from (2.10) a new control $\nu^\perp(t)$, such that

$$\nu^\perp(t) = \nu(t), \text{ if } t \in (t_0, t^\wedge) \quad \text{and} \quad \nu^\perp(t) = 0 \text{ if } t \in [t^\wedge, T], \qquad (2.12)$$

The phase vector, $\xi^\perp(t)$, corresponding to the control $\nu^\perp(t)$, is a solution of equations

$$\frac{d\xi^\perp}{dt} = A\xi^\perp + B\nu^\perp, \quad \xi^\perp(t_0) = \xi^0.$$

Therefore, $\xi^{\perp}(t)=0$, if $t\in[t^{\wedge},T]$. Compare values $J^{\wedge}\{\nu(\cdot)\}$ and $J^{\wedge}\{\nu^{\perp}(\cdot)\}$. The value of the functional (2.2) for $\nu=\nu^{\perp}(t)$ is equal to

$$J^{\wedge}\{\nu^{\perp}(\cdot)\}=\frac{1}{2}\int_{t_0}^{t^{\wedge}}[\langle\xi^{\perp},R\xi^{\perp}\rangle+\langle\nu^{\perp},P\nu^{\perp}\rangle]\,\mathrm{d}t+\frac{1}{2}\int_{t^{\wedge}}^{T}[\langle\xi^{\perp},R\xi^{\perp}\rangle+\langle\nu^{\perp},P\nu^{\perp}\rangle]\,\mathrm{d}t.$$

The second integral is equal to 0 because $\nu^{\perp}(t)=0$ and $\xi^{\perp}(t)=0$ on the interval $t\in[t^{\wedge},T]$. For $J^{\wedge}\{\nu(\cdot)\}$ we have:

$$J^{\wedge}\{\nu(\cdot)\}=\frac{1}{2}\int_{t_0}^{t^{\wedge}}[\langle\xi,R\xi\rangle+\langle\nu,P\nu\rangle]\,\mathrm{d}t+\frac{1}{2}\int_{t^{\wedge}}^{T}[\langle\xi,R\xi\rangle+\langle\nu,P\nu\rangle]\,\mathrm{d}t.$$

The second integral here is equal to 0 as well. Indeed using (2.10) and (2.11), we have

$$\frac{\mathrm{d}\langle\xi,p\rangle}{\mathrm{d}t}=\langle\xi,R\xi\rangle+\langle\nu,P\nu\rangle.$$

By integrating the last equality on the interval $[t^{\wedge},T]$ and using conditions $\xi(t^{\wedge})=0$ and $p(T)=0$, we obtain that

$$\int_{t^{\wedge}}^{T}[\langle\xi,R\xi\rangle+\langle\nu,P\nu\rangle]\,\mathrm{d}t=0.$$

Taking into account that $\nu^{\perp}(t)=\nu(t),\xi^{\perp}(t)=\xi(t)$, if $t\in(t_0,t^{\wedge})$ we receive that $J^{\wedge}\{\nu^{\perp}(\cdot)\}=J^{\wedge}\{\nu(\cdot)\}$. Note that the conditions of the maximum principle are not fulfilled for $\nu^{\wedge}$ on all interval $[t_0,T]$ because it were possible only if $B'p(t)$ would be equal to 0 at each point $t\in[t^{\wedge},T]$. But it is impossible for a controllable system. Therefore $\nu^{\perp}$ is not an optimal one. We can conclude that $\nu(\cdot)$ is not optimal as well. However, if each control for which necessary conditions are fulfilled is not optimal one, then optimal control does not exist at all. Lemma 2 is proved.

