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Forced Vibration of a Ball Attached to a Cable

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This work describes an analytic solution to predict the forced oscillation of a suspended cable and an attached ball. The oscillations are driven by a sinusoidal movement at the fixed end of the cable. This problem may be used in the verification of numerical software which is commonly used to design systems with suspended cables.

Keywords: CAD code verification; Dynamic systems; Cables vibrations; Waves

INTRODUCTION

Suspension cables are common elements of large-scale structures such as stadiums and bridges. With the advent of computer-aided-design (CAD) software packages it has become common that such structures are designed and analyzed with CAD software. The value of using such software packages in the analysis of structures is well known.

To insure that these CAD software programs accurately predict the behavior of the structure caused by both internal and external forces it is important that each software package be tested in each area in which the software is used for structural analysis. One of the problems associated with extensive use of CAD is that the accuracy of many software packages is not adequately tested for certain situations for which it is used.

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The intent of this work is to present an analytic solution to predict the vibration in a suspended cable which results from the forced oscillation at the upper-end of the cable. It is expected that this solution may be used as a verification problem for CAD software. A practical physical example of this system is a wrecking ball attached to a chain which is being forced to oscillate at the upper end of the chain.

ANALYSIS

Problem Statement

The problem is illustrated in Fig. 1. A ball with a mass of m_p is suspended from a flexible cable of mass m_c. The upper end of the cable is forced to oscillate in the lateral direction according to the relation:

$$y(L, t) = \epsilon \sin(\omega t). \tag{1}$$

Some important assumptions used in this analysis are:

- Attributes associated with the cable:
  (1) The cable has a fixed length of L;
  (2) It is infinitely flexible (it may sustain no moment-arm);
  (3) The measure of the angle between the x-axis and the cable, $\alpha(x)$, is always small such that $\alpha(x) \ll 1$;

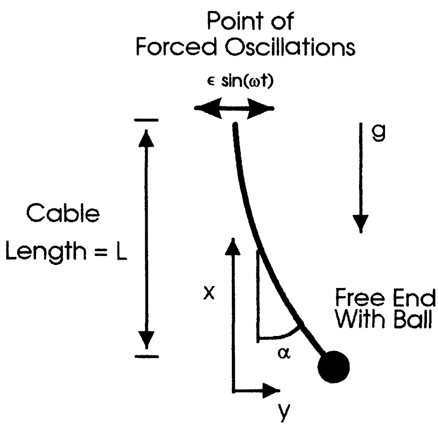

FIGURE 1 Schematic of a cable with an attached ball.

(4) The cable has a mass of m_c which is uniformly distributed over the cable length;
(5) At $x = L$, the cable is forced to oscillate with $y(L, t) = \epsilon \sin(\omega t)$;
(6) Cable motion is only in the lateral direction with $y(x, t)$ being the dependent variable (no variation in the z-direction).

- Attributes associated with the ball:
  (1) The suspended ball is attached to the cable at $x = 0$;
  (2) The ball has a mass of m_p but no volume (infinite density).
- Attributes associated with both the cable and ball system:
  (1) The system is in a constant gravitational field g which acts in the opposite direction of the x-axis (normally $g = 9.81$ N/kg);
  (2) There are no frictional effects;
  (3) The system is initially at rest with $y(x, 0) = \dot{y}(x, 0) = 0$.

Mathematical Formulation

The partial differential equation associated with the lateral vibration of a cable of tension T is derived in many texts on applied differential equations and is given by the following equation:

$$\frac{m_c}{L}\frac{\partial^2 y}{\partial t^2} = \frac{\partial}{\partial x}\left(T\frac{\partial y}{\partial x}\right) \quad \text{for } 0 < x < L, \tag{2}$$

where T is the tension in the cable. Normally, the tension is treated as a constant. In this situation, the tension varies with x, since as x increases the total weight carried by the cable increases linearly:

$$T(x) = g\left(m_p + m_c\frac{x}{L}\right) \quad \text{for } 0 < x < L.$$

Substituting this relation into Eq. (2) yields the following partial differential equation for the motion for the cable:

$$\frac{m_c}{L}\frac{\partial^2 y}{\partial t^2} = g\frac{\partial}{\partial x}\left[\left(m_p + m_c\frac{x}{L}\right)\frac{\partial y}{\partial x}\right] \quad \text{for } 0 < x < L. \tag{3}$$

