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A River Water Quality Model for Time Varying BOD Discharge Concentration

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We consider a model for biochemical oxygen demand (BOD) in a semi-infinite river where the BOD is prescribed by a time varying function at the left endpoint. That is, we study the problem with a time varying boundary loading. We obtain the well-posedness for the model when the boundary loading is smooth in time. We also obtain various qualitative results such as ordering, positivity, and boundedness. Of greatest interest, we show that a periodic loading function admits a unique asymptotically attracting periodic solution. For non-smooth loading functions, we obtain weak solutions. Finally, for certain special cases, we show how to obtain explicit solutions in the form of infinite series.

Keywords: River model; Environmental modeling; Weak solutions; Quasi-steady-state; Convection–diffusion equations

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1 INTRODUCTION

In this work we carefully examine a standard model for water quality in a river with a time varying boundary loading [2]. In the second section we obtain a full well-posedness result for smooth loading

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functions. In the third section, qualitative properties such as ordering, positivity, and boundedness are obtained. We also show that a periodic forcing function admits a unique periodic solution (quasi-steady-state) and that this solution is globally asymptotically stable. In Section 4, we permit loading functions that are merely bounded and locally square integrable. We obtain the existence and uniqueness of weak solutions. Finally, in Section 5, we obtain explicit analytic solutions to a water quality model with periodic boundary loading and constant initial data. We shall do this by finding such solutions in the form of infinite series.

We will consider the amount of organic waste in a river, as measured by biochemical oxygen demand (BOD). We assume that symmetry across the width of the river holds so that we may model the river as being one-dimensional. The river is flowing with a uniform velocity of $u > 0$ and a waste treatment facility is assumed to discharge at a point of the river. The location of the treatment facility's discharge will be given the x spatial coordinate of 0 and the river will flow downstream in an increasing x direction. The function of position, x, and time, t, $w(x, t)$ in units of mass(length)$^{-3}$ will measure BOD. We assume that the organic material diffuses with diffusivity $D > 0$ with units of (length)^2(time)$^{-1}$. Furthermore, the organic material will decay with a rate proportional to its concentration with proportionality constant $k_1 > 0$ with units of (time)$^{-1}$. Finally, lands bordering the river will discharge BOD into the river at a rate of $c \geq 0$ in units of mass(length)$^{-3}$(time)$^{-1}$.

Putting the above physical mechanisms together yields the following partial differential equation:

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} + D\frac{\partial^2 w}{\partial x^2} - k_1 w + c, \quad \text{for } t > 0, \text{ and } x \in (0, \infty). \tag{1}$$

We assume that as we move away from the waste discharge location, the BOD approaches its natural equilibrium. That is

$$\lim_{x \to \infty} w(x, t) = \frac{c}{k_1} \quad \text{for all } t > 0. \tag{2}$$

If the BOD discharge is given by a function $f(t)$, with units of mass(length)$^{-3}$ and the initial distribution of BOD at $t = 0$, $w_0(x)$, is known, we obtain the last two equations of the model:

$$w(0, t) = f(t) \tag{3}$$

and

$$w(x,0) = w_0(x), \quad x \in (0,\infty). \tag{4}$$

2 CLASSICAL SOLUTIONS

We will first transform (1)–(4) into a system with homogeneous boundary conditions. Then we will examine the relevant spatial operator and show that it generates an analytic semigroup on an appropriate Hilbert space. This will allow us to obtain our well-posedness result.

First the transformation:

We will consider the following system:

$$\frac{\partial v}{\partial t} = -u\frac{\partial v}{\partial x} + D\frac{\partial^2 v}{\partial x^2} - k_1 v + (u + D - k_1)\left(f(t) - \frac{c}{k_1}\right)\mathrm{e}^{-x} - f'(t)\mathrm{e}^{-x},$$

$$\text{for } t > 0, \text{ and } x \in (0,\infty), \tag{5}$$

$$\lim_{x\to\infty} v(x,t) = 0 \quad \text{for all } t > 0, \tag{6}$$

$$v(0,t) = 0, \tag{7}$$

and

$$v(x,0) = w_0(x) - \left(f(0) - \frac{c}{k_1}\right)\mathrm{e}^{-x} - \frac{c}{k_1}. \tag{8}$$

A simple computation will show that if v satisfies (5)–(8), then

$$w(x,t) = v(x,t) + \left(f(t) - \frac{c}{k_1}\right)\mathrm{e}^{-x} + \frac{c}{k_1} \tag{9}$$

will be a solution to (1)–(4).

Our mathematical tools may be found in [4] and [5]. The following is actually the concatenation of two theorems found in [5].

THEOREM 1 *Let V and H be Hilbert spaces for which the identity $V \hookrightarrow H$ is continuous. Let $a\colon V \times V \to \mathbb{C}$ be continuous, sesquilinear, and V-elliptic. In particular, there are constants c and K such that $0 < c \le K$ and*

$$a(w,v) \le K\|w\|_V\,\|v\|_V, \quad \textit{for all } w \textit{ and } v \textit{ in } V$$

and

$$\operatorname{Re} a(v,v) \geq c\|v\|_V^2, \quad \textit{for all } v \textit{ in } V.$$

Define

$$D(A) = \{w \in V\colon\ |a(w,v)| \leq K_w|v|_H, \quad v \in V\},$$

where K_w depends only on w and let A, a linear function from $D(A)$ to H, be given by

$$(Aw, v)_H = a(w,v), \quad w \in D(A),\ v \in V.$$

Then $D(A)$ is dense in H and $-A$ generates an analytic semigroup on H.

In order to apply this theorem to problem (5)–(8), we must choose appropriate spaces H and V, and pick a sesquilinear form a so that the operator we define from it is given by $Av = u(\partial v/\partial x) - D(\partial^2 v/\partial x^2) + k_1 v$. The space H will be the space of measurable functions on $[0,\infty)$ such that $\int_0^\infty |v|^2\,\mathrm{d}x < \infty$ for each v in H. That is $H = L^2([0,\infty))$. The inner product on H is given by $(w,v)_H = \int_0^\infty w(x)\bar{v}(x)\,\mathrm{d}x$. The space V will consist of those elements of H, v, whose distributional derivatives $\partial v/\partial x$ are also in H and satisfy $v(0) = 0$ and $\lim_{x\to\infty} v(x) = 0$. The inner product on V is $(w,v)_V = \int_0^\infty (\partial w(x)/\partial x)(\overline{\partial v}(x)/\partial x) + w(x)\bar{v}(x)\,\mathrm{d}x$; that is $V = W_0^{1,2}([0,\infty))$. The required imbedding is a standard theorem of Sobolev spaces [1]. We will also find the following metric spaces useful: $M_j = \{v + (c/k_1)\colon v \in W^{2,j}[0,\infty)\}$ with metric $d(w_1,w_2) = \|w_1 - w_2\|_{2,j}$. We define the form a on $V \times V$ by

$$a(w,v) = \int_0^\infty u\frac{\partial w}{\partial x}(x)\bar{v}(x) + D\frac{\partial w}{\partial x}(x)\frac{\overline{\partial v}}{\partial x}(x) + k_1 w(x)\bar{v}(x)\,\mathrm{d}x.$$

According to Theorem 1, we need to obtain two estimates. The first of these is the continuity estimate. We obtain this now,

$$\begin{aligned}
|a(w,v)| &\leq \int_0^\infty u\left|\frac{\partial w}{\partial x}\right||v| + D\left|\frac{\partial w}{\partial x}\right|\left|\frac{\partial v}{\partial x}\right| + k_1|w||v|\,\mathrm{d}x \\
&\leq u\left\|\frac{\partial w}{\partial x}\right\|_H \|v\|_H + D\left\|\frac{\partial w}{\partial x}\right\|_H\left\|\frac{\partial v}{\partial x}\right\|_H + k_1\|w\|_H\|v\|_H \\
&\leq u\|w\|_V\|v\|_V + D\|w\|_V\|v\|_V + k_1\|w\|_V\|v\|_V \\
&= (u + D + k_1)\|w\|_V\|v\|_V,
\end{aligned}$$

where we have used Hölder's inequality, the fact that

$$\sqrt{\left\|\frac{\partial w}{\partial x}\right\|_H^2 + \|w\|_H^2} = \|w\|_V,$$

and the inequality $2\alpha\beta \le \alpha^2 + \beta^2$.

The second estimate that we need is the coercivity estimate. This is obtained using the following estimates:

$$\begin{aligned}
\operatorname{Re} a(w,w) &= \int_0^\infty u \operatorname{Re}\frac{\partial w}{\partial x}(x)\overline{w}(x) + D\operatorname{Re}\frac{\partial w}{\partial x}(x)\overline{\frac{\partial w}{\partial x}}(x) \\
&\quad + k_1 \operatorname{Re} w(x)\overline{w}(x)\,\mathrm{d}x \\
&= \int_0^\infty \frac{u}{2}\frac{\partial}{\partial x}|w|^2 + D\left|\frac{\partial w}{\partial x}\right|^2 + k_1|w|^2\,\mathrm{d}x \\
&\ge \frac{u}{2}[|w(x)|^2]_0^\infty + \min(D,k_1)\|w\|_V^2 \\
&= \min(D,k_1)\|w\|_V^2.
\end{aligned}$$

We have now shown that the operator $-A$ defined in Theorem 1 generates an analytic semigroup on H.

