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The Simplex Method for Nonlinear Sliding Mode Control

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General nonlinear control systems described by ordinary differential equations with a prescribed sliding manifold are considered. A method of designing a feedback control law such that the state variable fulfills the sliding condition in finite time is based on the construction of a suitable simplex of vectors in the tangent space of the manifold. The convergence of the method is proved under an obtuse angle condition and a way to build the required simplex is indicated. An example of engineering interest is presented.

Keywords: Variable structure control; Nonlinear control systems

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1. INTRODUCTION

We consider general nonlinear control systems described by ordinary differential equations, with a fixed sliding manifold which has been designed in order to fulfill given control aims. An effective way to control such systems is the method based on variable structure techniques. We refer the reader to [1] for a survey.

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The standard approach to drive and maintain the system state vector on the sliding manifold requires designing a feedback control law which is obtained componentwise, i.e. if

$$s_i(x) = 0, \quad i = 1, \dots, M$$

are the equations representing the sliding manifold, the design of a pair of control laws is required for each $i = 1, \dots, M$.

The simplex control method was firstly introduced in [2] for control systems with control variable appearing affinely in the dynamics. The approach is simpler than the standard one since the required number of control variables is decreased to $M + 1$. Applications of the simplex control method to particular control systems has been reported in [3,4] showing interesting properties of robustness, simplicity and good performance even under deterministic uncertainty acting on the plant. Therefore it is of interest to extend the applicability of the methods to more general classes of systems than those originally considered in [2].

In this paper we generalize the method to nonlinear control systems with available states and known dynamics. Under an obtuse angle condition on the simplex vectors we prove the convergence of the method in finite time. The proof requires to verify some geometric properties of the simplex vectors, and a detailed analysis of the dynamic behavior of the system under the discontinuous feedback provided by the simplex algorithm we consider.

An outline of the paper follows. In Section 2 the control problem is described. In Section 3 we list the assumptions under which the simplex method will be shown to converge in finite time. In Section 4 we describe the choice of the feedback control law. Moreover the geometrical meaning of the main convergence condition is explained. Several elementary properties of a simplex of vectors (used in the method) are collected in Section 5. Section 6 is devoted to the convergence proof. In Section 7 we present three examples of control systems which can be handled by the simplex method. Computer simulations are included. The last example describes a real control system of engineering interest which is operated by using this method. In Section 8 we indicate an explicit construction of a fixed simplex as required by our convergence theorem.

2. PROBLEM STATEMENT

We consider the control system described by the state equations

$$\dot{x} = f(t, x, u), \quad t \geq 0, \tag{1}$$

the control constraint

$$u \in U, \tag{2}$$

the sliding manifold

$$s(x) = 0. \tag{3}$$

Here the state variable $x \in R^N$ and the control constraint $U \subset R^K$. The sliding manifold is defined by the mapping $s : R^N \to R^M$ and is fixed in order to fulfill prescribed control objectives.

We assume that the dynamics given by

$$f\colon [0, +\infty) \times \Omega \times U \to R^N,$$

where Ω is an open bounded set, are completely known, moreover every state variable $x = x(t)$, $t \geq 0$, is available to the controller. A known constant $L > 0$ is fixed such that Ω contains the ball in R^N of center 0 and radius L. We want to control the state variables $x = x(t)$ of the system which fulfill the condition

$$|x(t)| \leq L, \quad t \geq 0 \tag{4}$$

in order to verify the sliding property

$$s[x(t)] = 0 \tag{5}$$

for every t sufficiently large. Condition (4) is reasonable on physical grounds; we want to control those states which are contained in a (sufficiently large) fixed ball.

Remark We shall employ in this paper the notion of Filippov's solutions x of (1) corresponding to discontinuous feedbacks u (see [5] and [6]). According to this definition, every system's state x fulfills almost everywhere the condition

$$\dot{x}(t) \in \operatorname{cl}\operatorname{co} f(t, x(t), U),$$

where cl co denotes closed convex hull. For general nonlinear dynamics (1), the sliding motion equation is not unique and the choice of a suitable feedback control which guarantees (5) is not obvious, see [1, Chapter 3]. Methods which select the appropriate control law are therefore of interest in this context.

3. ASSUMPTIONS

We employ standard notations. A prime denotes transpose and $|\cdot|$ is the Euclidean norm. The following conditions will be assumed throughout the paper.

$N \geq M$, $s \in C^1(R^N, R^M)$ and the Jacobian matrix

$$\frac{\partial s}{\partial x}(x) \text{ has maximum rank } M \text{ for almost every } x. \tag{6}$$

U is a nonempty closed set in R^K; f is a Carathéodory mapping such that there exists $A \geq 0$ with

$$|f(t, x, u)| \leq A \quad \text{if } t \geq 0, \;\; x \in \Omega, \;\; u \in U. \tag{7}$$

The key assumption is the following.

There exist constants $a > 0$ and $c \neq 0$, and for every $t \geq 0$, $x \in \Omega$ with $|x| \leq L$ and $s(x) \neq 0$ there exist points

$$u_1(t, x), \ldots, u_{M+1}(t, x)$$

in U such that each u_i is a Carathéodory function and, writing

$$g_i = g_i(t, x) = \frac{\partial s}{\partial x}(x) f[t, x, u_i(t, x)], \quad i = 1, \ldots, M+1,$$

we have $0 < a \leq |g_i|$ for every i and

$$g_i' g_h \leq -c^2 |g_i||g_h| \quad \text{if } i \neq h. \tag{8}$$

By the following Lemma 6 in Section 5 it will follow from (8) that for every $q \in R^M$ there exists a proper subset I of $\{1, \dots, M+1\}$ such that

$$q \in \operatorname{cone}(g_i \colon i \in I). \tag{9}$$

In (9) we denote by $\operatorname{cone}(g_i \colon i \in I)$ the set of all points

$$\sum \{\alpha_i g_i \colon i \in I\} \quad \text{with } \alpha_i \geq 0, \;\; i \in I.$$

4. THE SIMPLEX CONTROL ALGORITHM

By (9), for every $t \geq 0$ and x fulfilling (4) with $s(x) \neq 0$ there exist coefficients $\alpha_i = \alpha_i(t, x) \geq 0$, $i = 1, \dots, M+1$, such that

$$s(x) = \sum_{i=1}^{M+1} \alpha_i g_i(t, x) \tag{10}$$

and some $\alpha_h = 0$.

Let $h = h(t, x)$ denote the smallest index k between 1 and $M+1$ such that

$$s(x) = \sum \{\alpha_i g_i(t, x) \colon \; i = 1, \dots, M+1, i \neq k\}.$$

Then define

$$u^*(t, x) = u_h(t, x). \tag{11}$$

The feedback control law defined by (11) will be referred to as the simplex control algorithm for the control system (1)–(3).