Equation (3) may be set into a non-dimensional form using the following dimensionless parameters:

$$M = \frac{m_{\mathrm{p}}}{m_{\mathrm{c}}},$$

$$r = 2\sqrt{M + \frac{x}{L}},$$

$$\tau = t\sqrt{\frac{g}{L}},$$

and

$$h(r, \tau) = \frac{y(x, t)}{\epsilon}.$$

The non-dimensional partial differential equation is

$$\frac{\partial^2 h}{\partial \tau^2} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial h}{\partial r}\right] \quad \text{for } 2\sqrt{M} < r < 2\sqrt{M+1}. \tag{4}$$

This may be restated as the wave equation in cylindrical coordinates:

$$\frac{\partial^2 h}{\partial \tau^2} = \nabla^2 h$$

where ∇^2 is the one-dimensional laplacian:

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial}{\partial r}\right].$$

The boundary condition where the cable is forced to oscillate, at $x = L$ (or $r = 2\sqrt{M+1}$) is

$$y(x = L, t) = \epsilon \sin(\omega t). \tag{5}$$

In dimensionless parameters, this boundary condition is

$$h(r = 2\sqrt{M+1}, \tau) = \sin(\omega^* \tau) \tag{6}$$

where $\omega^* = \sqrt{L/g}\,\omega$. The boundary condition where the ball is connected to the cable (at $x=0$ or $r = 2\sqrt{M}$) may be derived from Newton's second law of motion:

$$m_{\mathrm{p}} \frac{\partial^2 y}{\partial t^2} = F_y$$

or

$$m_{\mathrm{p}} \left.\frac{\partial^2 y}{\partial t^2}\right|_{x=0} = m_{\mathrm{p}} g \left.\frac{\partial y}{\partial x}\right|_{x=0}. \tag{7}$$

Using dimensionless parameters, this boundary condition is

$$\left.\frac{\partial^2 h}{\partial \tau^2}\right|_{r=2\sqrt{M}} = \frac{1}{\sqrt{M}} \left.\frac{\partial h}{\partial r}\right|_{r=2\sqrt{M}}. \tag{8}$$

The initial conditions imposed on $h(r, \tau)$ are

$$\overset{\text{no displacement}}{h(r,0)} = 0 \quad \text{and} \quad \overbrace{\left.\frac{\partial h}{\partial \tau}\right|_{\tau=0}}^{\text{no velocity}} = 0. \tag{9}$$

The solution to Eq. (4) may be obtained with an analysis using a convolution integral with Duhamel's principle:

$$h(r, \tau) = \int_0^{\tau} F(\tau - \gamma) \frac{\partial \Phi(r, \gamma)}{\partial \gamma}\, \mathrm{d}\gamma. \tag{10}$$

In Eq. (10) $\Phi(r, \tau)$ is the solution to Eq. (4);

$$\frac{\partial^2 \Phi}{\partial \tau^2} = \overbrace{\frac{1}{r}\frac{\partial}{\partial r}\left[r \frac{\partial \Phi}{\partial r}\right]}^{\nabla^2 \Phi} \quad \text{for } 2\sqrt{M} < r < 2\sqrt{M+1}, \tag{11}$$

with the following boundary conditions and initial conditions:

$$\Phi(2\sqrt{M+1}, \tau) = 1, \tag{12}$$

$$\left.\frac{\partial^2 \Phi}{\partial \tau^2}\right|_{r=2\sqrt{M}} = \frac{1}{\sqrt{M}} \left.\frac{\partial \Phi}{\partial r}\right|_{r=2\sqrt{M}}, \tag{13}$$

$$\Phi(r, 0) = 0, \tag{14}$$

and

$$\left.\frac{\partial\Phi}{\partial\tau}\right|_{\tau=0} = 0.$$

The function $F(\tau)$ in Eq. (10) is the boundary condition at the top of the cable, at $r = 2\sqrt{M+1}$:

$$F(\tau) = \sin(\omega^{*}\tau). \tag{15}$$

Solution for $\Phi(r, \tau)$

The solution for $\Phi(r, \tau)$ may be found with the use of separation of variables. The function is assumed to be separable in r and τ:

$$\Phi(r, \tau) = \sum_{n} R_n(r)\Upsilon_n(\tau), \tag{16}$$

where for at least some values of n, $R_n \Upsilon_n \neq 0$. Following the traditional 'separation of variables' procedure, Eq. (11) leads to the following sets of ordinary differential equations:

$$\frac{\ddot{\Upsilon}_n}{\Upsilon_n} = \frac{R_n'' + (1/r)R_n'}{R_n} = -\lambda_n^2. \tag{17}$$