To see that the operator A obtained above is the one we desire, we will integrate against some test functions.

First, let ϕ be any compactly supported infinitely differentiable function on $[0,\infty]$. Then we have for any w in $D(A)$ there is a constant $K_w > 0$ so that

$$\begin{aligned}
K_w\|\phi\| &\ge |a(w,\phi)| \\
&= \left|\int_0^\infty u\frac{\partial w}{\partial x}(x)\overline{\phi}(x) + D\frac{\partial w}{\partial x}(x)\overline{\frac{\partial \phi}{\partial x}}(x) + k_1 w(x)\overline{\phi}(x)\,\mathrm{d}x\right| \\
&= \left|\int_0^\infty \left(u\frac{\partial w}{\partial x} - D\frac{\partial^2 w}{\partial x^2} + k_1 w\right)\overline{\phi}\,\mathrm{d}x\right|,
\end{aligned}$$

where we have used the definition of distributional derivative. Since $u(\partial w/\partial x) - D(\partial^2 w/\partial x^2) + k_1 w$ defines a continuous linear functional on H, it must be in H. Since $u(\partial w/\partial x) + k_1 w$ is already in H, this implies that $\partial^2 w/\partial x^2$ is in H. Furthermore, let B be defined as function from $D(A)$ into H defined by

$$Bw = u\frac{\partial w}{\partial x}(x) + D\frac{\partial^2 w}{\partial x^2}(x) + k_1 w(x).$$

Then

$$\begin{aligned}(Aw,\phi)_H - (Bw,\phi)_H &= a(w,\phi) - (Bw,\phi)_H \\ &= \int_0^\infty u\frac{\partial w}{\partial x}(x)\overline{\phi}(x) + D\frac{\partial w}{\partial x}(x)\overline{\frac{\partial \phi}{\partial x}}(x) \\ &\quad + k_1 w(x)\overline{\phi}(x)\,\mathrm{d}x - \int_0^\infty\left(u\frac{\partial w}{\partial x} - D\frac{\partial^2 w}{\partial x^2} + k_1 w\right)\overline{\phi}\,\mathrm{d}x \\ &= 0\end{aligned}$$

where we have integrated by parts. Since $C_0^\infty[0,\infty)$ is dense in H, we have $Aw = Bw = u(\partial w(x)/\partial x) + D(\partial^2 w(x)/\partial x^2) + k_1 w(x)$ on $D(A) = \{w \in V\colon (\partial^2 w/\partial x^2) \in H\}$. Notice that we automatically obtain that elements of $D(A)$ are continuously differentiable on $(0,\infty)$.

All of the above and the properties of analytic semigroups found in [4] yields the following theorem.

THEOREM 2 *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be differentiable with Hölder continuous derivative. Then for each* w_0 *in* M_0, *Eqs.* (5)–(9) *has a unique classical solution.*

Above, by classical solution, we mean a function $v\colon [0,\infty) \to H$ which is continuous on $[0,\infty)$, continuously differentiable on $(0,\infty)$, $v(t)$ is in $D(A)$ for $t > 0$, and v satisfies (5) for $t > 0$.

It is easy to see that the following corollary is true.

COROLLARY 3 *Let* $f\colon [0,\infty) \to \mathbb{R}$ *be differentiable with Hölder continuous derivative and* v *the solution to Eqs.* (5)–(9) *given by Theorem* 2 *for some* w_0 *in* M_0. *If we define* w *by* $w(x,t) = v(x,t) + (f(t) - c/k_1)\mathrm{e}^{-x} + c/k_1$, *then* w *is the* (*unique*) *solution to* (1)–(4).

By solution here, we mean that $w\colon [0,\infty) \to M_0$ is continuous on $[0,\infty)$ and continuously differentiable on $(0,\infty)$. Note also that for $t > 0$, $\partial^2 w/\partial x^2$ is in M_2.

3 QUALITATIVE PROPERTIES OF SOLUTIONS

We now give some qualitative results on solutions. In particular we will obtain ordering, positivity, and boundedness results.

THEOREM 4 *Let* w^1 *and* w^2 *be solutions to problem* (1)–(4) *with data* f^1, w_0^1 *and* f^2, w_0^2 *respectively. If* $f^1 \leq f^2$ *a.e. and* $w_0^1 \leq w_0^2$ *a.e., then for each* $t > 0$, $w^1(t) \leq w^2(t)$ *a.e. on* $[0, \infty)$.

Proof We will use two approximate functions to help us. The first is an approximation of

$$sgn_0^+(t) = \begin{cases} 1 & t > 0, \\ 0 & t \leq 0, \end{cases}$$

and is defined by

$$\rho_k(t) = \begin{cases} 1 & t > 1/k, \\ kt & 0 < t \leq 1/k, \\ 0 & t \leq 0 \end{cases}$$

for each natural number k. The second is an approximation to t^+ and is defined by

$$\sigma_k(t) = \int_0^t \rho_k(\tau)\,\mathrm{d}\tau$$

Then by subtracting the differential equations satisfied by w^1 and w^2 we obtain

$$\frac{\partial}{\partial t}(w^1 - w^2) = -u\frac{\partial}{\partial x}(w^1 - w^2) + D\frac{\partial^2}{\partial x^2}(w^1 - w^2) - k_1(w^1 - w^2)\cdot$$

Multiplying by $\rho_k(w^1 - w^2)$ and integrating in x from 0 to ∞ yields

$$\begin{aligned}
&\int_0^\infty \left[\frac{\partial}{\partial t}(w^1 - w^2)\right]\rho_k(w^1 - w^2)\,\mathrm{d}x \\
&\quad = -u\int_0^\infty \frac{\partial}{\partial x}(w^1 - w^2)\rho_k(w^1 - w^2)\,\mathrm{d}x \\
&\qquad + D\int_0^\infty \frac{\partial^2}{\partial x^2}(w^1 - w^2)\rho_k(w^1 - w^2)\,\mathrm{d}x \\
&\qquad - k_1\int_0^\infty (w^1 - w^2)\rho_k(w^1 - w^2)\,\mathrm{d}x
\end{aligned}$$

Simplifying and integrating by parts, we obtain

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^\infty \sigma_k(w^1-w^2)\,\mathrm{d}x &= -u\sigma_k(w^1-w^2)|_{x=0}^\infty \\ &\quad + D\left[\frac{\partial}{\partial x}(w^1-w^2)\rho_k(w^1-w^2)\right]_{x=0}^\infty \\ &\quad - D\int_0^\infty\left[\frac{\partial}{\partial x}(w^1-w^2)\right]^2\rho_k'(w^1-w^2)\,\mathrm{d}x \\ &\quad - k_1\int_0^\infty (w^1-w^2)\rho_k(w^1-w^2)\,\mathrm{d}x\end{aligned}$$

Noting that the first two terms on the right are 0 and the integrals are nonnegative, we see that

$$\frac{\partial}{\partial t}\int_0^\infty \sigma_k(w^1-w^2)\,\mathrm{d}x \le 0.$$

Integrating from 0 to t yields

$$\int_0^\infty \sigma_k(w^1(x,t)-w^2(x,t))\,\mathrm{d}x - \int_0^\infty \sigma_k(w^1(x,0)-w^2(x,0))\,\mathrm{d}x \le 0.$$

Since $\sigma_k(w^1(x{,}0)-w^2(x{,}0))=0$ a.e., we see that

$$\int_0^\infty \sigma_k(w^1(x,t)-w^2(x,t))\,\mathrm{d}x \le 0.$$

Letting k go to ∞ and applying the Lebesgue dominated convergence theorem yields

$$\int_0^\infty (w^1(x,t)-w^2(x,t))^+\,\mathrm{d}x \le 0$$

and hence $w^1(x,t)\le w^2(x,t)$ for a.e. x and all $t\ge 0$.