The basic condition (8) means the following. For every point (t, x) we can choose control laws $u_i = u_i(t, x)$, $i = 1, \dots, M+1$ such that the corresponding vectors $g_i = g_i(t, x)$ define a simplex in R^M whose edges form uniformly obtuse angles. More precisely, the scalar products of the normalized edges are bounded from above by a fixed negative constant, as required in (8). It follows that the space R^M is partitioned in $M+1$ cones

$$Q_h = \left\{ \sum \alpha_i g_i \colon \alpha_i \geq 0, \; i = 1, \dots, M+1, \; i \neq h \right\}, \quad h = 1, \dots, M+1,$$

with pairwise disjoint interiors. Given $x \in R^N$, the vector $s(x)$ will belong to some well-defined Q_h (with the smallest index h). Then g_h corresponds to u_h and points outside Q_h. The choice of the simplex control $u^*(t,x)$ is then exactly $u_h(t,x)$. It is reasonable, on intuitive grounds, to guess that the simplex control algorithm fulfills the sliding condition (5). The main mathematical problem is to prove this basic convergence property.

5. GEOMETRY OF THE SIMPLEX

In this section we collect some lemmas which will be used in the proof of the main result of the paper. These lemmas, of elementary character, provide a number of geometrical properties of the simplex obtained by considering the vectors g_i as in condition (8).

In this section g_i denote fixed vectors of R^M.

LEMMA 1 *Let $g_1, \ldots, g_{N+1}$ be such that*

$$g_i' g_h < 0 \quad \text{if } i \neq h, \tag{12}$$

and $g_1, \ldots, g_N$ are linearly independent. If $g_{N+1} = \sum_{i=1}^N y_i g_i$ with real coefficients y_i, then $y_i < 0$ for all i.

Proof By (12) we see that

$$\begin{aligned}
g_{N+1}' g_1 &= y_1 |g_1|^2 + y_2 g_1' g_2 + \cdots + y_N g_1' g_N < 0,\\
g_{N+1}' g_2 &= y_1 g_1' g_2 + y_2 |g_2|^2 + \cdots + y_N g_2' g_N < 0,\\
&\vdots\\
g_{N+1}' g_N &= y_1 g_1' g_N + y_2 g_2' g_N + \cdots + y_N |g_N|^2 < 0.
\end{aligned}$$

This gives the system of linear inequalities

$$\sum_{k=1}^{N} a_{ik} y_k < 0, \quad i = 1, \ldots, N, \tag{13}$$

where the coefficients $a_{ik} < 0$ if $i \neq k$, $a_{ii} > 0$ and the Gram matrix $A = (a_{ik})$ is symmetric and positive definite by linear independence.

Solving (13) by Gaussian elimination, the inequalities are preserved at each step and we get $D_N y_N / D_{N-1} < 0$, where D_N, D_{N-1} are principal minors of A (see [7, pp. 238–239]), hence $D_N > 0$, $D_{N-1} > 0$ whence $y_N < 0$, and recursively $y_i < 0$ for all i. □

LEMMA 2 *Let $g_1, \ldots, g_N, g_{N+1}, g_{N+2}$ be such that $g_i' g_h < 0$ if $i \neq h$ and $g_1, \ldots, g_N$ are linearly independent. Then $g_1, \ldots, g_N, g_{N+1}$ are linearly independent as well.*

Proof The equality $g_{N+1} = \sum_{i=1}^{N} y_i g_i$ for some real coefficients y_i implies by Lemma 1 that $y_i < 0$, contradicting $g_{N+1}' g_{N+2} < 0$. □

LEMMA 3 *If $g_1, \ldots, g_K$ fulfill* (12) *then $K \leq M + 1$.*

Proof Let N be the maximum number of linearly independent g_i, $N \leq K$ and $N \leq M$. If $K > M + 1$, by Lemma 2 we get the linear independence of $g_1, \ldots, g_{N+1}$ contradicting the definition of N. □

LEMMA 4 *Let $g_1, \ldots, g_{M+1}$ fulfill* (12). *Then $g_1, \ldots, g_M$ are linearly independent.*

Proof Let N be the maximum number of linearly independent g_i. If $N < M$, by considering $g_1, \ldots, g_N, g_{N+1}, g_{N+2}$ we get a contradiction by Lemma 2. □

LEMMA 5 *Let $g_1, \ldots, g_{M+1}$ fulfill* (12). *Then there exist coefficients $a_i > 0$ such that*

$$\sum_{i=1}^{M+1} a_i g_i = 0.$$

This obvious consequence of Lemmas 1 and 4 justifies the name of the simplex method.

LEMMA 6 *Let $g_1, \ldots, g_{M+1}$ fulfill* (12). *Consider*

$$C_i = \operatorname{cone}(g_k \colon k \neq i).$$

Then

$$\bigcup \{C_i \colon i = 1, \ldots, M+1\} = R^M.$$

Proof By Lemmas 1 and 4, $g_1, \ldots, g_M$ are linearly independent and

$$g_{M+1} = \sum_{k=1}^{M} a_k g_k,$$

where each $a_k < 0$. Fix any $y \in R^M$. Then $y = \sum_{i=1}^{M} h_i g_i$ (again by Lemma 4) with suitable coefficients h_i. Let

$$h_p/a_p = \max\{h_i/a_i : i = 1, \ldots, M\}.$$

If $h_p/a_p \leq 0$, then each $h_i \geq 0$ hence $y \in C_{M+1}$. If $h_p/a_p > 0$ then we have

$$y = a_1\left(\frac{h_1}{a_1} - \frac{h_p}{a_p}\right)g_1 + \cdots + a_M\left(\frac{h_M}{a_M} - \frac{h_p}{a_p}\right)g_p + \frac{h_p}{a_p} g_{M+1},$$

moreover $a_i(h_i/a_i - h_p/a_p) \geq 0$ for every i, hence $y \in C_p$. □

As claimed before (end of Section 3), Lemma 6 shows that (9) follows from (8). The proof of Lemma 6 can be used to build an algorithm which locates a cone C_p to which any given point in R^M belongs. This is of significance in order to implement the simplex control algorithm, which according to (10) requires knowing a cone to which $s(x)$ belongs.

6. CONVERGENCE OF THE SIMPLEX CONTROL ALGORITHM

This section is devoted to the proof of the convergence theorem. Given u^* defined by (1), a vector $x = x(t) \in R^N$ will be called a state corresponding to u^* if x is a Filippov solution to (1) in $[0, +\infty)$ with $u = u^*$. For the notion of Filippov solution to (1) (a standard one in variable structure control theory, see [1]) we refer to [5] and [6].

THEOREM 1 *Suppose that the assumptions* (6)–(8) *hold. Then every state x corresponding to* u^* *and fulfilling* (4) *verifies the sliding condition* (5) *for every t sufficiently large.*

In the proof we shall need the following

LEMMA 7 *The mapping* $(t, x) \to f[t, x, u^*(t, x)]$ *is* (*Lebesgue*) *measurable.*

Proof Step 1: Given $t \geq 0$ and x with $|x| \leq L$ we can choose the coefficients $\alpha_i(t, x)$ in (10) in a measurable way.