The Pontryagin's maximum principle yields the following necessary conditions:

$$w = P^{-1}B'p, \tag{2.13}$$

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + \Gamma p + \phi, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = Rz - A'p,$$
$$z(t_0) = z^0, \; p(T) = 0, \; \Gamma = BP^{-1}B'. \tag{2.14}$$

Write down a matrix Riccati's equation for the problem (2.4), (2.5):

$$\frac{\mathrm{d}S}{\mathrm{d}t} + SA + A'S + S\Gamma S = R, \quad S(T) = 0, \; t \in (t_0, T). \tag{2.15}$$

Let for $w(t)$, $z(t)$ the necessary conditions (2.13), (2.14) hold. Consider increments, $\Delta w(t), \Delta z(t)$, of $w(t), z(t)$. Using the solution of the Riccati's equation, $S(t)$, one can write down the next formula for the functional (2.5) increment, if the control $w(t)$ varies:

$$\Delta J = \int_{t_0}^{T} \|\Delta w - P^{-1}B'S\Delta z\|_P^2 \, \mathrm{d}t. \tag{2.16}$$

Here $\|\pi\|_P^2$ stands for the square of a norm $\langle \pi, P\pi \rangle$. For increments $\Delta w(t), \Delta z(t)$ the following equation holds:

$$\frac{\mathrm{d}\Delta z}{\mathrm{d}} t = A\Delta z + B\Delta w, \quad \Delta z(t_0) = 0, \tag{2.17}$$

Proof of Formula (2.16) Consider a function $y(t) = \frac{1}{2}\langle \Delta z(t), S(t)\Delta z(t) \rangle$, where for $\Delta z(t)$ Eq. (2.17) holds and for $S(t)$ (2.15) holds. For the derivative $\mathrm{d}y/\mathrm{d}t$ we have, using (2.15) and (2.17):

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}\langle \Delta z, (-S\Gamma S + R)\Delta z \rangle + \langle \Delta z, SBw \rangle, \tag{2.18}$$

where $\Gamma = BP^{-1}B'$. After some transformations we deduce

$$\frac{\mathrm{d}y}{\mathrm{d}t} = \frac{1}{2}\langle z, Rz \rangle + \frac{1}{2}\langle w, Pw \rangle + \|\Delta w - P^{-1}B'S\Delta z\|_P^2. \tag{2.19}$$

By integrating of the last equality on the interval $[t_0, T]$ and using conditions $\Delta z(t_0) = 0$ and $S(T) = 0$, we receive the formula (2.16).

Thus, if Riccati's equation (2.15) has a solution on the interval $[t_0, T]$, then the formula (2.16) holds and, therefore, $\Delta J \geq 0$. The following statement is proved.

LEMMA 3 *The control* (2.12), (2.13) *is optimal, if Riccati's equation* (2.15) *has a solution on the interval* $[t_0, T]$.

To compare the conditions of existence (Lemma 2) with necessary Pontryagin's conditions (2.13), (2.14) and sufficient conditions of optimality (Lemma 3), we need a lemma which establishes a connection between the canonical Eq. (2.11) and Riccati's equation (2.15).

Consider the canonical system

$$\frac{\mathrm{d}z}{\mathrm{d}t} = Az + \Gamma p, \quad \frac{\mathrm{d}p}{\mathrm{d}t} = Rz - A'p, \quad z(\tau) = z^0, \; p(T) = 0, \tag{2.20}$$

where $\tau \in (t_0, T)$, matrix equations

$$\frac{\mathrm{d}\Phi}{\mathrm{d}t} = A\Phi + \Gamma\Psi, \quad \frac{\mathrm{d}\Psi}{\mathrm{d}t} = R\Phi - A'\Psi, \quad \Phi(T) = I, \; \Psi(T) = 0, \tag{2.21}$$

where the I is the unity matrix, and Riccati's equation

$$\frac{\mathrm{d}S}{\mathrm{d}t} + SA + A'S + S\Gamma S = R, \quad S(T) = 0, \; t \in (t_0, T). \tag{2.22}$$

LEMMA 4 *The following four propositions are equivalent:*

(1) *Equations* (2.20) *for* $\tau = t_0$ *have a solution for each* z^0 *and a nontrivial solution* $z(t)$ *is not equal to* 0 *in inner points of the interval* (t_0, T).

(2) *For solutions of Eqs.* (2.21)

$$\det \Phi(t) \neq 0, \quad t \in (t_0, T).$$

(3) *The Riccati's equation* (2.22) *has a solution on* $[t_0, T]$ *and* $p(t) = S(t)z(t)$.