COROLLARY 5 *Let w be the solution to problem* (1)–(4) *where $f\ge 0$ on* $[0,\infty)$ *and $w_0\ge 0$ on* $[0,\infty)$. *Then $w(x,t)\ge 0$ for $t\ge 0$ and a.e. $x\in[0,\infty)$.*

Proof In light of the previous result it will suffice to show that $w(x,t) \geq 0$ for $t \geq 0$ and a.e. $x \in [0,\infty)$ when $f=0$ on $[0,\infty)$ and $w_0 \in [0, c/k_1]$ on $[0,\infty)$. We will, in fact, show that $c/k_1 \geq w(x,t) \geq 0$. Multiply

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} + D\frac{\partial^2 w}{\partial x^2} - k_1 w + c$$

by $\rho_k(w - c/k_1)$, where ρ_k is as defined in the previous proof, and integrate with respect to x from 0 to ∞. After integrating by parts, this will yield

$$\begin{aligned}\frac{\partial}{\partial t}\int_0^\infty \sigma_k\left(w - \frac{c}{k_1}\right)\mathrm{d}x = {} & -u\sigma_k\left(w - \frac{c}{k_1}\right)\Bigg|_{x=0}^{\infty} + D\left[\frac{\partial w}{\partial x}\rho_k\left(w - \frac{c}{k_1}\right)\right]_{x=0}^{\infty} \\ & - D\int_0^\infty \left[\frac{\partial w}{\partial x}\right]^2 \rho_k'\left(w - \frac{c}{k_1}\right)\mathrm{d}x \\ & - k_1\int_0^\infty \left(w - \frac{c}{k_1}\right)\rho_k\left(w - \frac{c}{k_1}\right)\mathrm{d}x.\end{aligned}$$

This first and second terms on the right are 0, the two integrals are both nonnegative. Thus

$$\frac{\partial}{\partial t}\int_0^\infty \sigma_k\left(w - \frac{c}{k_1}\right)\mathrm{d}x \leq 0$$

and

$$\int_0^\infty \sigma_k\left(w(x,t) - \frac{c}{k_1}\right)\mathrm{d}x \leq \int_0^\infty \sigma_k\left(w(x,0) - \frac{c}{k_1}\right)\mathrm{d}x = 0.$$

Letting k go to ∞ yields $\int_0^\infty (w(x,t) - (c/k_1))^+\,\mathrm{d}x \leq 0$ and thus $w(x,t) \leq c/k_1$ for $t \geq 0$ and a.e. $x \in [0,\infty)$. Now multiply

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} + D\frac{\partial^2 w}{\partial x^2} - k_1 w + c$$

by $\rho_k(-w)$ and integrate with respect to x from 0 to ∞. After integrating by parts this will yield

$$-\frac{\partial}{\partial t}\int_0^\infty \sigma_k(-w)\,\mathrm{d}x = u\sigma_k(-w)|_{x=0}^\infty + D\left[\frac{\partial w}{\partial x}\rho_k(-w)\right]_{x=0}^\infty + D\int_0^\infty \left[\frac{\partial w}{\partial x}\right]^2 \rho_k'(-w)\,\mathrm{d}x - k_1\int_0^\infty \left(w - \frac{c}{k_1}\right)\rho_k(-w)\,\mathrm{d}x.$$

The first two terms on the right are 0. Then the first integral $\int_0^\infty [\partial w/\partial x]^2 \rho_k'(-w)\,\mathrm{d}x$ is positive and since $w(x,t) \leq c/k_1$, $(w - c/k_1)\rho_k(-w)$ is nonpositive. Thus

$$-\frac{\partial}{\partial t}\int_0^\infty \sigma_k(-w)\,\mathrm{d}x \geq 0$$

and

$$\int_0^\infty \sigma_k(-w(x,t))\,\mathrm{d}x \leq \int_0^\infty \sigma_k(-w(x,0))\,\mathrm{d}x = 0.$$

Letting k tend to ∞ yields

$$\int_0^\infty (-w(x,t))^+\,\mathrm{d}x \leq 0,$$

and thus $w(x,t) \geq 0$ for $t \geq 0$ and a.e. $x \in [0,\infty)$.

THEOREM 6 *Let w be the solution to problem* (1)–(4) *where $f \geq 0$ on $[0,\infty)$ is in $L^\infty(0,\infty)$ and $w_0 \geq 0$ on $[0,\infty)$. Then $w(x,t)$ is essentially bounded on $[0,\infty) \times [0,\infty)$ with bound*

$$\|w(\cdot,t)\|_{L^\infty(0,\infty)} \leq \frac{c}{k_1} + \max\left\{\left\|w_0(\cdot) - \frac{c}{k_1}\right\|_{L^\infty(0,\infty)}, \left\|f(\cdot) - \frac{c}{k_1}\right\|_{L^\infty(0,t)}\right\}.$$

Proof We will use two approximate functions to help us. The first is an approximation of

$$sgn_0(t) = \begin{cases} 1 & t > 0, \\ 0 & t = 0, \\ -1 & t < 0 \end{cases}$$

and is defined by

$$r_k(t) = \begin{cases} 1 & t > 1/k, \\ kt & -1/k < t \leq 1/k, \\ -1 & t \leq -1 \end{cases}$$

for each natural number k. The second is an approximation to $|t|$ and is defined by

$$s_k(t) = \int_0^t r_k(\tau)\,\mathrm{d}\tau.$$

Let $p > 1$ and multiply

$$\frac{\partial w}{\partial t} = -u\frac{\partial w}{\partial x} + D\frac{\partial^2 w}{\partial x^2} - k_1 w + c$$

by $r_k(w - c/k_1)[s_k(w - c/k_1)]^{p-1}$, and integrate with respect to x from 0 to ∞. After integrating by parts, this will yield

$$\begin{aligned}
&\frac{1}{p}\frac{\partial}{\partial t}\int_0^\infty \left[s_k\left(w - \frac{c}{k_1}\right)\right]^p \mathrm{d}x \\
&\quad = -\frac{u}{p}\left[s_k\left(w - \frac{c}{k_1}\right)\right]^p\Bigg|_{x=0}^{\infty} + D\left[\frac{\partial w}{\partial x} r_k\left(w - \frac{c}{k_1}\right)\left[s_k\left(w - \frac{c}{k_1}\right)\right]^{p-1}\right]_{x=0}^{\infty} \\
&\qquad - D\int_0^\infty \left[\frac{\partial w}{\partial x}\right]^2 \left(r_k'\left(w - \frac{c}{k_1}\right)\left[s_k\left(w - \frac{c}{k_1}\right)\right]^{p-1}\right. \\
&\qquad \left. + \left(r_k'\left(w - \frac{c}{k_1}\right)\right)^2 \left[s_k\left(w - \frac{c}{k_1}\right)\right]^{p-2}\right)\mathrm{d}x \\
&\qquad - k_1\int_0^\infty \left(w - \frac{c}{k_1}\right) r_k\left(w - \frac{c}{k_1}\right)\left[s_k\left(w - \frac{c}{k_1}\right)\right]^{p-1}\mathrm{d}x.
\end{aligned}$$

Noting that the first two terms are zero at ∞ and that the integrals are positive, we obtain

$$
\begin{aligned}
&\frac{1}{p}\frac{\partial}{\partial t}\int_0^\infty \left[s_k\left(w-\frac{c}{k_1}\right)\right]^p \mathrm{d}x \\
&\quad \leq \frac{u}{p}\left[s_k\left(f(t)-\frac{c}{k_1}\right)\right]^p - D\frac{\partial w}{\partial x} r_k\left(f(t)-\frac{c}{k_1}\right)\left[s_k\left(f(t)-\frac{c}{k_1}\right)\right]^{p-1} \\
&\quad \leq \frac{u}{p}\left[s_k\left(f(t)-\frac{c}{k_1}\right)\right]^p + D\left|\frac{\partial w}{\partial x}(0,t)\right|\left[s_k\left(f(t)-\frac{c}{k_1}\right)\right]^{p-1}.
\end{aligned}
$$

Integrating with respect to t from 0 to t (abusing the notation) we obtain

$$
\begin{aligned}
\int_0^\infty \left[s_k\left(w(x,t)-\frac{c}{k_1}\right)\right]^p \mathrm{d}x \leq{}& \int_0^\infty \left[s_k\left(w(x,0)-\frac{c}{k_1}\right)\right]^p \mathrm{d}x \\
&+ u\int_0^t \left[s_k\left(f(\tau)-\frac{c}{k_1}\right)\right]^p \mathrm{d}\tau \\
&+ pD\int_0^t \left|\frac{\partial w}{\partial x}(0,\tau)\right|\left[s_k\left(f(\tau)-\frac{c}{k_1}\right)\right]^{p-1} \mathrm{d}\tau.
\end{aligned}
$$

Now letting k tend towards ∞ and applying the Lebesgue dominated convergence theorem yields

$$
\begin{aligned}
\int_0^\infty \left|w(x,t)-\frac{c}{k_1}\right|^p \mathrm{d}x \leq{}& \int_0^\infty \left|w_0(x)-\frac{c}{k_1}\right|^p \mathrm{d}x \\
&+ u\int_0^t \left|f(\tau)-\frac{c}{k_1}\right|^p \mathrm{d}\tau \\
&+ pD\int_0^t \left|\frac{\partial w}{\partial x}(0,\tau)\right|\left|f(\tau)-\frac{c}{k_1}\right|^{p-1} \mathrm{d}\tau.
\end{aligned}
$$

Equivalently,

$$
\begin{aligned}
\left\|w(\cdot,t)-\frac{c}{k_1}\right\|_{L^p(0,\infty)}^p \leq{}& \left\|w_0(\cdot)-\frac{c}{k_1}\right\|_{L^p(0,\infty)}^p + u\left\|f(\cdot)-\frac{c}{k_1}\right\|_{L^p(0,t)}^p \\
&+ pD\left\|\frac{\partial w}{\partial x}(0,\cdot)\right\|_{L^\infty(0,t)}\left\|f(\cdot)-\frac{c}{k_1}\right\|_{L^p(0,t)}^{p-1}.
\end{aligned}
$$

Taking pth roots and letting p go to infinity yields

$$\left\| w(\cdot,t) - \frac{c}{k_1} \right\|^p_{L^\infty(0,\infty)} \le \max\left\{ \left\| w_0(\cdot) - \frac{c}{k_1} \right\|_{L^\infty(0,\infty)}, \left\| f(\cdot) - \frac{c}{k_1} \right\|_{L^\infty(0,t)} \right\}$$

and we are done.