Consider the multifunction Γ defined by

$$\Gamma(t, x) = \{w \in R^{M+1} : \sum_{i=1}^{M+1} w_i g_i(t, x) = s(x), w_i \geq 0 \text{ for every } i\}.$$

By (9), $\Gamma(t, x)$ is nonempty for every (t, x). The R^M-valued function

$$G(t, x, w) = \sum_{i=1}^{M+1} w_i g_i(t, x) - s(x)$$

is measurable by the assumptions, hence Step 1 is proved by an application of the implicit measurable function theorem ([8, Theorem 2J, p. 178]).

Step 2: u^* defined by (11) is measurable.

Let $h(t, x) = \min\{i \in \{1, \ldots, M+1\} : \alpha_i(t, x) = 0\}$ (where α_i are as in (10)), which is well defined by Lemma 6. We show measurability of h by proving measurability of

$$H(a) = \{(t, x) \in R^{N+1} : t \geq 0, x \in \Omega, h(t, x) \leq a\},$$

for every $a \in R$. If $H(a)$ is nonempty then

$$H(a) = \bigcup \{(t, x) \in R^{N+1} : t \geq 0, x \in \Omega, \alpha_j(t, x) = 0 \text{ for some } j \leq a\}$$

a finite union of measurable sets by Step 1. To prove Step 2 we consider

$$A_i = \{(t, y) \in R^{N+1} : t \geq 0,\ y \in \Omega, h(t, y) = i\}, \quad i = 1, \ldots, M+1.$$

Each A_i is measurable, and we have

$$u^*(t, x) = \sum_{i=1}^{M+1} u_i(t, x) \ \text{char}(A_i, t, x),$$

where char (A, t, x) denotes the value at t, x of the characteristic function of the set A (which takes on the value 1 at the points of A,0 elsewhere). Hence u^* is measurable, whence $(t, x) \to f[t, x, u^*(t, x)]$ is since f is Carathéodory ([8, Corollary 2B, p. 174]). The proof of Lemma 7 is now complete. □

Proof of Theorem 1 Let $x = x(t)$ be a Filippov solution in $[0, +\infty)$ of (1) corresponding to u^*. Write

$$f^*(t, x) = f[t, x, u^*(t, x)].$$

By Lemma 7 and Eq. (7) we can apply a characterization of the Filippov solutions ([5, p. 202]).

Write co for the convex hull, cl for the closure and $B(x, r)$ for the open ball of center x and radius r in R^N.

We get

$$\dot{x}(t) \in \mathrm{cl}\,\mathrm{co} f^*(t, B(x(t), 1/p) \backslash T) \tag{14}$$

for every $p = 1, 2, \ldots$, almost every $t \geq 0$ and any subset T of R^N of N-dimensional Lebesgue measure 0. Now fix t such that there exists $\dot{x}(t)$, $s[x(t)] \neq 0$ and (14) holds. Write for short notation

$$s = s[x(t)], \quad \dot{s} = \dot{s}[x(t)], \quad g_i = g_i[t, x(t)].$$

Given T, by (14) for every p there exists a point

$$y_p \in \mathrm{co} f^*(t, B(x(t), 1/p) \backslash T)$$

such that $|y_p - \dot{x}(t)| \leq 1/p$. By the Carathéodory convexity theorem ([9, Theorem 17.1, p. 155]) for every p we can find numbers $\lambda_{jp} \geq 0$, $j = 1, \ldots, N + 1$, which sum up to 1, and points $z_{jp} \notin T$ such that

$$|z_{jp} - x(t)| \leq 1/p, \tag{15}$$

$$y_p = \sum_{j=1}^{N+1} \lambda_{jp} f^*(t, z_{jp}) \to \dot{x}(t) \quad \text{as } p \to +\infty. \tag{16}$$

Now only two possibilities arise. Let us write int for the interior and bdry for the boundary. Recall the definition of h (before (11)).

Case 1: $s \in \text{int cone}\ (g_i: i \neq h)$.
Case 2: $s \in \text{bdry cone}\ (g_i: i \neq h)$.

Suppose that Case 1 occurs. Since x is differentiable at t and s is smooth,

$$s'\dot{s} = s'\frac{\partial s}{\partial x}[x(t)]\dot{x}(t)$$

is the limit (by (16) and (10)) as $p \to +\infty$ of

$$\sum_{i\neq h} \alpha_i g_i' \frac{\partial s}{\partial x}[x(t)] \sum_{j=1}^{N+1} \lambda_{jp} f^*(t, z_{jp}). \tag{17}$$

For every j and every sufficiently large p, (15) yields

$$s(z_{jp}) \in \text{int cone}(g_i : i \neq h).$$

By the assumptions, $g_i(t, \cdot)$ is continuous, hence

$$s(z_{jp}) \in \text{int cone}(g_i(t, z_{jp}) \colon i \neq h)$$

whence

$$u^*(t, z_{jp}) = u_h(t, z_{jp}). \tag{18}$$

Then (17) and (18) show that $s'\dot{s}$ is the limit as $p \to +\infty$ of

$$\sum_{i\neq h} \alpha_i g_i' \frac{\partial s}{\partial x}[x(t)] \sum_{j=1}^{N+1} \lambda_{jp} f[t, z_{jp}, u_h(t, z_{jp})]. \tag{19}$$

Passing to a subsequence we can assume that

$$\lambda_{jp} \to \lambda_j \geq 0, \quad \sum_{j=1}^{N+1} \lambda_j = 1. \tag{20}$$

Then the corresponding subsequence of (19) converges towards

$$\sum_{i\neq h} \alpha_i g_i' \frac{\partial s}{\partial x}[x(t)]f(t, x(t), u_h[t, x(t)])$$

due to the continuity of $u_h(t,\cdot)$ and $f(t,\cdot,\cdot)$. Summarizing,

$$s'\dot{s}/|s| = (1/|s|)\sum_{i\neq h} \alpha_i g_i' g_h. \tag{21}$$

Suppose that Case 2 occurs. Write

$$I = \{1, \ldots, M+1\}\backslash\{h\}.$$

Then s belongs to some face of cone(g_i: $i \neq h$). Then there exists a set $J \subset I$ such that

$$s \in \text{bdry cone}(g_i\colon i \in J).$$

Write

$$C = \text{cone}(g_i\colon i \neq h), \quad T = s^{-1}(\text{bdry}\, C).$$

We shall need the following.

LEMMA 8 *The N-dimensional Lebesgue measure of T is* 0.

Taking Lemma 8 for granted, it follows by (14) that there exist points z_{jp} fulfilling (15) such that

$$s(z_{jp}) \notin \text{bdry}\, C \quad \text{for every } j \text{ and } p,$$

moreover there exist numbers λ_{jp} fulfilling (16). Now we have only three possibilities.

Subcase 1: For some subsequence, $s(z_{jp}) \in int\ C$.