(4) *Equations* (2.20) *for each* $\tau \in (t_0, T)$ *have the unique solution for all* z^0, *and in particular there is only the trivial solution,* $z(t) = 0$ *for* $z^0 = 0$.

Proof of Lemma 4 At the beginning, we prove the equivalence of propositions 1, 2 and 3. It is sufficient for this purpose to prove that

1 implies 2, 2 implies 3, and 3 implies 1. After that we prove that 2 implies 4 and 4 implies 2.

If 1 is true, then it is possible to write down the general solution of Eqs. (2.20) through the boundary values $z(T)=z^\wedge, p(T)=p^\wedge$:

$$z(t) = \Phi(t)z^\wedge + \Phi'(t)p^\wedge, \quad p(t) = \Psi(t)z^\wedge + \Psi'(t)p^\wedge,$$

where Φ, Φ', Ψ, Ψ' are the corresponding submatrices of the fundamental matrix. Taking into account that $p^\wedge = 0$ and that the fundamental matrix is unity for $t = T$ and, therefore, $\Phi(T) = I, \Psi(T) = 0$, we have

$$z(t) = \Phi(t)z^\wedge, \quad p(t) = \Psi(t)z^\wedge, \quad \Phi(T) = I, \quad \Psi(T) = 0 \qquad (2.23)$$

Thus, any solution of Eqs. (2.20) with $p(T) = 0$ has the form (2.23). Substituting (2.23) into (2.20) and taking into account that the vector $z^\wedge$ is an arbitrary one, we obtain Eqs. (2.21). If $\det \Phi(t)$ were 0 for some inner point t, then such $z^\wedge$ would exist that for this t we would have $z(t) = \Phi(t)z^\wedge = 0$. It is a contradiction to proposition 1 of the lemma. It is proved that 1 implies 2.

If 2 is true, consider the matrix $S = \Psi\Phi^{-1}$. Calculating the derivative $\mathrm{d}S/\mathrm{d}t$, by virtue (2.21) we receive (2.22). From $z^\wedge = \Phi^{-1}z$, we conclude that $p = \Psi z^\wedge = \Psi\Phi^{-1}z = Sz$. Thus, 2 implies 3.

If 3 is true, then $p(t) = S(t)z(t)$, where $z(t)$ is the solution of the equation

$$\frac{\mathrm{d}z}{\mathrm{d}t} = (A + \Gamma S)z, \quad z(t_0) = z^0$$

and $S(t)$ is the solution of the Riccati's equation (2.22). This pair $z(t), p(t)$ is a solution of Eqs. (2.20). If the solution is nontrivial, then $z(t)$ cannot be equal to 0 at a point t, because in this case $p(t)$ would be equal to 0 at the same point, and the solution could not be nontrivial. Thus 3 implies 1. The equivalence of 1, 2 and 3 is proved.

Now we prove that 2 implies 4. Let 2 be fulfilled. The uniqueness of a solution is obviously deduced from the uniqueness of the trivial solution in the case $z^0 = 0$. We prove this last statement. Each solution $z(t)$ can be represented in the form (2.23): $z(t) = \Phi(t)z^\wedge$. The boundary condition $z(\tau) = 0$ for $\tau \in [t_0, T]$ gives $\Phi(\tau)z^\wedge = 0$ and, by virtue of the condition 2, we have $z^\wedge = 0$, i.e., the solution is trivial. The implication "2 implies 4" is proved.

At last we prove that 4 implies 2. If 4 is fulfilled then only trivial solution $z^\wedge = 0$ is possible as a solution of linear algebraic equations $\Phi(t)z^\wedge = 0$. It means that $\det \Phi(t) \neq 0$ for $t \in [t_0, T]$ and therefore 2 is fulfilled. The lemma is proved.

The next statement is a corollary of Lemma 4.

LEMMA 5 *If an optimal control exists for each initial data, then Riccati's equation* (2.6) *has a solution for* $t \in [t_0, T]$.