It may be shown that all characteristic values, λ_n^2, are real and R_n may be taken to be real.* It is demonstrated in the Appendix that if λ_n^2 is negative then, $R_n(r) = 0$. There are non-trivial solutions associated with $\lambda_n^2 = 0$ and $\lambda_n^2 > 0$. Each of these two cases are investigated separately.

λ_n^2 *is Zero* When $\lambda_n^2 = 0$, Eq. (17) corresponds to

$$\frac{\ddot{\Upsilon}_0}{\Upsilon_0} = \frac{R_0'' + (1/r)R_0'}{R_0} = 0. \tag{18}$$

where R_0 and Υ_0 correspond to the distinct value $\lambda_0 = 0$.

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* See Problem 28 of Chapter 5 of Hildebrand.

The general solutions to Eq. (18) are

$$\Upsilon_0(\tau) = c_1\tau + c_2$$

and

$$R_0 = d_1 + d_2 \ln(r).$$

The boundary condition for Φ at $r = 2\sqrt{M+1}$, Eq. (12), may be satisfied completely by the product $\Upsilon_0(\tau)R_0(2\sqrt{M+1})$ if

$$\Upsilon_0(\tau) = R_0(r) = 1.$$

Since $R_0(r)$ satisfies completely the boundary condition at $r = 2\sqrt{M+1}$, all other solution components must have the following boundary condition at $r = 2\sqrt{M+1}$:

$$R_n(2\sqrt{M+1}) = 0 \quad \text{for } n \neq 0. \tag{19}$$

λ_n^2 *is Positive* When $\lambda_n^2 > 0$, Eq. (17) corresponds to

$$\frac{\ddot{\Upsilon}_n}{\Upsilon_n} = \frac{R_n'' + (1/r)R_n'}{R_n} = -\lambda_n^2. \tag{20}$$

In this case, the boundary condition at $r = 2\sqrt{M+1}$ results in the solutions for $R_n(r)$ and $\Upsilon_n(\tau)$ being

$$R_n(r) = Y_0(2\lambda_n\sqrt{M+1})J_0(\lambda_n r) - J_0(2\lambda_n\sqrt{M+1})Y_0(\lambda_n r) \tag{21}$$

and

$$\Upsilon_n(\tau) = A_n\cos(\lambda_n\tau) + B_n\sin(\lambda_n\tau).$$

The value of λ_n may be obtained using Eq. (13). In this case, λ_n is the nth positive root of

$$-\lambda_n^2 R_n(2\sqrt{M}) = \frac{1}{\sqrt{M}}\left.\frac{\mathrm{d}R_n}{\mathrm{d}r}\right|_{r=2\sqrt{M}} \tag{22}$$

or

$$\lambda_n R_n(2\sqrt{M}) - \frac{S_n(2\sqrt{M})}{\sqrt{M}} = 0, \tag{23}$$

where

$$S_n(r) = Y_0(2\lambda_n\sqrt{M+1})J_1(\lambda_n r) - J_0(2\lambda_n\sqrt{M+1})Y_1(\lambda_n r). \tag{24}$$

The complete solution for $\Phi(r,\tau)$ is of the form:

$$\Phi(r,\tau) = 1 + \sum_{n=1}^{\infty} R_n(r)[A_n \cos(\lambda_n\tau) + B_n \sin(\lambda_n\tau)]. \tag{25}$$

The unknown coefficients A_n and B_n may be found considering the initial conditions on $\Phi(r,\tau)$:

$$\left.\frac{\partial\Phi}{\partial\tau}\right|_{\tau=0} = 0 \Rightarrow B_n = 0. \tag{26}$$

With $B_n = 0$, Eq. (25) reduces to

$$\Phi(r,\tau) = 1 + \sum_{n=1}^{\infty} A_n R_n(r)\cos(\lambda_n\tau). \tag{27}$$

The coefficients A_n may be obtained by use of the relation

$$\Phi(r,0) = 0 \Rightarrow 1 + \sum_{n=1}^{\infty} A_n R_n(r) = 0. \tag{28}$$

Equation (28) does not give the values of the coefficients A_n in an explicit form. Normally, differential equations, and the associated boundary conditions used in a separation of variable analysis involve a 'proper Sturm–Liouville' problem (see Ref. [2]). With a proper Sturm–Liouville problem the functions $R_n(r)$ would be orthogonal on the interval $2\sqrt{M} < r < 2\sqrt{M+1}$ with respect to the weighting function r. However, this problem is not a proper Sturm–Liouville problem and the functions $R_n(r)$ are not orthogonal.