If the loading function f is periodic, it is natural to ask whether or not there are periodic solutions to problem (1)–(4). Such periodic solutions, called quasi-steady-state solutions in the engineering literature [2], not only exist, but are unique and attracting.

PROPOSITION 7 *Let f have period T. Then there is a unique w_0 in M_0 so that the solution to problem* (1)–(4) *with this initial condition satisfies $w(x,T) = w_0(x)$ in M_0. Furthermore, if $\widehat{w}_0$ is any other initial condition with solution $\widehat{w}(x,t)$, then* $\lim_{t\to\infty} \|w(\cdot,t) - \widehat{w}(\cdot,t)\|_{L^2[0,\infty)} = 0$.

Proof We will obtain the existence of a periodic solution using the contraction mapping theorem. Define the map Ψ from M_0 into itself by $\Psi(\widehat{w}_0)(x) = \widehat{w}(x,T)$, where $\widehat{w}$ is the solution to problem (1)–(4) with initial condition $\widehat{w}_0$. If there is a positive α strictly less than one such that

$$\|\Psi(w_0) - \Psi(\widehat{w}_0)\|_{L^2[0,\infty)} \le \alpha \|w_0 - \widehat{w}_0\|_{L^2[0,\infty)}$$

then Ψ will have a unique fixed point which will be our desired periodic solution. We proceed as in previous proofs by subtracting the equations for w and $\widehat{w}$ from each other, multiplying by $w - \widehat{w}$ and integrating with respect to x,

$$\begin{aligned}
&\frac{1}{2}\frac{\partial}{\partial t}\|w(\cdot,t) - \widehat{w}(\cdot,t)\|^2_{L^2[0,\infty)} \\
&\quad = -u\int_0^\infty (w(x,t) - \widehat{w}(x,t))_x (w(x,t) - \widehat{w}(x,t))\,\mathrm{d}x \\
&\quad\quad + D\int_0^\infty (w(x,t) - \widehat{w}(x,t))_{xx}(w(x,t) - \widehat{w}(x,t))\,\mathrm{d}x \\
&\quad\quad - k_1\int_0^\infty (w(x,t) - \widehat{w}(x,t))^2\,\mathrm{d}x
\end{aligned}$$

$$
\begin{aligned}
&= -\frac{u}{2}[(w(x,t)-\widehat{w}(x,t))^2]_0^\infty \\
&\quad + D[(w(x,t)-\widehat{w}(x,t))_x(w(x,t)-\widehat{w}(x,t))]_0^\infty \\
&\quad - D\int_0^\infty [(w(x,t)-\widehat{w}(x,t))_x]^2\,\mathrm{d}x - k_1\|w(\cdot,t)-\widehat{w}(\cdot,t)\|^2_{L^2[0,\infty)} \\
&\le -k_1\|w(\cdot,t)-\widehat{w}(\cdot,t)\|^2_{L^2[0,\infty)}.
\end{aligned}
$$

Thus

$$\frac{1}{2}\frac{\partial}{\partial t}\|w(\cdot,t)-\widehat{w}(\cdot,t)\|^2_{L^2[0,\infty)} \le -k_1\|w(\cdot,t)-\widehat{w}(\cdot,t)\|^2_{L^2[0,\infty)}.$$

Solving the differential inequality yields

$$\|w(\cdot,t)-\widehat{w}(\cdot,t)\|^2_{L^2[0,\infty)} \le \mathrm{e}^{-2k_1 t}\|w_0-\widehat{w}_0\|^2_{L^2[0,\infty)}.$$

Thus, letting $t=T$ gives the required contraction estimate

$$\|\Psi(w_0)-\Psi(\widehat{w}_0)\|_{L^2[0,\infty)} \le \mathrm{e}^{-k_1 T}\|w_0-\widehat{w}_0\|_{L^2[0,\infty)}.$$

To see that the unique periodic solution is globally asymptotically stable, let w_0 be such that the solution to problem (1)–(4) with this initial condition satisfies $w(x,T)=w_0(x)$ in M_0. Furthermore, if $\widehat{w}_0$ is any other initial condition with solution $\widehat{w}(x,t)$, then the inequality we have already obtained

$$\|w(\cdot,t)-\widehat{w}(\cdot,t)\|^2_{L^2[0,\infty)} \le \mathrm{e}^{-2k_1 t}\|w_0-\widehat{w}_0\|^2_{L^2[0,\infty)}$$

proves that $\lim_{t\to\infty}\|w(\cdot,t)-\widehat{w}(\cdot,t)\|_{L^2[0,\infty)}=0$.

4 WEAK SOLUTIONS

In order to obtain the existence of classical solutions to our problem, we had to put strong continuity conditions on the loading function f. However, we can obtain unique weak solutions for any $L^2_{\text{loc}}([0,\infty))$ loading function f. In order to define weak solutions, we recall that the Sobolev space $W^{1,2}[0,\infty)$ is the closure of the infinitely differentiable functions on $[0,\infty)$ with compact support under the norm

$$\|u\|_{1,2} = \sqrt{\int_0^\infty |u|^2\,\mathrm{d}x + \int_0^\infty \left|\frac{\mathrm{d}u}{\mathrm{d}x}\right|^2 \mathrm{d}x}.$$

We call a function $w(x,t)$ such that $(w(x,t)-c/k_1)$ in $C([0,\infty)\to L^2(0,\infty))$ a weak solution of (1)–(4) if $(w(x,t)-c/k_1)(1-\mathrm{e}^{-Kx})$ is in $L^2_{\mathrm{loc}}([0,\infty)\to\{u\in W^{1,2}(0,\infty)\colon u(0)=0\})$ for sufficiently large K, $w(x,t)$ is in $L^2_{\mathrm{loc}}([0,\infty)\to C[0,X))$ for each $X>0$, its distributional derivative $\partial w/\partial t$ is in the continuous dual of $L^2([0,T)\to\{u\in W^{1,2}(0,\infty)\colon u(0)=0\})$, $\lim_{x\downarrow 0}\int_0^T |w(x,t)-f(t)|^2\,\mathrm{d}t$ for each $T>0$, $w(x,0)=w_0(x)$, and the following holds for every v in $L^2([0,T)\to\{u\in W^{1,2}(0,\infty)\colon u(0)=0\})$:

$$\int_0^\infty\int_0^T \frac{\partial w}{\partial t} v\,\mathrm{d}t\,\mathrm{d}x = \int_0^T\int_0^\infty -u\frac{\partial w}{\partial x}(x)v(x) - D\frac{\partial w}{\partial x}(x)\frac{\partial v}{\partial x}(x) - k_1 w(x)v(x) + cv\,\mathrm{d}x\,\mathrm{d}t.$$

We will start by showing that classical solutions are weak solutions. Then we will show that weak solutions are unique and obtain our existence and approximation results. For the rest of this section we will take K be a constant greater than u/D.

LEMMA 8 *A classical solution to problem* (1)–(4) *is a weak solution to* (1)–(4).

Proof Let w be a classical solution to (1)–(4). Then $(w(x,t)-c/k_1)$ is in $C([0,\infty)\to L^2(0,\infty))$, $(w(x,t)-c/k_1)$ is also in $L^2_{\mathrm{loc}}([0,\infty)\to W^{1,2}(0,\infty))$, $w(x,t)$ is in $L^2_{\mathrm{loc}}([0,\infty)\to C[0,X))$ for each $X>0$, its derivative $\partial w/\partial t$ is continuous into $L^2[0,\infty)$ and hence defines a bounded linear functional on $L^2([0,T)\to\{u\in W^{1,2}(0,\infty)\colon u(0)=0\})$, $w(0,t)=f(t)$, $w(x,0)=w_0(x)$, and the following holds for every u in $L^2([0,T)\to\{u\in W^{1,2}(0,\infty)\colon u(0)=0\})$

$$\frac{\partial w}{\partial t}v = -u\frac{\partial w}{\partial x}v + D\frac{\partial^2 w}{\partial x^2}v - k_1 wv + cv.$$

Integrating with respect to x from 0 to ∞ and integrating by parts yields

$$\begin{aligned}\int_0^\infty \frac{\partial w}{\partial t} v\,\mathrm{d}x &= -u\int_0^\infty \frac{\partial w}{\partial x} v\,\mathrm{d}x + D\int_0^\infty \frac{\partial^2 w}{\partial x^2} v\,\mathrm{d}x - k_1\int_0^\infty \left(w - \frac{c}{k_1}\right) v\,\mathrm{d}x \\ &= -u\int_0^\infty \frac{\partial w}{\partial x} v\,\mathrm{d}x + D\left[\frac{\partial w}{\partial x} v\right]_{x=0}^\infty - D\int_0^\infty \frac{\partial w}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x \\ &\quad - k_1\int_0^\infty \left(w - \frac{c}{k_1}\right) v\,\mathrm{d}x \\ &= -u\int_0^\infty \frac{\partial w}{\partial x} v\,\mathrm{d}x - D\int_0^\infty \frac{\partial w}{\partial x}\frac{\partial v}{\partial x}\,\mathrm{d}x - k_1\int_0^\infty \left(w - \frac{c}{k_1}\right) v\,\mathrm{d}x.\end{aligned}$$

Intergrating with respect to t completes the proof.