As before, for every j and p

$$u^*(t, z_{jp}) = u_h(t, z_{jp}), \tag{18}$$

then, by (17) and (18), $s'\dot{s}$ is the limit as $p \to +\infty$ of

$$\sum_{i \neq h} \alpha_i g_i' \sum_{j=1}^{N+1} \lambda_{jp} \left\{ \left[\frac{\partial s}{\partial x}[x(t)] - \frac{\partial s}{\partial x}(z_{jp}) \right] f[t, z_{jp}, u_h(t, z_{jp})] \right.$$
$$\left. + \frac{\partial s}{\partial x}(z_{jp}) f[t, z_{jp}, u_h(t, z_{jp})] \right\}. \qquad (22)$$

Since $|x(t)| \leq L$, by (15) we see that $|z_{jp}| \leq L + 1$ for every j and p. It follows by (7) that there exists a constant A (independent of t) such that

$$|f(t, z_{jp}, u_h(t, z_{jp})| \leq A.$$

Since $\partial s / \partial x$ is uniformly continuous on compact sets, given $\epsilon > 0$ we get

$$\left| \frac{\partial s}{\partial x}[x(t)] - \frac{\partial s}{\partial x}(z_{jp}) \right| \leq \epsilon$$

for every j and sufficiently large p. It follows that the first term of (22) is bounded above by

$$A\epsilon \left| \sum_{i \neq h} \alpha_i g_i \right|,$$

while the second term may be written as

$$\sum_{i \neq h} \sum_{j=1}^{N+1} \lambda_{jp} \alpha_i \{ [g_i - g_i(t, z_{jp})]' g_h(t, z_{jp}) + g_i(t, z_{jp})' g_h(t, z_{jp}) \}. \qquad (23)$$

Given $\epsilon > 0$, the uniform continuity of $g_i(t, \cdot)$ and the equiboundedness of $g_h(t, z_{jp})$ imply that the first term of (23) is bounded above by $W \epsilon A \sum_{i \neq h} \alpha_i$, where the constant $W \geq |\partial s / \partial x(z_{jp})|$ for every j and p. Assumption (8) implies that the second term of (23) is bounded above by

$$-c^2 \sum_{i \neq h} \sum_{j=1}^{N+1} \alpha_i \lambda_{jp} |g_i(t, z_{jp})| |g_h(t, z_{jp})|.$$

Then $s'\dot{s}/|s|$ is the limit as $p \to +\infty$ of a sequence of points, each term being bounded from above by

$$A\epsilon\left(1 + W\sum_{i\neq h}\frac{\alpha_i}{|s|}\right) - c^2\sum_{i\neq h}\sum_{j=1}^{N+1}\frac{\alpha_i\lambda_{jp}|g_i(t, z_{jp})||g_h(t, z_{jp})|}{|s|}. \tag{24}$$

Subcase 2: For a subsequence $s(z_{jp})$ is exterior to C.

Since s belongs to some face of C, there exists an index set J containing at most $M-1$ points, such that

$$s \in \text{rel int cone}(g_i \colon i \in J),$$

where rel int denotes the relative interior. Thus we can write

$$s = \sum_{i\in J}\alpha_i g_i,$$

where $\alpha_i > 0$ for every $i \in J$, moreover the number of indices in J is the smallest possible. Since $s(z_{jp}) \to s$ as $p \to +\infty$, for every j and p we can find a set H of indices such that $J \subset H \subset \{1, \ldots, M+1\}$ and $s(z_{jp}) = \sum_{i\in H}\beta_i g_i$, where $\beta_i > 0$ for every $i \in H$. Indeed, let $s(z_{jp})$ be represented (for a suitable subsequence) as a positive linear combination of g_i, $i \in P$ with P strictly included in J. Then $s(z_{jp}) \in \text{cone}(g_i \colon i \in P)$ which is closed, hence s would belong to the same cone, contradicting the minimality of J. It follows that for every $j = 1, \ldots, N+1$ and every p there exists an index $k = k(j, p) \notin H$, hence $k \notin J$ as well, such that

$$u^*(t, z_{jp}) = u_k(t, z_{jp}).$$

By taking a subsequence of z_{jp} we can choose $k = k_j$ independent of p. Then $s'\dot{s}$ is the limit as $p \to +\infty$ of the sequence

$$\sum_{i\in J}\sum_{j=1}^{N+1}\left\{\alpha_i\lambda_{jp}\left[g_i'\frac{\partial s}{\partial x}[x(t)] - g_i(t, z_{jp})'\frac{\partial s}{\partial x}(z_{jp})\right]f[t, z_{jp}, u_{k_j}(t, z_{jp})]\right.$$
$$\left. + g_i(t, z_{jp})'\frac{\partial s}{\partial x}(z_{jp})f[t, z_{jp}, u_{k_j}(t, z_{jp})]\right\}. \tag{25}$$

As in the calculations yielding (24), given $\epsilon > 0$, $s'\dot{s}/|s|$ is the limit as $p \to +\infty$ of a sequence, each term of which is bounded above by

$$\frac{\epsilon A \sum_{i \in J} \alpha_i}{|s|} - c^2 \sum_{i \in J} \sum_{j=1}^{N+1} \lambda_{jp} \alpha_i \frac{|g_i(t, z_{jp})||g_{k_j}(t, z_{jp})|}{|\sum_{i \in J} \alpha_i g_i|}. \tag{26}$$

Having fixed the vectors $g_i = g_i[t, x(t)]$, $i \neq h$, consider

$$Z = \{w \in R^M\colon w_i \geq 0 \text{ for every } i \text{ and } w \neq 0\},$$

$$y(w) = \sum_{i \neq h} w_i g_i' g_h \Big/ \left| \sum_{i \neq h} w_i g_i \right|, w \in Z. \tag{27}$$

Then y is continuous on Z, positively homogeneous of degree 0, and by (8) $y(w) \leq -c^2 a$, $w \in Z$. Consider $D = \{w \in Z\colon |w| = 1\}$. Since D is compact, it follows that there exists a constant $k \neq 0$ (independent of t) such that

$$\max y(D) = -k^2 = \max y(Z). \tag{28}$$

In Case 1 we have, by (21) and (28)

$$s'\dot{s} \leq -k^2 |s|. \tag{29}$$

Now we show that (29) holds also in Case 2. Remembering (20), (24), (26) and (28) we get in each subcase

$$\frac{s'\dot{s}}{|s|} \leq \epsilon A \left(1 + W \sum_{i \in J} \frac{\alpha_i}{|s|}\right) - c^2 \sum_{i \in J} \sum_{j=1}^{N+1} \frac{\lambda_j \alpha_i |g_i||g_{k_j}|}{|s|}, \tag{30}$$

where $W \geq 0$ and each $k_j \notin J$, a suitable subset (abusing notation) of $\{1, \ldots, M+1\}$. Given any nonzero point w such that $w_i \geq 0$, $i \in J$, consider

$$q(w) = \sum_{i \in J} w_i \Big/ \left| \sum_{i \in J} w_i g_i \right|.$$

Then q is everywhere positive and positively homogeneous of degree 0. Hence $q(w) \leq \sup\{q(y): y_i \geq 0 \text{ for every } i, |y| = 1\}$ which is finite, as we prove in the following lines. It suffices to show that there exists a constant $b > 0$ such that