Approximate Solution for A_n An approximate solution for the coefficients A_n may be obtained by truncation of Eq. (28) with appropriate substitutions:

$$\sum_{n=1}^{n_{\max}} A_n R_n(r) \simeq -1. \tag{29}$$

A system of equations for the coefficients A_n may be found by taking the inner product of each side of Eq. (29) with the function $R_m(r)$ (with $1 \leq m \leq n_{\max}$):

$$\sum_{n=1}^{n_{\max}} A_n [R_n(r) \circ R_m(r)] \simeq -[1 \circ R_m(r)], \tag{30}$$

where the inner product between two functions $f(r)$ and $g(r)$ is defined as:

$$f(r) \circ g(r) \equiv \int_{2\sqrt{M}}^{2\sqrt{M+1}} [r f(r) g(r)]\, \mathrm{d}r.$$

In matrix format, the system of equations implicit in Eq. (30) is

$$\begin{bmatrix} (R_1 \circ R_1) & (R_2 \circ R_1) & \cdots & (R_{n_{\max}} \circ R_1) \\ (R_1 \circ R_2) & (R_2 \circ R_2) & & (R_{n_{\max}} \circ R_2) \\ \vdots & \vdots & \ddots & \vdots \\ (R_1 \circ R_{n_{\max}}) & (R_2 \circ R_{n_{\max}}) & & (R_{n_{\max}} \circ R_{n_{\max}}) \end{bmatrix} \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_{n_{\max}} \end{pmatrix}$$

$$\simeq \begin{pmatrix} -1 \circ R_1 \\ -1 \circ R_2 \\ \vdots \\ -1 \circ R_{n_{\max}} \end{pmatrix}. \tag{31}$$

There are three types of inner products used in Eq. (31), each of which must be treated differently:

(1) The inner product $R_i \circ R_i$ is given by:†

$$
\begin{aligned}
R_i \circ R_i &= \int_{2\sqrt{M}}^{2\sqrt{M+1}} r R_i^2(r)\,\mathrm{d}r \\
&= \left\{ \frac{r^2}{2} [R_i^2(r) + S_i^2(r)] \right\} \cdot \Big|_{2\sqrt{M}}^{2\sqrt{M+1}} \\
&= 2(M+1)\, S_i^2(2\sqrt{M+1}) - 2M\big[R_i^2(2\sqrt{M}) + S_i^2(2\sqrt{M})\big].
\end{aligned}
$$

(2) $R_i \circ R_j$ with $i \neq j$:‡

$$
\begin{aligned}
R_i \circ R_j &= \int_{2\sqrt{M}}^{2\sqrt{M+1}} r R_i(r) R_j(r)\,\mathrm{d}r \\
&= \left(\frac{2\sqrt{M}}{\lambda_j^2 - \lambda_i^2}\right)\big[\lambda_i R_j(2\sqrt{M}) S_i(2\sqrt{M}) - \lambda_j R_i(2\sqrt{M}) S_j(2\sqrt{M})\big].
\end{aligned}
$$

(3) $1 \circ R_i$:¶

$$
\begin{aligned}
1 \circ R_i &= \int_{2\sqrt{M}}^{2\sqrt{M+1}} r R_i(r)\,\mathrm{d}r \\
&= \frac{2}{\lambda_i}\big[\sqrt{M+1}\, S_i(2\sqrt{M+1}) - \sqrt{M} S_i(2\sqrt{M})\big].
\end{aligned}
$$

Implementation of Duhamel's Principle Equations (27) and (15) may be substituted into Eq. (10) to obtain the solution for $h(r, \tau)$:

$$
\begin{aligned}
h(r,\tau) &= \int_0^{\tau} F(\tau - \gamma) \frac{\partial \Phi(r,\gamma)}{\partial \gamma}\,\mathrm{d}\gamma \\
&= \int_0^{\tau} \sin[(\omega^*(\tau - \gamma)] \left\{ \sum_{n=1}^{\infty} -\lambda_n A_n R_n(r) \sin(\lambda_n \gamma) \right\} \mathrm{d}\gamma
\end{aligned}
$$

† Equation (11.3.31) with $\mu = \nu = 0$ of Ref. [1].