THEOREM 9 *Weak solutions to* (1)–(4) *are unique.*

Proof Let w^1 and w^2 be weak solutions to (1)–(4). Let $h(x,t) = w^1(x,t) - w^2(x,t)$. Notice that $h(1-\mathrm{e}^{-Kx})$ is in $L^2([0,T) \to \{u \in W^{1,2}(0,\infty)\colon u(0) = 0\})$. Therefore

$$\begin{aligned}&\frac{1}{2}\int_0^\infty h^2(x,T)(1-\mathrm{e}^{-Kx})\,\mathrm{d}t \\ &= \int_0^T\int_0^\infty \frac{\partial h}{\partial t} h(1-\mathrm{e}^{-Kx})\,\mathrm{d}x \\ &= \int_0^T\int_0^\infty -u\frac{\partial h}{\partial x}(x,t)h(x,t)(1-\mathrm{e}^{-Kx}) \\ &\quad - D\frac{\partial h}{\partial x}(x,t)\frac{\partial h}{\partial x}(x,t)(1-\mathrm{e}^{-Kx}) - D\frac{\partial h}{\partial x}(x,t)h(x,t)K\mathrm{e}^{-Kx} \\ &\quad - k_1 h(x,t)h(x,t)(1-\mathrm{e}^{-Kx})\,\mathrm{d}x\,\mathrm{d}t \\ &\le \int_0^T\int_0^\infty \frac{u}{2}h^2(x,t)K\mathrm{e}^{-Kx} - \frac{D}{2}h^2(x,t)K^2\mathrm{e}^{-Kx} \\ &\quad - k_1 h(x,t)h(x,t)(1-\mathrm{e}^{-Kx})\,\mathrm{d}x\,\mathrm{d}t \\ &\le 0,\end{aligned}$$

where we have integrated by parts in the x variable. The above implies that $\frac{1}{2}\int_0^\infty h^2(x,T)(1-\mathrm{e}^{-Kx})\,\mathrm{d}t = 0$ for each $T > 0$ and we are done.

The following result will give the existence of weak solutions as well as the approximation result which will allow us to write down explicit analytic representations of weak solutions in some cases.

THEOREM 10 *For every f in $L^2_{\text{loc}}([0,\infty)) \cap L^\infty([0,\infty))$, there is a (unique) weak solution w to (1)–(4). Furthermore, if $\{f^n\}_{n=1}^\infty$ is any sequence of differentiable functions with Hölder continuous derivatives on $[0,\infty)$ that converges to f in $L^2_{\text{loc}}([0,\infty))$, then the associated solutions to problem (1)–(4), w^n, converge to w in $C([0,\infty)) \to L^2(0,\infty))$, uniformly on bounded t-intervals. If $f^n \to f$ in $L^\infty([0,\infty))$, then $w^n \to w$ in $L^\infty([0,\infty) \times [0,\infty))$.*

Proof Let f be in $L^2_{\text{loc}}([0,\infty)) \cap L^\infty([0,\infty))and\{f^n\}_{n=1}^\infty$ any sequence of differentiable functions with Hölder continuous derivative on $[0,\infty)$ that converges of f in $L^2_{\text{loc}}([0,\infty))$ which is bounded in $L^\infty([0,\infty))$ by M. Let w^n be the associated solutions to problem (1)–(4), w^n. Let $T > 0$. Let $K > u/D$. Let $h = w^n - w^m$. Then

$$
\begin{aligned}
&\frac{1}{2}\int_0^\infty h^2(x,T)(1-\mathrm{e}^{-Kx})\,\mathrm{d}x \\
&\quad = \int_0^T \int_0^\infty \frac{\partial h}{\partial t} h(1-\mathrm{e}^{-Kx})\,\mathrm{d}x \\
&\quad = \int_0^T \int_0^\infty -u\frac{\partial h}{\partial x}(x,t)h(x,t)(1-\mathrm{e}^{-Kx}) \\
&\qquad + D\frac{\partial^2 h}{\partial x^2}(x,t)h(x,t)(1-\mathrm{e}^{-Kx}) - k_1 h(x,t)h(x,t)(1-\mathrm{e}^{-Kx})\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_0^T \int_0^\infty \frac{u}{2}h^2(x,t)K\mathrm{e}^{-Kx} - D\frac{\partial h}{\partial x}(x,t)\frac{\partial h}{\partial x}(x,t)(1-\mathrm{e}^{-Kx}) \\
&\qquad - D\frac{\partial h}{\partial x}(x,t)h(x,t)K\mathrm{e}^{-Kx}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_0^T \int_0^\infty \frac{u}{2}h^2(x,t)K\mathrm{e}^{-Kx}\,\mathrm{d}x - \frac{D}{2}h^2(x,t)K\mathrm{e}^{-Kx}\Big|_0^\infty \\
&\qquad - \int_0^\infty \frac{D}{2}h^2(x,t)K^2\mathrm{e}^{-Kx}\,\mathrm{d}x\,\mathrm{d}t \\
&\quad \le \int_0^T \frac{DK}{2}h^2(0,t)\,\mathrm{d}t \\
&\quad = \frac{DK}{2}\|f^n - f^m\|_{L^2[0,T]},
\end{aligned}
$$

where we have integrated by parts and used the fact that $\int_0^T \int_0^\infty (u/2)h^2(x,t)K\mathrm{e}^{-Kx}\,\mathrm{d}x - \int_0^\infty (D/2)h^2(x,t)K^2\mathrm{e}^{-Kx}\,\mathrm{d}x \le 0$. Thus

$$\int_0^\infty |w^n(x,T) - w^m(x,T)|^2(1-\mathrm{e}^{-Kx})\,\mathrm{d}x \le DK\|f^n - f^m\|_{L^2[0,T]}.$$

Let $\varepsilon > 0$ and choose $l > 0$ so that $2lM < \varepsilon/2$ and choose N so that if $m, n \ge N$ then

$$\|f^n - f^m\|_{L^2[0,T]} < \frac{\varepsilon}{2}\frac{1-\mathrm{e}^{-Kl}}{KD}.$$

Then

$$\begin{aligned}
&\int_0^\infty |w^n(x,T) - w^m(x,T)|^2\,\mathrm{d}x \\
&\quad = \int_0^l |w^n(x,T) - w^m(x,T)|^2\,\mathrm{d}x + \int_l^\infty |w^n(x,T) - w^m(x,T)|^2\,\mathrm{d}x \\
&\quad \le 2lM + \frac{1}{1-\mathrm{e}^{-Kl}}\int_l^\infty |w^n(x,T) - w^m(x,T)|^2(1-\mathrm{e}^{-Kx})\,\mathrm{d}x \\
&\quad < \frac{\varepsilon}{2} + \frac{1}{1-\mathrm{e}^{-Kl}}DK\|f^n - f^m\|_{L^2[0,T]} \\
&\quad < \varepsilon.
\end{aligned}$$

Since this holds for every $T \ge 0$, we have that $\{w^n - c/k_1\}$ is uniformly Cauchy in $C([0,T) \to L^2(0,\infty))$ for each $t > 0$ and hence converges to a function w such that $(w(x,t) - c/k_1)$ is in $C([0,T) \to L^2(0,\infty))$.

Returning to the computation above and not discarding the term $D\int_0^\infty \int_0^\infty (\partial h(x,t)/\partial x)(\partial h(x,t)/\partial x)(1-\mathrm{e}^{-Kx})\,\mathrm{d}x\,\mathrm{d}t$ yields the inequality

$$\begin{aligned}
&\int_0^\infty |w^n(x,T) - w^m(x,T)|^2(1-\mathrm{e}^{-Kx})\,\mathrm{d}x \\
&\quad \le DK\|f^n - f^m\|_{L^2[0,T]} \\
&\qquad - D\int_0^2 \int_0^\infty \left(\frac{\partial w^n}{\partial x}(x,t) - \frac{\partial w^m}{\partial x}(x,t)\right)^2 (1-\mathrm{e}^{-Kx})\,\mathrm{d}x\,\mathrm{d}t
\end{aligned}$$

which implies that $\{(\partial w_n/\partial x)^2(x,t)(1-\mathrm{e}^{-Kx})\}$ is Cauchy in $L^2([0,T) \to L^2([0,\infty))$. Now

$$\int_0^T \int_0^\infty [((w^n(x,t) - w^m(x,t))(1 - \mathrm{e}^{-Kx}))_x]^2 \,\mathrm{d}x\,\mathrm{d}t$$
$$= \int_0^T \int_0^\infty \left(\frac{\partial(w^n - w^m)}{\partial x}(x,t)(1 - \mathrm{e}^{-Kx})\right)^2$$
$$+ ((w^n(x,t) - w^m(x,t))K\mathrm{e}^{-Kx})^2$$
$$- 2\frac{\partial(w^n - w^m)}{\partial x}(x,t)(1 - \mathrm{e}^{-Kx})\left(w^n(x,t) - \frac{c}{k_1}\right)K\mathrm{e}^{-Kx}\,\mathrm{d}x\,\mathrm{d}t.$$

Since each term on the right hand side converges to 0, $(w(x,t) - c/k_1)(1 - \mathrm{e}^{-Kx})$ is in $L^2_{\mathrm{loc}}([0,\infty) \to \{u \in W^{1,2}(0,\infty):\ u(0) = 0\})$.