$$\left|\sum_{i=1}^{M+1} w_i y_i\right| \geq b \tag{31}$$

for every choice of the vectors $y_1, \ldots, y_{M+1}$ in R^M fulfilling $|y_i| \geq a$ for each i, $y_i' y_j \leq -c^2 |y_i||y_j|$ if $i \neq j$, and for every choice of the nonnegative numbers $w_1, \ldots, w_{M+1}$ fulfilling $w_h = 0$ for some index h, with $\sum_{i=1}^{M+1} w_i^2 = 1$. Arguing by contradiction, suppose that (31) fails. Then we can find sequences y_i^n, w_i^n, $i = 1, \ldots, M+1$ such that $w_i^n \geq 0$,

$$\sum_{i=1}^{M+1} w_i^n y_i^n \to 0 \quad \text{as } n \to +\infty, \tag{32}$$

$$\sum_{i=1}^{M+1} (w_i^n)^2 = 1, \tag{33}$$

$w_h^n = 0$ for some index $h = h(n)$, and $|y_i^n| \geq a$, $y_i^{n\prime} y_j^n \leq -c^2 |y_i^n||y_j^n|$ if $i \neq j$. Passing to some subsequence we can assume that h does not depend of n. Consider

$$z_i^n = y_i^n / a.$$

By (32), $\sum_{i=1}^{M+1} w_i^n z_i^n \to 0$, hence $\sum_{i=1}^{M+1} w_i^n z_i^{n\prime} z_h^n \to 0$. However $z_i^{n\prime} z_h^n \leq -c^2 |z_i^n||z_h^n|$ and $|z_i^n| \geq 1$, whence

$$\sum_{i=1}^{M+1} w_i^n z_i^{n\prime} z_h^n \leq -c^2 \sum_{i=1}^{M+1} w_i^n.$$

Then we get $\limsup \sum_{i=1}^{M+1} w_i^n \leq 0$, hence $\sum_{i=1}^{M+1} w_i^n \to 0$, whence $w_i^n \to 0$ for each i, contradicting (33). In particular

$$\sum_{i \in J} \alpha_i / |s| \leq Q \tag{34}$$

for a suitable constant Q (independent of t), for every α_i and $t \geq 0$. Since $|g_{k_j}| \geq a$ by assumption, the last term of (30) is bounded above by

$$-(c^2 a/|s|) \sum_{i \in J} \alpha_i |g_i| \leq -c^2 a. \tag{35}$$

Then by (34) and (35) the right-hand side of (30) is bounded above by $\epsilon A(1 + WQ) - c^2 a \leq -k^2$ for some $k \neq 0$, provided ϵ is sufficiently small. This shows that (29) holds in the previous subcases.

Subcase 3: We can partition $\{1, \ldots, N+1\}$ in two nonempty subsets F, G such that for a subsequence $s(z_{jp}) \in \operatorname{int} C$ if $j \in F$, and $s(z_{jp})$ is exterior to C if $j \in G$. Following the reasoning of the previous subcases, we have

$$u^*(t, z_{jp}) = u_h(t, z_{jp}) \text{ if } j \in F, u^*(t, z_{jp}) = u_{k_j}(t, z_{jp}) \quad \text{if } j \in G,$$

where h and each k_j do not belong to J. By (17), $s'\dot{s}$ is the limit as $p \to +\infty$ of

$$\sum_{i \in J} \alpha_i g_i' \frac{\partial s}{\partial x}[x(t)] \left(\sum_{j \in F} \lambda_{jp} f[t, z_{jp}, u_h(t, z_{jp})] + \sum_{j \in G} \lambda_{jp} f[t, z_{jp}, u_{k_j}(t, z_{jp})] \right).$$

Using the same estimates as in the previous subcases, by (24), (26), (34) and (35) we get again (29).

Thus (29) holds in all cases, and as well known, (29) yields the conclusion. □

Proof of Lemma 8 Let $E = \operatorname{bdry} C$. Let A be any bounded measurable subset of $s^{-1}(E)$. Then the restriction of s to A is Lipschitz, and it can be extended to a Lipschitz function on the whole R^N. The coarea formula ([10, Theorem 1, p. 112]) yields

$$\begin{aligned}\int_A |Ds(x)|\, dx &= \int_{R^M} H^{N-M}[A \cap s^{-1}(y)]\, dy \\ &\leq \int_E H^{N-M}[s^{-1}(E) \cap s^{-1}(y)]\, dy = 0\end{aligned}$$

since meas $E=0$. By arbitrariness of A we get $\int_{s^{-1}(E)}|Ds(x)|\,dx=0$. Since $|Ds(x)|>0$ almost everywhere by (6), it follows that $s^{-1}(E)$ is of measure 0. □

Remarks (1) Existence of Filippov solutions to (1) in $(0,+\infty)$ corresponding to u^* is assured provided f is a Carathéodory function in $[0,+\infty)\times R^N\times U$ and for every compact set $B\subset R^N$ there exists $A>0$ such that $|f(t,x,u)|\le A$ if $t\ge 0$, $x\in B$, $u\in U$ ([5, Theorem 4, p. 212]).

(2) In order to check the boundedness condition (4) is often possible to rely on a priori estimates also involving bounds on the initial states we want to control. As a particular case, let f be as in the above Remark 1. Suppose there exists a function B such that $\int_0^{+\infty} B(t)\,dt$ converges and $|f(t,x,u)|\le B(t)$ for almost every t, every x,u. Then (4) is fulfilled if the initial states are uniformly bounded. (General criteria yielding bounded states can be obtained from [6, Theorem 5, p. 151].)

7. EXAMPLES

(1) Let $M=1$ so that the sliding manifold (3) (modulo condition (6)) is of dimension $N-1$ in R^N. Then the key condition (8) reduces to the following. For every $t\ge 0$ and $x\in\Omega$ with $s(x)\ne 0$ there exist $u_1,u_2\in U$ such that $g_1(t,x)>0$, $g_2(t,x)<0$. In this case the simplex control law is the standard one in variable structure control theory (see [1, Chapter I]), namely $u^*=u_2$ if $s(x)>0$, $u^*=u_1$ if $s(x)<0$.

(2) Consider the scalar input system

$$\dot{x}_1 = u - x_1,\ \ \dot{x}_2 = x_1 - u^3,\ \ \dot{x}_3 = u,\ \ U=[-2,2]$$
$$s_1(x) = x_1 + x_2,\ \ s_2(x) = x_3.$$

Here $N=3$, $M=2$, $K=1$. We have $(\partial s/\partial x)f=(u-u^3,u)'$ and by taking $u_1=1$, $u_2=-2$, $u_3=-1/\sqrt{3}$ we get a (fixed) simplex with obtuse angles (Fig. 1). The dynamic behavior of some states x corresponding to the simplex control is shown in Fig. 2, where $x(0)=(10,20,30)'$ or $(100,200,300)'$.