‡ Equation (11.3.29) with $\mu = \nu = 0$ of Ref. [1].

¶ Equation (11.3.5) with $p = \nu = 0$ and $\mu = 1$ of Ref. [1].

$$
\begin{aligned}
&= -\sum_{n=1}^{\infty} \lambda_n A_n R_n(r) \int_0^{\tau} \sin[\omega^*(\tau-\gamma)] \sin(\lambda_n \gamma)\, \mathrm{d}\gamma \\
&= \sum_{n=1}^{\infty} A_n R_n(r) \left[\frac{\lambda_n \omega^* \sin(\lambda_n \tau) - \lambda_n^2 \sin(\omega^* \tau)}{\lambda_n^2 - (\omega^*)^2} \right].
\end{aligned} \tag{32}
$$

Using dimensional parameters the solution is

$$
\begin{aligned}
y(x,t) = \epsilon \sum_{n=1}^{\infty} & A_n R_n \left(2\sqrt{\frac{m_\mathrm{p}}{m_\mathrm{c}} + \frac{x}{L}} \right) \\
& \times \left[\frac{\lambda_n \omega \sqrt{L/g} \sin(\sqrt{g/L}\, \lambda_n t) - \lambda_n^2 \sin(\omega t)}{\lambda_n^2 - ((L\omega^2)/g)} \right].
\end{aligned} \tag{33}
$$

Truncation and Convergence As a practical matter, Eq. (32) and its dimensional counter part Eq. (33) must be truncated. Approximations for the coefficients A_n were obtained from the system of equations implicit in Eq. (31) for $1 \le n \le n_{\mathrm{max}}$. The solution $h_{n_{\mathrm{max}}}(r,\tau)$ is defined to be the approximation for $h(r,\tau)$ when the series is truncated at n_{max}:

$$
h_{n_{\mathrm{max}}}(r,\tau) = \sum_{n=1}^{n_{\mathrm{max}}} A_n R_n(r) \left[\frac{\lambda_n \omega^* \sin(\lambda_n \tau) - \lambda_n^2 \sin(\omega^* \tau)}{\lambda_n^2 - (\omega^*)^2} \right]. \tag{34}
$$

Values of A_n are taken as zero for $n > n_{\mathrm{max}}$.

It was found that the convergence was fairly slow with respect to the specific value of n_{max}. The convergence rate for Eq. (34) is illustrated in Fig. 2 for values of n_{max} of 2, 3, 99, and 100. Notice that the curve for $n_{\mathrm{max}} = 99$ is nearly indistinguishable from the curve with $n_{\mathrm{max}} = 100$.

VERIFICATION EXERCISES

To help insure that Eq. (34) has been correctly derived and implemented, a set of three analytic verification problems have been developed. The first involves the time required for the initial disturbance to propagate downwards to the ball. The second is a comparison with the special case where the ratio of masses, M, is very small (a very light ball). The final test is for the special case where M is very large (a very heavy ball).

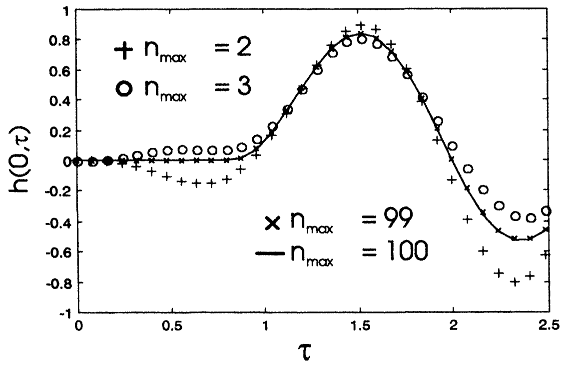

FIGURE 2 Convergence of the truncated series for the special case of $M=1$, $\omega^*=4$, for $n_{\max}=2$, 3, 99, and 100.