Let v be in $C_0^\infty((0,\infty) \times (0,\infty))$, which is dense in $L^2([0,T) \to \{u \in W^{1,2}(0,\infty):\ u(0) = 0\})$. Then we have

$$-\int_0^\infty \int_0^T \left(w(x,t) - \frac{c}{k_1}\right)\frac{\partial v}{\partial t}\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^\infty \int_0^T \frac{\partial w_n}{\partial t} v\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^T \int_0^\infty -u\frac{\partial w_n}{\partial x}(x)v(x) - D\frac{\partial w_n}{\partial x}(x)\frac{\partial v}{\partial x}(x)$$
$$- k_1\left(w_n(x,t) - \frac{c}{k_1}\right)v\,\mathrm{d}x\,\mathrm{d}t.$$

Letting n tend towards ∞ yields

$$-\int_0^\infty \int_0^T \left(w(x,t) - \frac{c}{k_1}\right)\frac{\partial v}{\partial t}\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^\infty \int_0^T g v\,\mathrm{d}t\,\mathrm{d}x$$
$$= \int_0^T \int_0^\infty -u\frac{\partial w}{\partial x}(x)v(x) - D\frac{\partial w}{\partial x}(x)\frac{\partial v}{\partial x}(x)$$
$$- k_1\left(w(x,t) - \frac{c}{k_1}\right)v\,\mathrm{d}x\,\mathrm{d}t$$

where g is the weak limit of $\partial w_n/\partial t$ in the continuous dual of $L^2([0,T) \to \{u \in W^{1,2}(0,\infty):\ u(0) = 0\})$ and

$$|g(v)| \leq (u + D + k_1)\left\|w - \frac{c}{k_1}\right\|_{L^2([0,T)\to W^{1,2}(0,\infty))}\|v\|_{L^2([0,T)\to W^{1,2}(0,\infty))}.$$

Furthermore, since

$$g(v) = \int_0^\infty \int_0^T gv\,\mathrm{d}t\,\mathrm{d}x = -\int_0^\infty \int_0^T \left(w(x,t) - \frac{c}{k_1}\right)\frac{\mathrm{d}v}{\mathrm{d}t}\,\mathrm{d}t\,\mathrm{d}x,$$

g is the distributional derivative of w, $\partial w/\partial t$.

Since for each natural number n, $w_n(x,0) = w_0(x)$ and w_n converges to w in $C([0,T) \to L^2(0,\infty))$, $w(x,0) = w_0(x)$ in $L^2(0,\infty)$.

All that remains in order to show existence of weak solutions is to show that $w(x,t)$ is in $L^2_{\mathrm{loc}}([0,\infty) \to C[0,X))$ for each $X > 0$ and that as x tends to zero, $\int_0^T |w(x,t) - f(t)|^2\,\mathrm{d}t$ tends to zero. Let $h = w^n - w^m$. Then

$$\begin{aligned}
\int_x^\infty h^2(\eta,t)\,\mathrm{d}\eta &= \int_0^t -u \int_0^x h_x h\,\mathrm{d}\eta\,\mathrm{d}\tau \\
&\quad + \int_0^t D \int_x^\infty h_{xx} h\,\mathrm{d}\eta\,\mathrm{d}\tau - k_1 \int_0^t \int_x^\infty h^2\,\mathrm{d}\eta\,\mathrm{d}\tau \\
&= u \int_0^t h^2(x,\tau)\,\mathrm{d}\tau - D \int_0^t h_x(x,\tau) h(x,\tau)\,\mathrm{d}\tau \\
&\quad - D \int_0^t \int_x^\infty (h_x(\eta,\tau))^2\,\mathrm{d}\eta\,\mathrm{d}\tau - k_1 \int_0^t \int_x^\infty h^2\,\mathrm{d}\eta\,\mathrm{d}\tau.
\end{aligned}$$

We now integrate from 0 to x in the spatial variable again,

$$\begin{aligned}
&\int_0^x \int_\beta^\infty h^2(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}\beta \\
&= u \int_0^t \int_0^x h^2(\eta,\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau - \frac{D}{2} \int_0^t (h(x,\tau))^2 - (f^n(\tau) - f^m(\tau))^2\,\mathrm{d}\tau \\
&\quad - D \int_0^t \int_0^x \int_\beta^\infty (h_x(\eta,\tau))^2\,\mathrm{d}\eta\,\mathrm{d}\beta\,\mathrm{d}\tau - k_1 \int_0^t \int_0^x \int_\beta^\infty h^2\,\mathrm{d}\eta\,\mathrm{d}\beta\,\mathrm{d}\tau \\
&= u \int_0^t \int_0^x h^2(\eta,\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau - \frac{D}{2} \int_0^t (h(x,\tau))^2 - (f^n(\tau) - f^m(\tau))^2\,\mathrm{d}\tau \\
&\quad - D \int_0^t x \int_x^\infty (h_x(\eta,\tau))^2\,\mathrm{d}\eta - \int_0^x \eta (h_x(\eta,\tau))^2\,\mathrm{d}\eta\,\mathrm{d}\tau \\
&\quad - k_1 \int_0^t \int_0^x \int_\beta^\infty h^2\,\mathrm{d}\eta\,\mathrm{d}\beta\,\mathrm{d}\tau.
\end{aligned}$$

Rearranging yields

$$\frac{D}{2}\int_0^t (h(x,\tau))^2\,\mathrm{d}\tau = -\int_0^x\int_\beta^\infty h^2(\eta,t)\,\mathrm{d}\eta\,\mathrm{d}\beta + u\int_0^t\int_0^x h^2(\eta,\tau)\,\mathrm{d}\eta\,\mathrm{d}\tau$$
$$+\frac{D}{2}\int_0^t (f^n(\tau)-f^m(\tau))^2\,\mathrm{d}\tau$$
$$-D\int_0^t x\int_x^\infty (h_x(\eta,\tau))^2\,\mathrm{d}\eta - \int_0^x \eta(h_x(\eta,\tau))^2\,\mathrm{d}\eta\,\mathrm{d}\tau$$
$$-k_1\int_0^t\int_0^x\int_\beta^\infty h^2\,\mathrm{d}\eta\,\mathrm{d}\beta\,\mathrm{d}\tau.$$

Since $\lim_{x\downarrow 0}(1-\mathrm{e}^{-Kx})/x = K$, we have that all of the terms on the right hand side are Cauchy, uniformly on bounded x intervals. Furthermore, each classical solution $w^n(x,t)$ satisfies that $\lim_{y\to x}\int_0^t (w^n(y,\tau)-w^n(x,\tau))^2\,\mathrm{d}\tau = 0$ for each positive t. Therefore, the same result holds for w. We have now shown that for every f in $L^2_{\mathrm{loc}}([0,\infty))\cap L^\infty([0,\infty))$, there is a (unique) weak solution w to (1)–(4). Furthermore, if $\{f^n\}_{n=1}^\infty$ is any sequence of differentiable functions with Hölder continuous derivatives on $[0,\infty)$ that converges to f in $L^2_{\mathrm{loc}}([0,\infty))$, then the associated solutions to problem (1)–(4), w^n, converge to w in $C([0,\infty)\to L^2(0,\infty))$, uniformly on bounded t-intervals. To see that if $f^n\to f$ in $L^\infty([0,\infty))$, Then $w^n\to w$ in $L^\infty([0,\infty)\times[0,\infty))$, let $h=w^n-w^m$ and using the same notation as in the proof of the boundedness of classical solutions, multiply the differential equation by $r_k(h)[s_k(h)]^{p-1}$ and integrate from 0 to ∞,

$$\frac{1}{p}\frac{\partial}{\partial t}\int_0^\infty [s_k(h)]^p\,\mathrm{d}x = -(u/p)[s_k(h)]^p|_{x=0}^\infty + D\left[\frac{\partial h}{\partial x} r_k(h)[s_k(h)]^{p-1}\right]_{x=0}^\infty$$
$$-D\int_0^\infty \left[\frac{\partial h}{\partial x}\right]^2 (r_k'(h)[s_k(h)]^{p-1}$$
$$+(r_k'(h))^2[s_k(h)]^{p-2})\,\mathrm{d}x$$
$$-k_1\int_0^\infty (h)r_k(h)[s_k(h)]^{p-1}\,\mathrm{d}x.$$