The statement of Lemma 5 is the implication of Lemma 2 and of the equivalence of the assertions 1 and 3 of the Lemma 4, if one takes into account that nonexistence of solutions of Eqs. (2.14), that is, the nonexistence of controls which obey Pontryagin's maximum principle, means the nonexistence of the optimal control.

Proof of Theorems 1 *and* 2 To prove Theorem 1, we use Lemma 1 for the reduction of the existence problem to the homogeneous case, then use Pontryagin's maximum principle, Lemma 5 and at last Lemma 3. To prove Theorem 2, we should in addition use the equivalence of assertions 3 and 4 of Lemma 4.

3 REFORMULATION OF MAIN RESULTS OF *LQ*-OPTIMIZATION

To reformulate results of *LQ*-optimization in general terms, we consider a general optimization problem with the free right end of the trajectory. The minimized functional has the form:

$$\int_{t_0}^{T} F(z, w, t)\, \mathrm{d}t, \tag{3.1}$$

where $w(t)$ is a control, which belongs to some closed set W, $z(t) = \{z_1(t), \ldots, z_n(t)\}$ is a phase trajectory, which obeys the equations

$$\frac{\mathrm{d}z_i}{\mathrm{d}t} = f_i(z, w, t),\ z_i(t_0) = z_i^0,\quad w(t) \in W,\ i = 1, \ldots, n. \tag{3.2}$$

Remind some well-known definitions. The function

$$H(z,p,w,t) = \Sigma p_i f_i(z,w,t) - F(z,w,t) \tag{3.3}$$

is called Hamiltonian. The system of equations

$$\frac{dz_i}{dt} = \frac{\partial H}{\partial p_i} \frac{dp_i}{dt} = -\frac{\partial H}{\partial x_i}, \quad z_i(\tau) = z_i^0, \; p_i(T) = 0 \tag{3.4}$$

is called the canonical one. Here $t_0 \leq \tau \leq T$. If a control $w^\wedge(t)$ is fixed for $t \in [\tau, T]$, then the canonical system with the boundary condition $z_i(\tau) = z_i^0$, determines for $t \in [\tau, T]$ the unique functions $z^\wedge(t)$ and $p^\wedge(t)$, which we call corresponding to the control $w^\wedge(t)$. We say that a control $w^*(t) \in W$ obeys the Pontryagin's maximum condition on the interval $[\tau, T]$ with the boundary condition $z_i(\tau) = z_i^0$ if

$$H(z^*(t), p^*(t), w^*(t), t) \geq H(z^*(t), p^*(t), w, t) \quad \text{for all } w \in W, \tag{3.5}$$

where functions $z^*(t), p^*(t)$ are corresponding to the control $w^*(t)$ for $t \in [\tau, T]$.

We write down now the Bellman's equation. Let $Q(x, \tau)$ be the Bellman's function, that is an optimal functional value in the problem with the initial data $z_i(\tau) = x_i$. Denote by $\nabla_x Q$ the vector which components are the partial derivatives $\partial Q / \partial x_i$. The Bellman's equation has the form:

$$\frac{\partial Q}{\partial \tau} + \max\{H(x, \nabla_x Q, w, \tau)\} = 0, \tag{3.6}$$

where the maximum is taken with respect to $w \in W$. The boundary value of the Bellman's function is:

$$Q(x, T) = 0. \tag{3.7}$$

For the LQ-optimization we have:

$$f_i(z, w, t) = \Sigma a_{ik}(t) z_k + \Sigma b_{is}(t) w_s, \tag{3.8}$$

where $a_{ik}(t)$ and $b_{is}(t)$ are elements of matrices $A(t)$ and $B(t)$ in (2.1). Besides

$$F(z, w, t) = \tfrac{1}{2}\Sigma\Sigma R_{ik} z_i z_k + \tfrac{1}{2}\Sigma\Sigma P_{sq} w_s w_q. \tag{3.9}$$