Propagation Time

The time required for the initial disturbance wave to travel through the cable may be calculated as the integral of the inverse of the local wave velocity:

$$t_{\mathrm{p}} = \int_L^0 \frac{\mathrm{d}x}{v(x)},$$

where t_{p} is the time required for the initial displacement wave to propagate down the cable, and $v(x)$ is the local wave propagation velocity.

The local wave propagation speed is

$$|v(x)| = \sqrt{\frac{T(x)}{\rho}}.$$

Initially, the wave is traveling in a direction which is opposite to that of the x-axis resulting in the following integral:

$$\begin{aligned} t_{\mathrm{p}} &= \int_L^0 \frac{\mathrm{d}x}{v(x)} = -\int_L^0 \sqrt{\frac{m_{\mathrm{c}}/L}{g(m_{\mathrm{p}} + m_{\mathrm{c}}(x/L))}}\,\mathrm{d}x \\ &= \sqrt{\frac{1}{Lg}} \int_0^L \frac{\mathrm{d}x}{\sqrt{M + (x/L)}} \\ &= 2\sqrt{\frac{L}{g}}\left(\sqrt{M+1} - \sqrt{M}\right). \end{aligned}$$

In terms of the dimensionless time this is

$$\tau_{\mathrm{p}} = 2(\sqrt{M+1} - \sqrt{M}). \tag{35}$$

Special Case Where $M \ll 1$

The second verification exercise is for the special case were the ball is much lighter than the cable, $M \ll 1$. For this special case, the non-dimensional partial differential equation is:

$$\frac{\partial^2 h}{\partial \tau^2} = \frac{1}{r}\frac{\partial}{\partial r}\left[r\frac{\partial h}{\partial r}\right] \quad \text{for } 0 < r < 2.$$

The boundary conditions and initial conditions are

$$h(2, \tau) = \sin(\omega^* \tau),$$

$$\overbrace{h(r, 0)}^{\text{no displacement}} = 0 \quad \text{and} \quad \overbrace{\left.\frac{\partial h}{\partial \tau}\right|_{\tau=0}}^{\text{no velocity}} = 0.$$

The solution to this system of equations may be obtained using Duhamel's principle and orthogonal Bessel functions with

$$\Phi(r, \tau) = 1 - \sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{\lambda_n J_1(2\lambda_n)} \cos(\lambda_n \tau),$$

where λ_n is the nth positive root of

$$J_0(2z) = 0.$$

The exact solution using dimensionless parameters is then

$$h(r, \tau) = -\sum_{n=1}^{\infty} \frac{J_0(\lambda_n r)}{J_1(2\lambda_n)} \left[\frac{\omega^* \sin(\lambda_n \tau) - \lambda_n \sin(\omega^* \tau)}{\lambda_n^2 - (\omega^*)^2}\right]. \tag{36}$$

As a practical matter, the summation in Eq. (36) must be truncated at some value of n, say $n_{\max}$. Using dimensional parameters, the solution is

$$y(x,t) = -\epsilon \sum_{n=1}^{\infty} \frac{J_0(2\lambda_n\sqrt{x/L})}{J_1(2\lambda_n)} \left[\frac{(\omega\sqrt{L/g}\,)\sin(\lambda_n\sqrt{g/L}\,t) - \lambda_n \sin(\omega t)}{(\lambda_n^2 - \omega^2(L/g))} \right].$$

Special Case Where $M \gg 1$

The third, and final verification exercise is for the special case were the ball is much heavier than the cable, $M \gg 1$. In this case, a second-order ordinary differential equation approximately describes the displacement of the ball, $y_p(t)$:

$$\frac{\mathrm{d}^2 y_\mathrm{p}}{\mathrm{d}t^2} + \left(\frac{g}{L}\right) y_\mathrm{p} = \epsilon \frac{g}{L} \sin(\omega t).$$

The initial conditions are

$$y_\mathrm{p}(0) = \dot{y}_\mathrm{p}(0) = 0.$$

The solution for this special case is simply

$$\lim_{M\to\infty} (y_\mathrm{p}) = \frac{\epsilon g}{g - L\omega^2} \left[\sin(\omega t) - \omega\sqrt{\frac{L}{g}} \sin\left(t\sqrt{\frac{g}{L}}\right) \right].$$

Using dimensionless parameters, this is equivalent to

$$\lim_{M\to\infty} (h_\mathrm{p}) = \frac{\sin(\omega^* \tau) - \omega^* \sin(\tau)}{1 - (\omega^*)^2}. \tag{37}$$

Test Problems

Two sample runs were performed to test Eq. (34) with the above special analytic solutions [Eqs. (35)–(37)].