Nothing that the first two terms are zero at ∞ and that the integrals are positive, we obtain

$$\begin{aligned}\frac{1}{p}\frac{\partial}{\partial t}\int_0^\infty [s_k(h)]^p\,\mathrm{d}x &\le \frac{u}{p}[s_k(f^n(t)-f^m(t))]^p \\ &\quad - D\frac{\partial h}{\partial x} r_k(f^n(t)-f^m(t))[s_k(f^n(t)-f^m(t))]^{p-1} \\ &\le \frac{u}{p}[s_k(f^n(t)-f^m(t))]^p \\ &\quad + D\left|\frac{\partial h}{\partial x}(0,t)\right|[s_k(f^n(t)-f^m(t))]^{p-1}.\end{aligned}$$

Integrating with respect to t from 0 to t (abusing the notation) we obtain

$$\begin{aligned}\int_0^\infty [s_k(h(x,t))]^p\,\mathrm{d}x &\le u\int_0^t [s_k(f^n(\tau)-f^m(\tau))]^p\,\mathrm{d}\tau \\ &\quad + pD\int_0^t \left|\frac{\partial h}{\partial x}(0,\tau)\right|\left[s_k(f^n(\tau)-f^m(\tau))\right]^{p-1}\mathrm{d}\tau.\end{aligned}$$

Now letting k tend towards ∞ and applying the Lebesgue dominated convergence theorem yields

$$\begin{aligned}\int_0^\infty |h(x,t)|^p\,\mathrm{d}x &\le u\int_0^t |f^n(t)-f^m(t)|^p\,\mathrm{d}\tau \\ &\quad + pD\int_0^t \left|\frac{\partial h}{\partial x}(0,\tau)\right||f^n(t)-f^m(t)|^{p-1}\,\mathrm{d}\tau.\end{aligned}$$

Equivalently,

$$\begin{aligned}\|h(\cdot,t)\|^p_{L^p(0,\infty)} &\le u\|f^n(t)-f^m(t)\|^p_{L^p(0,t)} \\ &\quad + pD\left\|\frac{\partial h}{\partial x}(0,\cdot)\right\|_{L_\infty(0,t)}\|f^n(t)-f^m(t)\|^{p-1}_{L^p(0,t)}.\end{aligned}$$

Taking pth roots and lettting p go to infinity yields

$$\|h(\cdot,t)\|_{L^\infty(0,\infty)} \le \|f^n(t)-f^m(t)\|_{L^\infty(0,t)}$$

and we are done.

Remark 1 Because a function in $L^2[0, T]$ can be approximated by increasing sequences of smooth functions, the results on ordering, positivity, and boundedness hold for weak as well as classical solutions.

5 FOURIER SERIES FOR WEAK SOLUTIONS

In [2], explicit solutions were given for problem (1)–(4) in the specific cases of f constant, f a sine function, or f a cosine function when w_0 is constant. In fact, the solutions for the foregoing were done by looking at the solutions for the real and imaginary parts of an exponential loading function f. It is this complex solution we will use here. We give them here and then use this information and Fourier series to give an explicit series representation for a particular weak solution.

If $w_0(x) = L_0$, a constant, $c = L_a$, and $f(t) = A$, the solution is given by

$$\begin{aligned}
w(x,t) &= \frac{1}{2}\left(A - \frac{L_a}{k_1}\right)\exp\left[\frac{ux}{2D} - \sqrt{\frac{\lambda}{D}}x\right]\operatorname{erf}c\left[\frac{x}{\sqrt{4Dt}} - \sqrt{\lambda t}\right] \\
&\quad + \frac{1}{2}\left(A - \frac{L_a}{k_1}\right)\exp\left[\frac{ux}{2D} + \sqrt{\frac{\lambda}{D}}x\right]\operatorname{erf}c\left[\frac{x}{\sqrt{4Dt}} - \sqrt{\lambda t}\right] \\
&\quad - \frac{1}{2}\left(L_0 - \frac{L_a}{k_1}\right)\exp\left[\frac{ux}{2D} - k_1 t\right] \\
&\quad \times \left[\exp\left(-x\sqrt{\frac{\beta}{D}}\right)\operatorname{erf}c\left(\frac{x}{\sqrt{4Dt}} - \sqrt{\beta t}\right)\right. \\
&\quad \left. + \exp\left(x\sqrt{\frac{\beta}{D}}\right)\operatorname{erf}c\left(\frac{x}{\sqrt{4Dt}} + \sqrt{\beta t}\right)\right] \\
&\quad + \left(L_0 - \frac{L_a}{k_1}\right)\exp(-k_1 t) + \frac{L_a}{k_1},
\end{aligned}$$

where $\beta = u^2/4D$.

For $f(t) = B\exp(\mathrm{i}wt)$, $c = 0$, and $w_0 \equiv 0$. In this case we have

$$w(x,t) = \frac{B}{2}\exp\left[\frac{ux}{2D} + \mathrm{i}wt\right]\left\{\exp\left(-x\sqrt{\frac{\alpha}{D}}\right)\operatorname{erf}c\left(\frac{x}{2\sqrt{Dt}} - \sqrt{\alpha t}\right)\right.$$
$$\left. + \exp\left(x\sqrt{\frac{\alpha}{D}}\right)\operatorname{erf}c\left(\frac{x}{2\sqrt{Dt}} + \sqrt{\alpha t}\right)\right\}$$

where $\lambda = u^2/4D + k_1$ and $\alpha = \lambda + \mathrm{i}w$.

We note that the only constant initial condition allowed by our existence theory is $w_0 = c/k_1$.

Example 1 We will take our unit of time to be one day, our unit of distance to be one meter, our unit of volume to be liters, and our unit of mass to be milligrams

$$\begin{aligned} D &= 17\,\frac{\mathrm{m}^2}{\mathrm{day}}, \\ u &= 25\,000\,\frac{\mathrm{m}}{\mathrm{day}}, \\ k_1 &= 0.5\,\mathrm{days}, \\ c &= 7.5\,\frac{\mathrm{mg}}{\mathrm{L-day}}, \\ w_0(x) &= 15\,\frac{\mathrm{mg}}{\mathrm{L}}. \end{aligned}$$

Our function f will be the period 1 day continuation of the following

$$f(t) = 15 + 26t\frac{\mathrm{mg}}{L}$$

which yields the sawtooth wave in Fig. 1. This will yield the fourier series for f of

$$f(t) \sim \sum_{n=-\infty}^{\infty} \frac{13\mathrm{i}}{n\pi}\exp[2\mathrm{i}n\pi t].$$

The truncated fourier series

$$f_N(t) \sim 28 + \sum_{n=-N, n\neq 0}^{N} \frac{13\mathrm{i}}{n\pi}\exp[2\mathrm{i}n\pi t]$$

with $N = 6$ is given in Fig. 2. We get a series representation for $w(x,t)$ by using superposition for the individual forcing functions 28 and

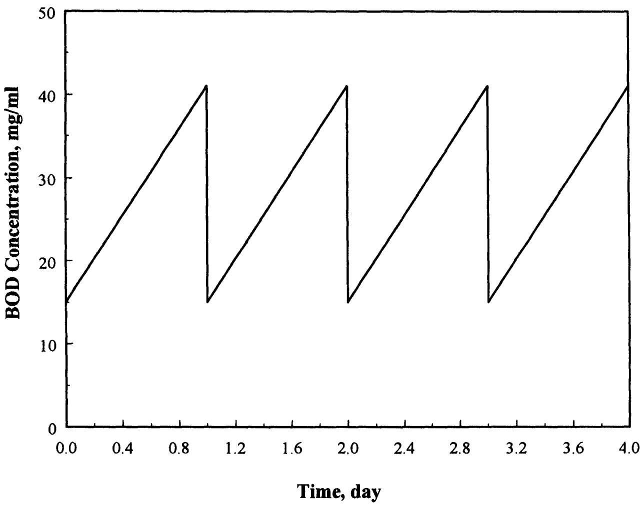

FIGURE 1 BOD loading function at $x = 0$.

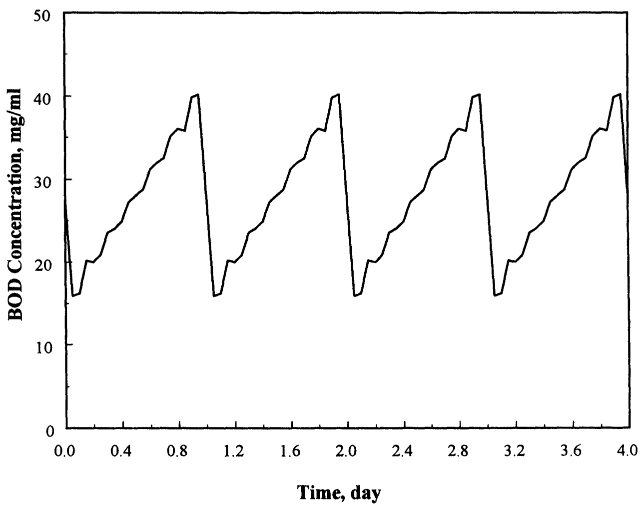

FIGURE 2 Approximate BOD loading function at $x = 0$.