(3) An interesting example of a real control system is given by the control of a finger of an artificial hand especially designed for

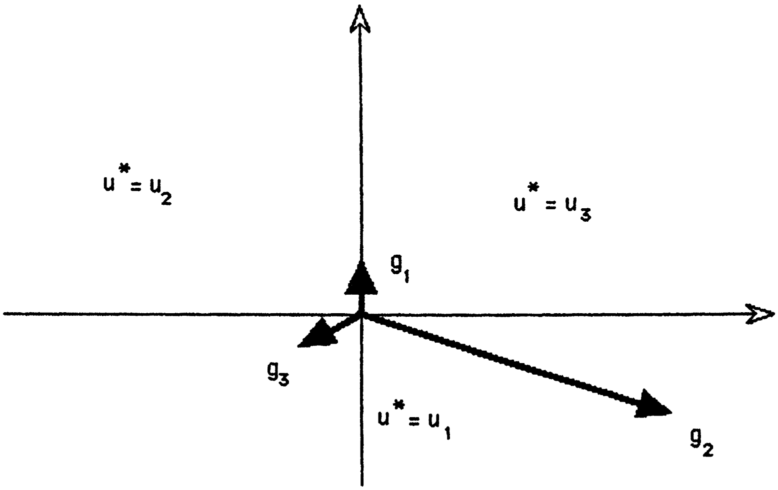


FIGURE 1 Simplex with obtuse angles.

underwater operations. (This system has been developed at DIST, University of Genova, European project AMADEUS.) The finger, see Fig. 3, is constituted by three bellows filled with oil and connected by two plates. The hand has three fingers, one plate is rigidly connected to the base of the hand and the other (the tip) is free. The relative position of the two plates is kinematically determined by a rigid stick connecting the centre of the tip to the centre of the base through a cardan joint, see Fig. 3. The bellows are coupled with other control bellows whose length is modified through the pressure exerted by a linear motor of the voice-coil type (2 kHz bandwith), see Fig. 3. The dynamics of the centre of mass of the tip are given by

$$\dot{x}_1 = x_2,\ \dot{x}_2 = -k_1(x_1 - x_4),\ \dot{x}_3 = x_4,\ \dot{x}_4 = k_1(x_1 - x_4) + \alpha u_1,$$
$$\dot{x}_5 = x_6,\ \dot{x}_6 = -k_2(x_5 - x_8),\ \dot{x}_7 = x_8,\ \dot{x}_8 = k_2(x_5 - x_8) + \alpha u_2. \tag{36}$$

The components of the control vector, given by

$$u_1 = (\sqrt{3}/2)d(F_2 - F_3), \quad u_2 = (d/2)(F_2 + F_3 - 2F_1), \tag{37}$$

are the components of the torque generated by the three forces with modulus F_1, F_2, F_3 respectively, developed by the three motors. In (37)

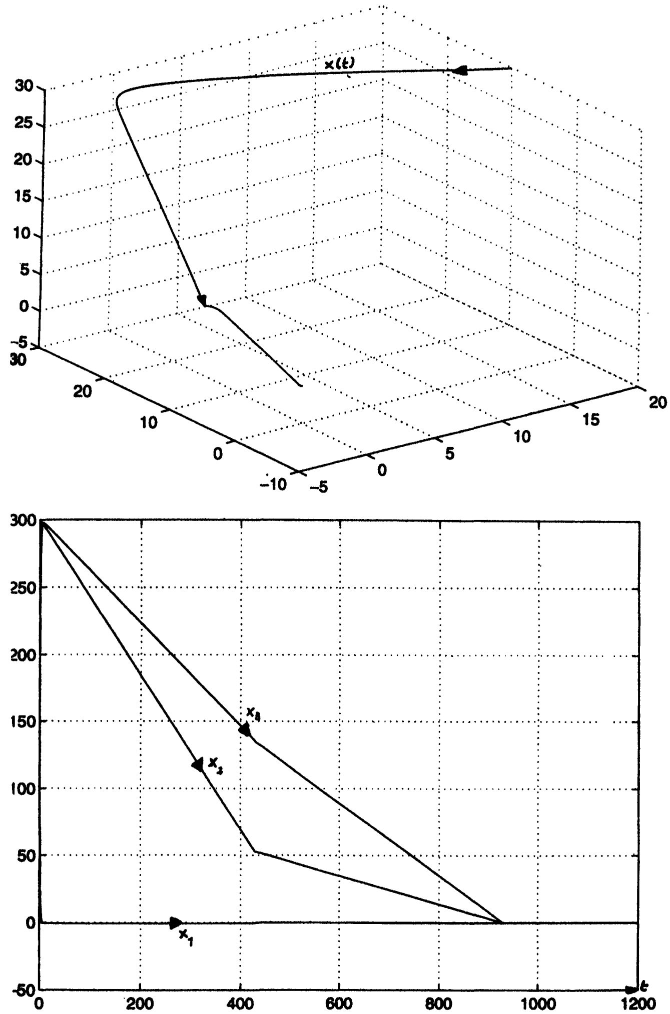
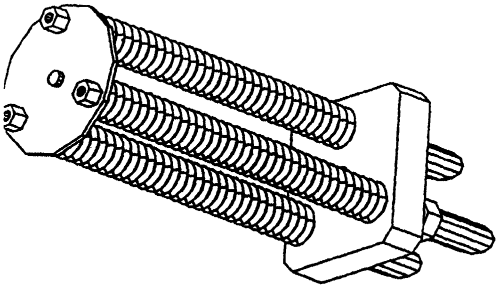
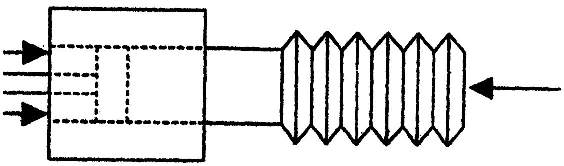


FIGURE 2

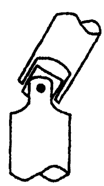
d is the constant distance between the bellows. In (36), (x_1, x_5) are the coordinates of the centre of gravity of the tip, (x_3, x_7) are the projections of the displacements of the control bellows on the reference frame and (x_4, x_8) are their velocities. All these quantities are measurable or computable by using the sensing devices. Moreover



The finger.



Control bellows.



Cardan joint.

FIGURE 3

α is a positive constant, k_1 and k_2 are elastic constants. We fix two constants x_1^*, x_5^*, the desired positions to be tracked. Then the sliding manifold is given by

$$\begin{aligned} s_1 &= x_4 + c_3 x_3 + c_1(x_1 - x_1^*) = 0, \\ s_2 &= x_8 + c_7 x_7 + c_5(x_5 - x_5^*) = 0. \end{aligned} \tag{38}$$

On the sliding manifold (38), the zero dynamics are those of a couple of third order asymptotically stable systems if the constants c_1, c_3, c_5, c_7 are suitably chosen. The nature of the control system (36)–(38) is well suited for the use of the simplex method. By activating the motors, one at a time in a fixed direction with a suitably chosen constant force modulus, due to the position of the bellows we obtain three vectors forming a simplex with obtuse angles.