The first test case was made with a very small value for M ($M = 10^{-6}$) for comparison purposes with Eqs. (35) and (36). For small

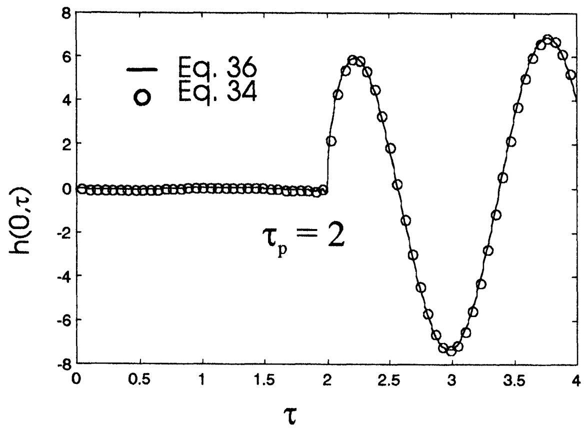

FIGURE 3 Test case no. 1 with $n_{max} = 100$, $\omega^* = 4$, and $M = 10^{-6}$.

values of M, the dimensionless propagation time is given by:

$$\lim_{M \to 0} \tau_p = 2.$$

Note in Fig. 3 that the dimensionless propagation time is very nearly $\tau_p = 2$. In addition, the displacement (at the lower end of the cable) predicted by Eq. (36) is very nearly that predicted by Eq. (34).

The second test case performed was with $M \gg 1$ which may be used to compare the predictions of Eq. (34) with those of Eq. (37). For this test the following parametric values were used: $M = 100$, $n_{max} = 100$, and $\omega^* = 4$ were used. The resulting displacement at the lower end of the cable is illustrated in Fig. 4.

For each of the above three criteria the propagation time, as well as the special cases where $M \ll 1$, and $M \gg 1$ the fit between Eq. (34) and the corresponding three analytic predictions was quite good.

CONCLUSION

An analytic solution has been presented which predicts the response of a ball suspended to a cable which is subject to a periodic forced oscillation. This solution may be used for verification purposes of numerical software which is intended to simulate the dynamic response of cables to forced oscillations.

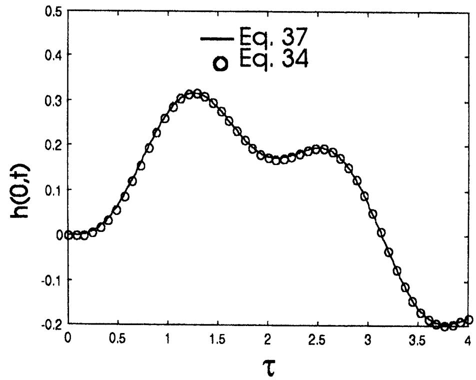

FIGURE 4 Test case no. 2 with $n_{\max} = 100$, $\omega^* = 4$, and $M = 100$.

NOMENCLATURE

g	gravitational field strength
L	length of the cable
$h(r, \tau)$	dimensionless displacement, $h(r, \tau) = y(x, t)/\epsilon$
m_c, m_p	mass of the cable and the ball respectively
M	mass ratio; $M = m_{\mathrm{p}}/m_{\mathrm{c}}$
r	dimensionless coordinate along the cable; $r = 2\sqrt{M + (x/L)}$
$R_n(r)$	function defined by Eq. (21)
$S_n(r)$	function defined by Eq. (24)
t_{p}	propagation time required for the initial displacement to travel to the ball
$T(x)$	tension in the cable
x	axis along the cable
$y(x, t)$	displacement of the cable
$y_{\mathrm{p}}(t)$	displacement for the ball if $M \gg 1$
ϵ	amplitude of oscillations at $x = L$
λ_n	nth positive root of Eq. (23)
τ	dimensionless time; $\tau = t\sqrt{g/L}$
ω	radial frequency of the forced oscillations
ω^*	dimensionless frequency of forced oscillations; $\omega^* = \omega\sqrt{L/g}$

References

[1] M. Abramowitz and I.A. Stegun, *Handbook of Mathematical Functions*, Dover Publications, New York, 1964.
[2] F.B. Hildebrand, *Advanced Calculus for Applications*, 2nd edn., Chapter 5, Prentice-Hall Publishing, Englewood Cliffs, NJ, 1976.