$(13\mathrm{i}/n)\exp[2\mathrm{i}n\pi t]$ for n a nonzero integer. This yields

$$\begin{aligned}
w(x,t) &= \frac{1}{2}\left(A - \frac{L_a}{k_1}\right)\exp\left[\frac{ux}{2D} - \sqrt{\frac{\lambda}{D}}x\right]\mathrm{erf}\,c\left[\frac{x}{\sqrt{4Dt}} - \sqrt{\lambda t}\right] \\
&+ \frac{1}{2}\left(A - \frac{L_a}{k_1}\right)\exp\left[\frac{ux}{2D} + \sqrt{\frac{\lambda}{D}}x\right]\mathrm{erf}\,c\left[\frac{x}{\sqrt{4Dt}} + \sqrt{\lambda t}\right] \\
&- \frac{1}{2}\left(L_0 - \frac{L_a}{k_1}\right)\exp\left[\frac{ux}{2D} - k_1 t\right] \\
&\times \left[\exp\left(-x\sqrt{\frac{\beta}{D}}\right)\mathrm{erf}\,c\left(\frac{x}{\sqrt{4Dt}} - \sqrt{\beta t}\right)\right. \\
&\left. + \exp\left(x\sqrt{\frac{\beta}{D}}\right)\mathrm{erf}\,c\left(\frac{x}{\sqrt{4Dt}} + \sqrt{\beta t}\right)\right] + \left(L_0 - \frac{L_a}{k_1}\right)\exp(-k_1 t) \\
&+ \frac{L_a}{k_1} + \sum_{n=-N, n\neq 0}^{N} \frac{13\mathrm{i}}{2n\pi}\exp\left[\frac{ux}{2D} + \mathrm{i}2\pi n t\right] \\
&\times \left\{\exp\left(-x\sqrt{\frac{\alpha_n}{D}}\right)\mathrm{erf}\,c\left(\frac{x}{2\sqrt{Dt}} - \sqrt{\alpha_n t}\right)\right. \\
&\left. + \exp\left(x\sqrt{\frac{\alpha_n}{D}}\right)\mathrm{erf}\,c\left(\frac{x}{2\sqrt{Dt}} + \sqrt{\alpha_n t}\right)\right\}.
\end{aligned}$$

Figures 3–7 will give the profile of the first 300 kilometers of the river at days 5, 10, 15, 20, and 25 respectively.

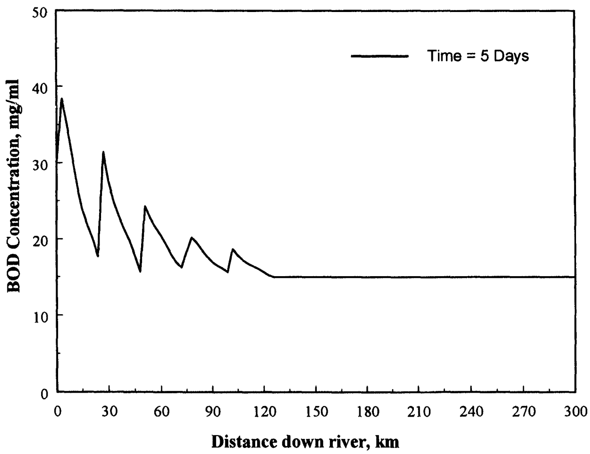

FIGURE 3 BOD concentration profile of the first 300 kilometers of the river after 5 days.

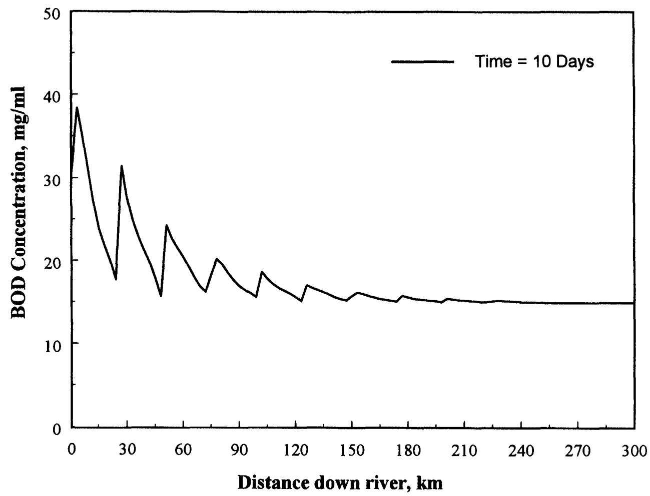

FIGURE 4 BOD concentration profile of the first 300 kilometers of the river after 10 days.

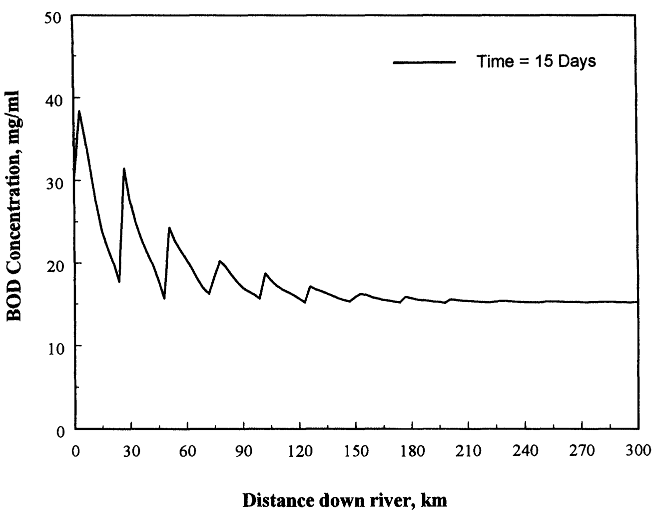

FIGURE 5 BOD concentration profile of the first 300 kilometers of the river after 15 days.

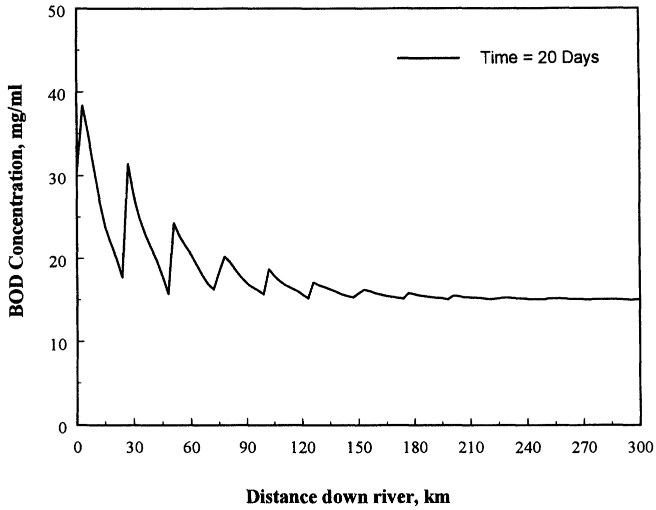

FIGURE 6 BOD concentration profile of the first 300 kilometers of the river after 20 days.

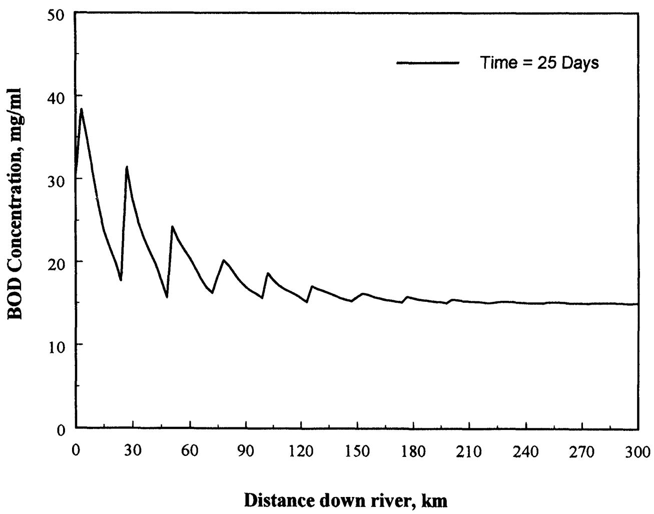

FIGURE 7 BOD concentration profile of the first 300 kilometers of the river after 25 days.

References

[1] R.A. Adams, *Sobolev Spaces*, Academic Press, New York, 1975.

[2] A. Alshawabkeh and D.D. Adrian Analytical water quality model for a sinusoidally varying BOD discharge concentration, *Wat. Res.* **31**(5), 1207–1215, 1997.

[3] M.G. Crandall, An Introduction to Evolution Governed by Accretive Operators, *International Symposium on Dynamical Systems*, Academic Press, New York (1976), pp. 131–165.

[4] A. Pazy, *Semigroups of Linear Operators and Applications to Partial Differential Equations*, Springer-Verlag, New York, 1983.

[5] R.E. Showalter, *Hilbert Space Methods for Partial Differential Equations*, Pitman, San Francisco, 1979.