If we calculate the maximum in (3.6) taking into account notations (3.3), (3.8), (3.9), we will come to the next equation in partial derivatives:

$$\frac{\partial Q}{\partial \tau} + \Sigma\Sigma \partial Q/\partial x_i a_{ik} x_k + \Sigma\Sigma G_{ik} \frac{\partial Q}{\partial x_i}\frac{\partial Q}{\partial x_k} + \tfrac{1}{2}\Sigma\Sigma R_{ik} x_i x_k = 0, \tag{3.10}$$

where G_{ik} are elements of the matrix $G = BP^{-1}B'$. It is easy to check that the quadratic form

$$Q = \tfrac{1}{2}\Sigma\Sigma S_{ik}(\tau) x_i x_k \tag{3.11}$$

gives a solution of the Bellman's equation (3.10) with the boundary value (3.7), if the matrix $S(\tau)$ obeys the Riccati's equation

$$\frac{\mathrm{d}S}{\mathrm{d}\tau} + SA + A'S + SGS = R, \quad S(T) = 0, \ \tau \in (t_0, T). \tag{3.12}$$

Thus, we come to the following statement:

STATEMENT 1 *If the Riccati's equation* (3.12) *for the LQ-problem has a solution for* $\tau \in (t_0, T)$, *then the Bellman's equation* (3.6) *for this problem has a solution in the region* $t_0 \le \tau \le T$.

The connection between Riccati equation and Hamilton–Jacobi (i.e. Bellman) equation was underlined by R.E. Kalman many years ago [3].

The statement asserts that the existence of a solution of the Riccati's equation means the existence of a solution of the Bellman's equation. Therefore Lemma 5 may be reformulated as follows:

STATEMENT 2 *If an optimal control in LQ-problem exists for each initial datum, then the Bellman's equation has a solution in the region* $t_0 \le \tau \le T$.

The equivalence of assertions 3 and 4 of Lemma 4 leads to the statement:

STATEMENT 3 *The Bellman's equation in LQ-problem has a solution in the region* $t_0 \leq \tau \leq T$ *iff for each boundary condition* $z(\tau) = z^0$ *(i.e., for each* τ, $t_0 \leq \tau \leq T$, *and each* $= z^0$*) there exists the unique control obeying the Pontryagin's maximum condition.*

Reformulation of Theorem 2 leads to the next statement.

STATEMENT 4 *If in LQ-problem for each boundary condition* $z(\tau) = z^0$ *there exists a unique control obeying Pontryagin's maximum condition, then such control is optimal.*

4 FUTURE DIRECTION: SOME GENERAL HYPOTHESES ON OPTIMAL CONTROL

Statements 1–4 on *LQ*-optimization and well-known results on sufficient condition of optimality in terms of Bellman's equation (cf. Lemma 3 above and theorems in [4]) lead to the following general hypotheses on optimal control:

HYPOTHESIS 1 *The necessary and sufficient condition for existence of optimal control for all initial data is the existence of a solution of Bellman's equation* (3.6), *with the boundary condition* (3.7) *in the region* $t_0 \leq \tau \leq T$.

HYPOTHESIS 2 *A solution of Bellman's equation* (3.6), *with the boundary condition* (3.7) *in the region* $t_0 \leq \tau \leq T$ *exists iff for each boundary condition* $z(\tau) = z^0$ *there exists a unique control, which obeys Pontryagin's maximum condition.*

HYPOTHESIS 3 *If for each boundary condition* $z(\tau) = z^0$ *there exists a unique control, which obeys Pontryagin's maximum condition, then such a control is optimal.*

Some refinements of these hypotheses must be, of course, made, in particular, relating to the tangent property (smoothness) of solutions of the Bellman's equation.

References

[1] B.D.O. Anderson and J.B. Moore. *Linear Optimal Control*, Englewood Cliffs, Prentice-Hall, 1971.
[2] T. Basar and P. Bernhard. *H^{∞}-Optimal Control and Related Minmax Design Problems – A Dynamic Game Approach*, Birkhauser, Boston MA, 1991.
[3] R.E. Kalman. Contributions to the theory of optimal control. *Boletin de la Sociedad Matematica Mexicana*, Vol. 5, pp. 102–119, 1960.
[4] V.G. Boltianskii. *Mathematical Methods of Optimal Control*, N.Y. Holt, Keinhart and Winston, 1971.