APPENDIX: PROOF THAT IF λ_n^2 IS NEGATIVE, THEN $R_n(r) = 0$, FOR CONTINUOUS REAL FUNCTIONS

Let μ_n be a positive number such that $\mu_n = -\lambda_n^2$. Multiplication of the Eq. (17) by $rR_n^2(r)$ yields

$$rR_n\left(R_n'' + \frac{1}{r}R_n'\right) = \mu_n r R_n^2; \tag{38}$$

thus

$$R_n \frac{\mathrm{d}}{\mathrm{d}r}\left(r\frac{\mathrm{d}R_n}{\mathrm{d}r}\right) = \mu_n r R_n^2.$$

Integration by parts from $2\sqrt{M}$ to $2\sqrt{M+1}$ results in the following:

$$\left[rR_n(r)\frac{\mathrm{d}R_n}{\mathrm{d}r}\right]\Bigg|_{2\sqrt{M}}^{2\sqrt{M+1}} - \int_{2\sqrt{M}}^{2\sqrt{M+1}} r\left(\frac{\mathrm{d}R_n}{\mathrm{d}r}\right)^2 \mathrm{d}r = \mu_n \int_{2\sqrt{M}}^{2\sqrt{M+1}} rR_n^2(r)\,\mathrm{d}r. \tag{39}$$

The first portion of the left side of Eq. (39) is

$$2\sqrt{M+1}\,R_n(2\sqrt{M+1})\frac{\mathrm{d}R_n}{\mathrm{d}r}\Bigg|_{r=2\sqrt{M+1}} - 2\sqrt{M}\,R_n(2\sqrt{M})\frac{\mathrm{d}R_n}{\mathrm{d}r}\Bigg|_{r=2\sqrt{M}}.$$

The boundary conditions imposed on $R_n(r)$ are

$$R_n(2\sqrt{M+1}) = 0$$

and Eq. (22)

$$\mu_n R_n(2\sqrt{M}) = \frac{1}{\sqrt{M}}\frac{\mathrm{d}R_n}{\mathrm{d}r}\Bigg|_{r=2\sqrt{M}}$$

or

$$\left.\frac{\mathrm{d}R_n}{\mathrm{d}r}\right|_{r=2\sqrt{M}} = \sqrt{M}\mu_n R_n(2\sqrt{M}).$$

Substituting these values into Eq. (39) yields

$$0 - 2\sqrt{M}R_n(2\sqrt{M})\,\overbrace{[\sqrt{M}\mu_n R_n(2\sqrt{M})]}^{(\mathrm{d}R_n/\mathrm{d}r)|_{r=2\sqrt{M}}}$$
$$= \int_{2\sqrt{M}}^{2\sqrt{M+1}} r\left(\frac{\mathrm{d}R_n}{\mathrm{d}r}\right)^2 \mathrm{d}r + \mu_n \int_{2\sqrt{M}}^{2\sqrt{M+1}} rR_n^2(r)\,\mathrm{d}r$$

or

$$-2M R_n^2(2\sqrt{m}) = \int_{2\sqrt{M}}^{2\sqrt{M+1}} r\left(\frac{\mathrm{d}R_n}{\mathrm{d}r}\right)^2 \mathrm{d}r + \int_{2\sqrt{M}}^{2\sqrt{M+1}} rR_n^2(r)\,\mathrm{d}r. \quad (40)$$

Clearly the right side of Eq. (40) is non-negative, for real values of R_n, and the left side is non-positive. Hence both sides of the equation must be zero:

$$\int_{2\sqrt{M}}^{2\sqrt{M+1}} r\left(\frac{\mathrm{d}R_n}{\mathrm{d}r}\right)^2 \mathrm{d}r + \int_{2\sqrt{M}}^{2\sqrt{M+1}} rR_n^2(r)\,\mathrm{d}r = 0. \quad (41)$$

For real continuous functions, Eq. (41) implies that $R_n(r) = 0$ if λ_n^2 is negative.