In the following we check the assumptions required by our convergence theorem. Our example has $N=8$, $M=K=2$. We fix arbitrarily $L>0$. Of course assumptions (6) and (7) are verified. To fulfill (8) we consider the following three control vectors

$$v=(0,-dF)',\ \ w=(\sqrt{3}dF/2,dF/2)',\ \ z=(-\sqrt{3}dF/2,dF/2)',\quad (39)$$

where $F=F_1=F_2=F_3$ is a positive constant to be chosen later. Thus v,w,z are the control laws corresponding to the activation of the motors one at the time, respectively. Now let $s=(s_1,s_2)'$ be given by (38) and consider

$$g_1=\frac{\partial s}{\partial x}f(t,x,v),\ \ g_2=\frac{\partial s}{\partial x}f(t,x,w),\ \ g_3=\frac{\partial s}{\partial x}f(t,x,z),$$

where, according to (36), $f(t,x,u)$ is the vector of components x_2, $-k_1(x_1-x_4)$, $x_4, k_1(x_1-x_4)+\alpha u_1$, $x_6, -k_2(x_5-x_8), x_8, k_2(x_5-x_8)+\alpha u_2$. Write

$$\phi_1(x)=c_1x_2+c_3x_4+k_1(x_1-x_4),$$
$$\phi_2(x)=c_5x_6+c_7x_8+k_2(x_5-x_8)$$

and fix any $c>0$. Elementary computations show the following properties. There exists a constant $H>0$ such that if $|x|\le L$ then

$$|g_1|+|g_2|+|g_3|\le H.$$

Then $g_i'g_h\le -c^2|g_i||g_h|$, as required by (8), if $g_i'g_h\le -c^2H^2$, $i\ne h$. On the other hand

$$g_1'g_2=-(\alpha^2/2)d^2F^2+(\alpha/2)d(\sqrt{3}\phi_1-\phi_2)F+\phi_1^2+\phi_2^2,$$
$$g_1'g_3=-(\alpha^2/2)d^2F^2-(\alpha/2)d(\phi_2+\sqrt{3}\phi_1)F+\phi_1^2+\phi_2^2,$$
$$g_2'g_3=-(\alpha^2/2)d^2F^2+\alpha d\phi_2F+\phi_1^2+\phi_2^2,$$

hence (8) is fulfilled if F is a suitably large constant. Finally we have

$$|g_1|^2 \geq \alpha^2 d^2 F^2 - 2\phi_2 \alpha d F,$$

$$|g_2|^2 \geq \alpha^2 d^2 F^2 + \alpha d(\phi_2 + \sqrt{3}\phi_1)F,$$

$$|g_3|^2 \geq \alpha^2 d^2 F^2 + \alpha d(\phi_2 - \sqrt{3}\phi_1)F.$$

Fix any $a > 0$, then $|g_i| \geq a$, $i = 1, 2, 3$, when $|x| \leq L$ if F is a suitably large constant.

In conclusion, all conditions required by the convergence theorem are satisfied by a proper choice of F. This choice can be made within the range of the forces allowing safe performance of the system. Examples of the real behavior of the finger tip when tracking a time-varying trajectory are shown in Fig. 4.

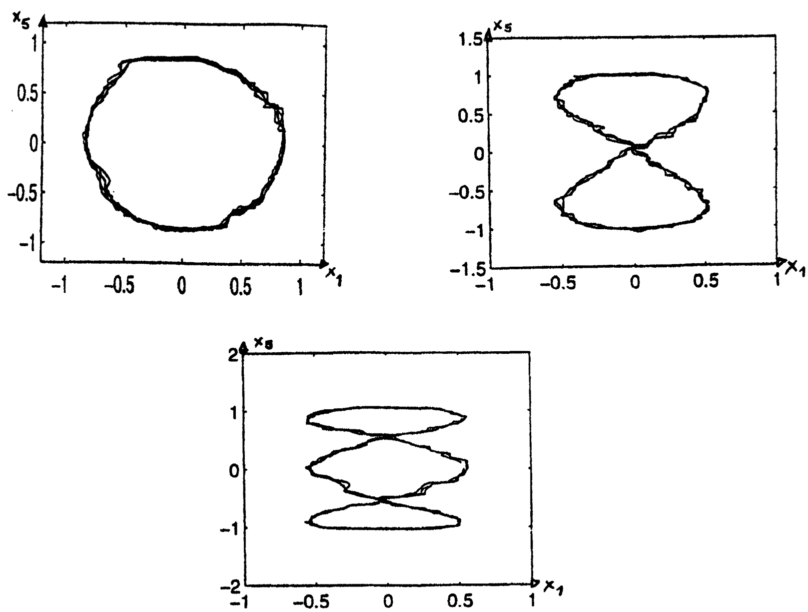


FIGURE 4

8. CONSTRUCTION OF A SIMPLEX

The simplex method was introduced in [2] for smooth dynamics of the form

$$f(t,x,u) = A(t,x) + B(t,x)u, \quad K = M$$

and time dependent sliding manifolds, based on considering a fixed simplex of constant vectors $u_i \in R^M$, $i = 1, \ldots, M+1$. The convergence of the method was proved assuming that $(\partial s/\partial x)B$ is uniformly nonsingular, without requiring the obtuse angle condition (8) about $(\partial s/\partial x)f(t,x,u_i)$, $i = 1, \ldots, M+1$.

In this section we show, in a particular case, how to build a simplex of vectors g_i, $i = 1, \ldots, M+1$, fulfilling the conditions required by our convergence theorem.

Step 1: In the following we describe, in a recursive fashion, the construction of a fixed simplex in R^M with obtuse angles. The procedure ends just at the Mth step.

(1) We choose any vector $E_1 \in R^M$, $|E_1| = 1$, and we consider $V_1 = -E_1$, thus $E_1' V_1 < 0$. Set $\alpha_1 = 1$.

(2) Choose a unit vector N_1 orthogonal to E_1 (certainly existing if $M \geq 2$) and consider

$$E_1, E_2 = (V_1 + N_1)/\sqrt{2}, \quad V_2 = -(E_1 + E_2)/\alpha_2$$

with $\alpha_2 > 0$ such that $|V_2| = 1$. Then

$$E_1' E_2 < 0, \;\; E_1' V_2 < 0, \;\; E_2' V_2 < 0, \;\; \alpha_2^2 = |V_1 - E_2|^2 = 2 - \sqrt{2}$$

thus $1/\sqrt{2} < \alpha_2 < \sqrt{2}$.

(3) Choose a unit vector N_{12} orthogonal to both E_1, E_2 with $|N_{12}| = 1$ (certainly existing if $M \geq 3$) and consider

$$E_1, E_2, E_3 = (V_2 + N_{12})/\sqrt{2}, \quad V_3 = -(E_1 + E_2 + E_3)/\alpha_3$$

with $\alpha_3 > 0$ such that $|V_3| = 1$. Then

$$E_i' E_3 < 0 \quad \text{if } i \neq 3, \qquad E_i' V_3 < 0 \quad \text{if } i = 1, 2, 3,$$

$\alpha_3^2 = |\alpha_2 V_2 - E_3|^2 = \alpha_2^2 - \sqrt{2}\alpha_2 + 1$, thus $1/\sqrt{2} < \alpha_3 < \sqrt{2}$.

(4) Let $E_1, E_2, \ldots, E_{K-1}$, V_{K-1} be K unit vectors of R^M, $K \le M$, such that $E_i' V_{K-1} < 0$, $i = 1, \ldots, K-1$, $V_{K-1} = -(1/\alpha_{K-1}) \sum_{i=1}^{K-1} E_i$, $1/\sqrt{2} < \alpha_{K-1} < \sqrt{2}$. Choose a unit vector $N_{1,2,\ldots,K-1}$ orthogonal to E_i, $i = 1, \ldots, K-1$ (certainly existing if $M \ge K$) and consider

$$E_1, E_2, \ldots, E_{K-1}, E_K = (1/\sqrt{2})(V_{K-1} + N_{1,2\ldots,K-1}),$$

$$V_K = -(1/\alpha_K) \sum_{i=1}^{K} E_i = (1/\alpha_K)(\alpha_{K-1} V_{K-1} - E_K),$$

with $\alpha_K > 0$ such that $|V_K| = 1$. Then

$$E_i' E_K = (1/\sqrt{2}) E_i' V_{K-1} < 0, \;\; E_i' V_K < 0$$
$$\text{if } i = 1, \ldots, K-1, \;\; E_K' V_K < 0,$$

$\alpha_K^2 = |\alpha_{K-1} V_{K-1} - E_K|^2 = \alpha_{K-1}^2 - \sqrt{2}\alpha_{K-1} + 1$, thus $1/\sqrt{2} < \alpha_K < \sqrt{2}$. In particular ($K = M$) we get a simplex with obtuse angles made up by the unit vectors $E_1, \ldots, E_M, E_{M+1} = V_M$ of R^M.

Step 2: The following procedure, of a local character, shows a way to build in a special case a simplex fulfilling (8). Assume that the control vector has M components (a significant case in variable structure control theory, see [1]), that the system is autonomous, 0 is an interior point of the control region U, $f(x, 0) = 0$ for all x with $|x| \le L$, $f(x, \cdot)$ is differentiable in a neighborhood of 0 if $|x| \le L$ with $\partial f/\partial u$ depending continuously on x, u, and

$$\frac{\partial g}{\partial u}(x, 0) \text{ is nonsingular} \quad \text{if } |x| \le L,$$

where $g(x, u) = (\partial s/\partial x)(x) f(x, u)$. Then (by continuity) there exists $r > 0$ such that $(\partial g/\partial u)(x, u)$ is nonsingular if $|x| \le L$ and $|u| \le r$. By the local inverse function theorem, $g(x, \cdot)$ is one-to-one on $B(0, r)$ the open ball in R^M of center 0 and radius r. Since $g(x, 0) = 0$ we have $|g(x, u)| > 0$ if $|u| = r$ and $|x| \le L$, whence

$$2m = \min\{|g(x, u)| : |u| = r, |x| \le L\} > 0.$$

We check that

$$B(0, m) \subset g[x, B(0, r)] \quad \text{if } |x| \le L. \tag{40}$$

Fix x with $|x| \le L, y \in R^M$ with $|y| < m$ and consider $h(u) = |g(x, u) - y|$, $|u| \le r$. If $|u| = r$ we have $h(u) \ge |g(x, u)| - |y| \ge m$, moreover $h(0) < m$, hence the global minimum value of h is attained at some interior point $\bar{u}$. It follows that $\bar{u}$ is a critical point of h^2, hence

$$\sum_{i=1}^{M} (g_i(x, \bar{u}) - y_i) \frac{\partial g_i}{\partial u_j}(x, \bar{u}) = 0, \quad j = 1, \ldots, M,$$

whence $g(x, \bar{u}) = y$. This yields (40). Now consider the vectors $E_1, \ldots, E_{M+1}$ obtained in the previous Step 1, and let

$$e_i = mE_i, \quad i = 1, \ldots, M+1.$$

By the local inverse function theorem, for every x with $|x| \le L$ we can find points $u_i = u_i(x)$ such that $|u_i(x)| \le r$ and $g[x, u_i(x)] = e_i$, $i = 1, \ldots, M+1$. Uniqueness of u_i implies their continuity. Finally (8) is obviously fulfilled.

Remarks (1) The proof above is nothing else than the standard variational proof of the local surjectivity theorem (see e.g. [11, Theorem 7.3, p. 141]) taking into account the uniformity with respect to the parameter x.

(2) Any simplex $E_1, \ldots, E_{M+1}$ fulfilling (8) will work in the previous construction.

(3) The same construction works in principle in the case when $g(x, 0) \ne 0$, provided we can apply a global inverse function theorem to $g(x, \cdot)$.

9. CONCLUSIONS AND FUTURE DIRECTIONS

In this paper we have extended the simplex control method of [2] to general nonlinear control systems, assuming that the plant and the state vector are completely known to the controller. A convergence theorem has been proved, and a method of constructing the required simplex has been described.

Due to the robustness properties inherited by the method, and taking into account the results of [3,4], it is natural to guess that a suitable version of the simplex method can be formulated to deal with nonlinear control systems subject to deterministic uncertainty, provided that the system states are available to the controller. We expect that convergence properties in the uncertain case hold along the lines presented here.

A second interesting development deals with the possibility to avoid the obtuse angle condition (8) by using a fixed simplex of vectors.

Work is in progress about the above topics.

Acknowledgements

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References

[1] V.I. Utkin, *Sliding Modes in Control Optimization*, Springer, 1992.
[2] S.V. Bajda and D.B. Izosimov, Vector method of design of sliding motion and simplex algorithms, *Automation Remote Control* **46** (1985), 830–837.
[3] G. Bartolini, A. Ferrara, V. Utkin and T. Zolezzi, A control vector simplex approach to variable structure control of nonlinear systems, *Internat. J. Robust Nonlin. Control* **7** (1997), 321–335.
[4] G. Bartolini and A. Ferrara, Multi-input sliding mode control of a class of uncertain nonlinear systems, *IEEE Trans. Automatic Control* **41** (1996), 1662–1666.
[5] A.F. Filippov, Differential equations with discontinuous right-hand side, *Amer. Math. Soc. Translations* **42** (1964), 199–231.
[6] A.F. Filippov, *Differential Equations with Discontinuous Righthand Sides*, Kluwer, 1988.
[7] G. Strang, *Linear Algebra and its Applications*, Academic Press, 1976.
[8] R.T. Rockafellar, Integral functionals, normal integrands and measurable selections, *Lecture Notes in Math.* 543, Springer, 1976, 157–207.
[9] R.T. Rockafellar, *Convex Analysis*, Princeton University Press, 1970.
[10] L.C. Evans and R.F. Gariepy, *Measure Theory and Fine Properties of Functions*, CRC Press, 1992.
[11] T.M. Apostol, *Mathematical Analysis*, Addison-Wesley, 1964.


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